Ek-8.3- Glauert İntegrali

$$G_n = \int_0^{\pi} \frac{\cos(n\theta_0)\, d\theta_0}{\cos\theta_0 - \cos\theta}$$

Burada *n* pozitif bir tamsayı olup, *n* parametresinin çeşitli değerleri için çözüm incelenirse:

n=0 için çözüm

$$G_0 = \int_0^{\pi} \frac{d\theta_0}{\cos\theta_0 - \cos\theta}$$

$\theta_0 \to \theta$ için tekillik söz konusu olup limit alınarak:

$$G_0 = Lim_{\varepsilon\to 0} \int_0^{\theta-\varepsilon} \frac{d\theta_0}{\cos\theta_0 - \cos\theta} + Lim_{\varepsilon\to 0} \int_{\theta+\varepsilon}^{\pi} \frac{d\theta_0}{\cos\theta_0 - \cos\theta}$$

$$G_0 = Lim_{\varepsilon\to 0} \left[\frac{1}{\sin\theta} Ln \frac{\sin\frac{\theta+\theta_0}{2}}{\sin\frac{\theta-\theta_0}{2}} \right]_0^{\theta-\varepsilon} + Lim_{\varepsilon\to 0} \left[\frac{1}{\sin\theta} Ln \frac{\sin\frac{\theta+\theta_0}{2}}{\sin\frac{\theta-\theta_0}{2}} \right]_{\theta+\varepsilon}^{\pi}$$

$$G_0 = Lim_{\varepsilon\to 0} \left[\frac{1}{\sin\theta} \left(Ln \frac{\sin\frac{\theta+\theta-\varepsilon}{2}}{\sin\frac{\theta-\theta+\varepsilon}{2}} - Ln \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}} \right) \right] + Lim_{\varepsilon\to 0} \left[\frac{1}{\sin\theta} \left(Ln \frac{\sin\frac{\theta+\pi}{2}}{\sin\frac{\theta-\pi}{2}} - Ln \frac{\sin\frac{\theta+\theta+\varepsilon}{2}}{\sin\frac{\theta-\theta-\varepsilon}{2}} \right) \right]$$

$$G_0 = Lim_{\varepsilon\to 0} \left\{ \frac{1}{\sin\theta} \left[Ln \frac{\sin\left(\theta-\frac{\varepsilon}{2}\right)}{\sin\frac{\varepsilon}{2}} + Ln \frac{\cos\frac{\theta}{2}}{-\cos\frac{\theta}{2}} - Ln \frac{\sin\left(\theta+\frac{\varepsilon}{2}\right)}{-\sin\frac{\varepsilon}{2}} \right] \right\}$$

Logaritma ifadeleri

$$G_0 = Lim_{\varepsilon\to 0} \left\{ \frac{Ln\left[\sin\left(\theta-\frac{\varepsilon}{2}\right)\right] - Ln\left(\sin\frac{\varepsilon}{2}\right) + Ln\left(\cos\frac{\theta}{2}\right) - Ln\left(-\cos\frac{\theta}{2}\right) - Ln\left[\sin\left(\theta+\frac{\varepsilon}{2}\right)\right] + Ln\left(-\sin\frac{\varepsilon}{2}\right)}{\sin\theta} \right\}$$

şeklinde açılıp yeniden

$$G_0 = Lim_{\varepsilon\to 0} \left\{ \frac{1}{\sin\theta} \left[Ln \frac{\sin\left(\theta-\frac{\varepsilon}{2}\right)}{\sin\left(\theta+\frac{\varepsilon}{2}\right)} + Ln \frac{\left(\cos\frac{\theta}{2}\right)\left(-\sin\frac{\varepsilon}{2}\right)}{\left(-\cos\frac{\theta}{2}\right)\left(\sin\frac{\varepsilon}{2}\right)} \right] \right\}$$

