

# ELE617E

## Lectures

Prof. Dr. Müştak E. Yalçın

Istanbul Technical University

mustak.yalcin@itu.edu.tr

# Algorithmic Strength Reduction

- It leads to a reduction in HW complexity by exploiting **substructure sharing**,
- reduce area or power,
- or iteration periode.

L-parallel circuit requires an  $L \times$  Area!

Question: to realize parallel FIR filtering structure that consume less area.

$$y(n) = h(n) \star x(n) \rightarrow Y(z) = H(z)X(z)$$

$$\begin{aligned} X(z) &= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + \dots \\ &= x(0) + x(2)z^{-2} + x(4)z^{-4} + \dots + z^{-1}\{x(1) + x(3)z^{-2} + x(5)z^{-4} + \dots\} \\ &= X_0(z^2) + z^{-1}X_1(z^2) \end{aligned}$$

where  $X_0(z^2) = \mathcal{Z}(x(2k))$  and  $X_1(z^2) = \mathcal{Z}(x(2k+1))$ .  $X(z)$  is decomposed into **two** poly phases.

$$H(z) = H_0(z^2) + z^{-1}H_1(z^2)$$

$$H_0(z) = h(0) + h(2)z^{-2} + h(4)z^{-4} + \dots$$

$$H_1(z) = h(1) + h(3)z^{-2} + h(5)z^{-4} + \dots$$

$$\begin{aligned} Y(z) &= (X_0(z^2) + z^{-1}X_1(z^2))(H_0(z^2) + z^{-1}H_1(z^2)) \\ &= \{X_0(z^2)H_0(z^2) + z^{-2}X_1(z^2)H_1(z^2)\} + \{X_0(z^2)H_1(z^2) + X_1(z^2)H_0(z^2)\} \end{aligned}$$

# Parallel FIR Filters

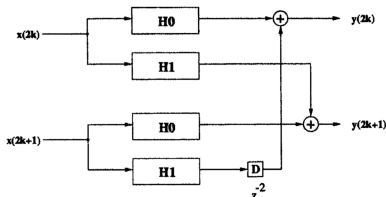
$$Y(z) = \{X_0(z^2)H_0(z^2) + z^{-2}X_1(z^2)H_1(z^2)\} + \{X_0(z^2)H_1(z^2) + X_1(z^2)H_0(z^2)\}$$

$$Y(z) = Y_0(z^2) + z^{-1}Y_1(z^2)$$

$$Y_0(z^2) = X_0(z^2)H_0(z^2) + z^{-2}X_1(z^2)H_1(z^2)$$

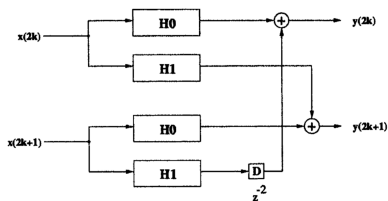
$$Y_1(z^2) = X_0(z^2)H_1(z^2) + X_1(z^2)H_0(z^2)$$

$$\begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix} = \begin{bmatrix} H_0 & z^{-2}H_1 \\ H_1 & H_0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \end{bmatrix}$$





$$\begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix} = \begin{bmatrix} H_0 & z^{-2}H_1 \\ H_1 & H_0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \end{bmatrix}$$



- 2-parallel FIR filtering structure.
- 2N MUL and 2(N-1) ADD

## L-parallel FIR

$$\begin{bmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{L-1} \end{bmatrix} = \begin{bmatrix} H_0 & z^{-L}H_{L-1} & \dots & z^{-L}H_1 \\ H_1 & H_0 & \dots & z^{-L}H_2 \\ \vdots & \vdots & \ddots & \vdots \\ H_{L-1} & H_{L-2} & \dots & H_0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{L-1} \end{bmatrix}$$

Polyphase Filters:

Filter	MUL	ADD	sub-filter
FIR	N	N-1	1
2-Parallel	2N	2(N-1)	4
3-Parallel	3N	3(N-1)	9
L-Parallel	LN	L(N-1)	L <sup>2</sup>

However, they give L samples each cycle

See 3-parallel FIR filter implementation Fig. 9.2 (page 259)

# Fast FIR Filters

$$Y(z) = Y_0(z^2) + z^{-1} Y_1(z^2)$$

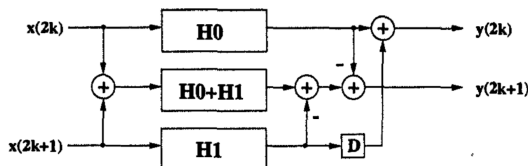
$$Y_0(z^2) = X_0(z^2)H_0(z^2) + z^{-2}X_1(z^2)H_1(z^2)$$

$$Y_1(z^2) = X_0(z^2)H_1(z^2) + X_1(z^2)H_0(z^2)$$
$$= \{X_0(z^2) + X_1(z^2)\} \{H_0(z^2) + H_1(z^2)\} - X_0(z^2)H_0(z^2) - X_1(z^2)H_1(z^2)$$

PS:  $Y_1 = f(X_0, X_1, H_0, H_1, H_0 + H_1, X_0 + X_1)$

Filters:  $H_0 = \{h_0, h_2, h_4, h_6\}$ ,  $H_1 = \{h_1, h_3, h_5, h_7\}$ ,

$H_0 + H_1 = \{h_0 + h_1, h_2 + h_3, h_4 + h_5, h_6 + h_7\}$



$D = z^{-2}$ ,  $3N/2$  MUL and  $3(N/2 - 1) + 4$  ADD operation.

# Fast FIR Filters

$$Y(z) = Y_0(z^2) + z^{-1}Y_1(z^2)$$

$$Y_0(z^2) = X_0(z^2)H_0(z^2) + z^{-2}X_1(z^2)H_1(z^2)$$

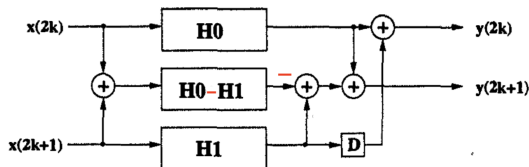
$$Y_1(z^2) = X_0(z^2)H_1(z^2) + X_1(z^2)H_0(z^2)$$
$$= X_0(z^2)H_0(z^2) + X_1(z^2)H_1(z^2) - \{X_0(z^2) - X_1(z^2)\}\{H_0(z^2) - H_1(z^2)\}$$

PS:  $Y_1 = f(X_0, X_1, H_0, H_1, H_0 - H_1, X_0 - X_1)$

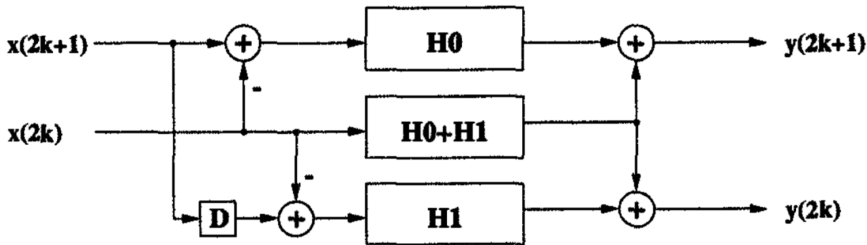
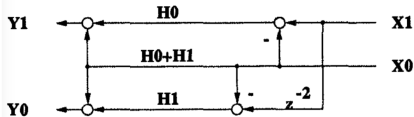
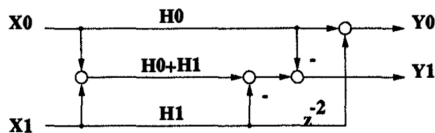
Filters:

$$H_0 = \{h_0, h_2, h_4, h_6\}, H_1 = \{h_1, h_3, h_5, h_7\},$$

$$H_0 + H_1 = \{h_0 - h_1, h_2 - h_3, h_4 - h_5, h_6 - h_7\}$$



# Transpose of Fast FIR

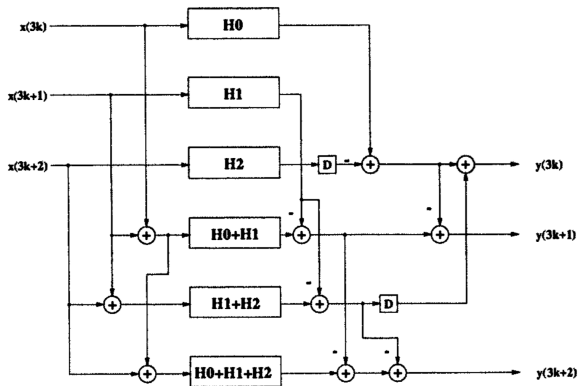


Transposition does not change the complexity of the filter algorithm.