şeklinde birleştirilirse, buradaki ikinci terimin değeri sıfır olup, bu durumda limit alınarak

$$G_0 = Lim_{\varepsilon\to 0} \left\{ \frac{1}{\sin\theta} \left[Ln \frac{\sin\left(\theta-\frac{\varepsilon}{2}\right)}{\sin\left(\theta+\frac{\varepsilon}{2}\right)} \right] \right\} = \left\{ \frac{1}{\sin\theta} \left[Ln \frac{\sin\theta}{\sin\theta} \right] \right\} = \left\{ \frac{1}{\sin\theta} [Ln 1] \right\} \quad \to \quad \boxed{G_0 = 0}$$

elde edilir.

n=1 için çözüm

$$G_1 = \int_0^{\pi} \frac{\cos\theta_0 \, d\theta_0}{\cos\theta_0 - \cos\theta}$$

$$G_1 = \int_0^{\pi} 1 + \frac{\cos\theta \, d\theta_0}{\cos\theta_0 - \cos\theta} = \int_0^{\pi} d\theta_0 + \cos\theta \int_0^{\pi} \frac{d\theta_0}{\cos\theta_0 - \cos\theta} = \pi + \cos\theta \, G_0 \quad \rightarrow \quad \boxed{G_1 = \pi}$$

n>1için genel çözüm

$$G_{n+1} + G_{n-1} = \int_0^{\pi} \frac{\cos[(n+1)\theta_0] + \cos[(n-1)\theta_0]}{\cos\theta_0 - \cos\theta} d\theta_0 = \int_0^{\pi} \frac{2\cos(n\theta_0)\cos\theta_0}{\cos\theta_0 - \cos\theta} d\theta_0$$

$$G_{n+1} + G_{n-1} = 2\int_0^{\pi} \cos(n\theta_0) \, d\theta_0 + 2\cos\theta \int_0^{\pi} \frac{\cos(n\theta_0)}{\cos\theta_0 - \cos\theta} d\theta_0 = 2\cos\theta \, G_n$$

$$G_{n+1} = 2\cos\theta \, G_n - G_{n-1}$$

n=1 alınarak $\quad G_2 = 2\cos\theta \, G_1 - G_0 = 2\cos\theta \, \pi = \frac{2\cos\theta \sin\theta}{\sin\theta}\pi \quad \rightarrow \quad \boxed{G_2 = \frac{\sin 2\theta}{\sin\theta}\pi}$

n=2 alınarak $\quad G_3 = 2\cos\theta \, G_2 - G_1 = 2\cos\theta \frac{\sin 2\theta}{\sin\theta}\pi - \pi = \frac{2\cos\theta \sin 2\theta - \sin\theta}{\sin\theta}\pi$

Ayrıca

$$\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos\theta + \sin\theta \cos 2\theta$$
$$\sin\theta = \sin(2\theta - \theta) = \sin 2\theta \cos\theta - \sin\theta \cos 2\theta$$

$$\sin 3\theta + \sin\theta = 2\sin 2\theta \cos\theta$$

olup yukarıda kullanılarak $\quad \boxed{G_3 = \frac{\sin 3\theta}{\sin\theta}\pi}$

n=3 alınarak $\quad G_4 = 2\cos\theta \, G_3 - G_2 = 2\cos\theta \frac{\sin 3\theta}{\sin\theta}\pi - \frac{\sin 2\theta}{\sin\theta}\pi = \frac{2\cos\theta \sin 3\theta - \sin 2\theta}{\sin\theta}\pi$

Ayrıca

$$\sin 4\theta = \sin(3\theta + \theta) = \sin 3\theta \cos\theta + \sin\theta \cos 3\theta$$
$$\sin 2\theta = \sin(3\theta - \theta) = \sin 3\theta \cos\theta - \sin\theta \cos 3\theta$$

$$\sin 4\theta + \sin 2\theta = 2\sin 3\theta \cos\theta$$

olup yukarıda kullanılarak $\quad \boxed{G_4 = \frac{\sin 4\theta}{\sin\theta}\pi}$

Bu işlemlerin aynı şekilde devam ettirilebileceği görülmektedir. Bu durumda bir genelleştirme yapılarak

$$\boxed{G_n = \int_0^{\pi} \frac{\cos(n\theta_0) \, d\theta_0}{\cos\theta_0 - \cos\theta} = \frac{\sin(n\theta)}{\sin\theta}\pi}$$

olacağı belirtilebilir.