$$Y_0 = X_0 H_0 - z^{-3} X_2 H_2 + z^{-3} [(H_1 + H_2)(X_1 + X_2) - H_1 X_1]$$

$$Y_1 = [(H_0 + H_1)(X_0 + X_1) - H_1 X_1] - [H_0 X_0 - z^{-3} H_2 X_2]$$

$$Y_3 = \dots$$



6(N/3) MUL and 6(N/3 - 1) + 10 ADD operation and %33,3 saving !

## Comparison Polyphase Filters

Filter	MUL	ADD	Reduced	
			MUL	ADD
FIR	N	N-1		
2-Parallel	2N	2(N-1)	3N/2	3N/2+1
3-Parallel	3N	3(N-1)	2N	2N+4
4-Parallel	4N	4(N-1)	9 N/4	20+9(N-1)

for 4-parallel case : MUL 4N to 9 N/4 hence %xxx

# 4-parallel FFA

Consider a 2-parallel FFA with a 2-parallel FFA.

$$X = X_0 + z^{-1}X_1 + z^{-2}X_2 + z^{-3}X_3$$

$$H_2 = H_0 + z^{-1}H_1 + z^{-2}H_2 + z^{-3}H_3$$

$$X = X_0 + z^{-2}X_2 + z^{-1}(X_1 + z^{-1}X_3) = X_0' + z^{-1}X_1'$$

$$H = H_0' + z^{-1}H_1'$$

$$HX = (H_0' + z^{-1}H_1')(X_0' + z^{-1}X_1') \\ = H_0'X_0' + z^{-2}H_1'X_1' + z^{-1}(H_0' + H_1')(X_0' + X_1') - X_0'H_0' - H_1'X_1'$$

$$H_0'X_0' = (H_0' + z^{-2}H_2)(X_0 + z^{-2}X_2) \\ = H_0X_0 + z^{-4}H_2X_2 + z^{-2}\{ (H_0 + H_2)(X_0 + X_2) - H_0X_0 - H_2X_2 \}$$

$$H_1'X_1' = H_1X_1 + z^{-1}H_3X_3 + z^{-1}\{ (H_1 + H_3)(X_1 + X_3) - H_1X_1 - H_3X_3 \}$$

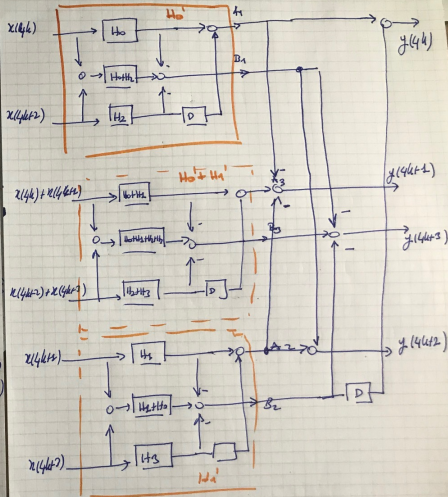
$$(H_0 + H_2)(X_0 + X_2) = (H_0 + H_2) + z^{-2}(H_2 + H_0) \{ (X_0 + X_2) + z^{-2}(X_2 + X_0) \} \\ = (H_0 + H_2)(X_0 + X_2) + z^{-2}\{ (H_0 + H_2)(X_2 + X_0) + (H_2 + H_0)(X_0 + X_2) \} \\ - (H_0 + H_2)(X_0 + X_2) - (H_2 + H_0)(X_2 + X_0)$$

$$H_0'X_0' = A_1 + z^{-2}B_1$$

$$H_1'X_1' = A_2 + z^{-2}B_2$$

$$(H_0 + H_2)(X_0 + X_2) = A_3 + z^{-2}B_3$$

$$HX = A_1 + z^{-2}B_1 + z^{-2}(A_2 + z^{-2}B_2) + z^{-1}(A_3 + z^{-2}B_3) - (A_1 + z^{-2}B_1) - (A_2 + z^{-2}B_2) \\ = A_1 + z^{-1}B_1 + z^{-1}A_2 + z^{-3}B_2 + z^{-1}A_3 + z^{-3}B_3 - A_1 - z^{-2}B_1 - A_2 - z^{-2}B_2 \\ = (A_1 + z^{-4}B_2) + z^{-1}(A_3 - A_1 - A_2) + z^{-2}(A_2 + B_2) + z^{-3}(B_3 - B_1 - B_2)$$





# 6-parallel FFA

$$X = X_0' z^{-1} A_1 e$$

$$H = H_0' z^{-2} A_2 A_1 e$$

$$X_0' = X_0 + z^{-1} X_1 + z^{-2} X_2 + \dots$$

$$X_1' = X_1 + z^{-1} X_2 + z^{-2} X_3 + \dots$$

$$H_0' = (H_0 + z^{-1} H_1) (X_0' z^{-1} A_1 e)$$

$$= H_0' X_0' + z^{-1} H_1 X_0' + z^{-2} H_2 (X_0 + X_1)$$

$$H_1' = (H_1 + z^{-1} H_2) (X_0 + z^{-1} X_1) (X_0 + z^{-1} X_1)$$

$$= (H_1 + z^{-1} H_2) (X_0 + z^{-1} X_1)$$

$$H_2' = H_2 X_0 + z^{-1} H_3 (X_0 + X_1) + z^{-2} (H_2 + H_3) (X_0 X_1) + z^{-3} (H_2 X_1 + H_3 X_0)$$

$$VW = (H_0 + z^{-1} H_1) (X_0 z^{-1} A_1 e)$$

$$= H_0 X_0 + z^{-1} H_1 X_0 + z^{-2} (H_2 + H_3) (X_0 + X_1) + z^{-3} (H_2 X_1 + H_3 X_0)$$

$$H_0 X_0' = A_1 + A_2 z^{-1} + A_3 z^{-2}$$

Fig. 9.4

$$X_1' \Rightarrow \begin{matrix} H_1' \\ H_2' \\ \vdots \end{matrix}$$

$$H_1' X_1' = (H_1 + z^{-1} H_2 + H_3 + H_4) (X_1 + z^{-1} X_2 + z^{-2} X_3)$$

$$H_1' X_1' = B_1 + z^{-1} B_2 + B_3 z^{-2}$$

$$(H_0 + H_1) (X_0' + X_1) = (H_0 + z^{-1} H_1) z^{-1} (H_0 + z^{-1} H_1) z^{-1} (X_0 + z^{-1} X_1) (X_0 + z^{-1} X_1)$$

$$= (H_0 + H_1) z^{-2} (H_2 + H_3) z^{-2} (H_4 + H_5) (\dots)$$

$$X(\omega) z^{-1} \Rightarrow \begin{matrix} H_0 + H_1 \\ H_2 + H_3 \\ H_4 + H_5 \end{matrix} \rightarrow \begin{matrix} B_1 \\ B_2 \\ B_3 \end{matrix}$$

Fig. 9.4.  $(H_0 + H_1) (\dots) = C_1 + C_2 z^{-1} + C_3 z^{-2}$

$$H_2' = H_2 X_0' + z^{-1} H_3 (X_0 + X_1) + z^{-2} (H_2 X_1 + H_3 X_0)$$

$$= H_2 X_0' + z^{-1} (H_2 X_1 + H_3 X_0) + z^{-2} (H_2 X_1 + H_3 X_0) + z^{-3} (H_2 X_1 + H_3 X_0)$$

$$= H_2 X_0' + z^{-1} (H_2 X_1 + H_3 X_0) + z^{-2} (H_2 X_1 + H_3 X_0) + z^{-3} (H_2 X_1 + H_3 X_0)$$

$$Y_0 = A_2 z^{-1} B_1$$

$$Y_1 = C_1 + C_2 z^{-1}$$