Example 6.4 Butterworth Step Invariant Filter Design

Problem: Determine the step invariant digital filter for a second-order normalized Butterworth approximation function, as shown below. Determine the differences that result for sampling periods of $T = 1.0$ sec and $T = 0.1$ sec.

$$H(s) = \frac{1}{s^2 + 1.4142 \cdot s + 1}$$

Solution: Again, we first determine the output of the analog system to an input step function and then use partial fraction expansion to determine the individual terms.

$$G(s) = \frac{1}{s} - \frac{0.5 + j0.5}{s + 0.7071 + j0.7071} - \frac{0.5 - j0.5}{s + 0.7071 - j0.7071}$$

The step response can then be determined by finding the inverse Laplace transform of $G(s)$, as indicated below:

$$g(t) = 1.0 - (0.5 + j0.5) \cdot e^{-(0.7071 + j0.7071)t} - (0.5 - j0.5) \cdot e^{-(0.7071 - j0.7071)t}$$

Then, after sampling at intervals of T, the discrete-time step response is determined to be

$$g(nT) = 1.0 - (0.5 + j0.5) \cdot e^{-(0.7071 + j0.7071)nT} - (0.5 - j0.5) \cdot e^{-(0.7071 - j0.7071)nT}$$

The discrete-time response to the step input can then be determined by using the z-transform.

$$G(z) = \frac{1}{1 - z^{-1}} - \frac{0.5 + j0.5}{1 - e^{(0.7071 + j0.7071)T} z^{-1}} - \frac{0.5 - j0.5}{1 - e^{(0.7071 - j0.7071)T} z^{-1}}$$

Now, by combining these terms over a common denominator (and performing a considerable amount of complex algebra), we have the system response to a step input:

$$G(z) = \frac{z^{-1} + e^{-0.707T}[(\sin(0.707 \cdot T) - \cos(0.707 \cdot T)] \cdot z^{-1}}{(1 - z^{-1}) \cdot [1 - 2 \cdot e^{-0.707 \cdot T} \cos(0.707T) \cdot z^{-1} + e^{-1.414 \cdot T} z^{-2}]}$$

$$+ \frac{e^{-1.414T} z^{-2} - e^{-0.707T}[\sin(0.707 \cdot T) + \cos(0.707 \cdot T)] \cdot z^{-2}}{(1 - z^{-1}) \cdot [1 - 2 \cdot e^{-0.707 \cdot T} \cos(0.707 \cdot T) \cdot z^{-1} + e^{-1.414 \cdot T} z^{-2}]}$$

The discrete-time transfer function $H(z)$ can now be determined by removing the $(1 - z^{-1})$ factor relating to the step input, and we can make the substitution of the different sampling periods in the general form to find the two distinct transfer functions:

$$H(z) = \frac{z^{-1} + e^{-0.707 \cdot T}[(\sin(0.707 \cdot T) - \cos(0.707 \cdot T)] \cdot z^{-1}}{[1 - 2 \cdot e^{-0.707 \cdot T} \cos(0.707 \cdot T) \cdot z^{-1} + e^{-1.414 \cdot T} z^{-2}]}$$

$$+ \frac{e^{-1.414 \cdot T} z^{-2} - e^{-0.707 \cdot T}[\sin(0.707 \cdot T) + \cos(0.707 \cdot T)] \cdot z^{-2}}{[1 - 2 \cdot e^{-0.707 \cdot T} \cos(0.707 \cdot T) \cdot z^{-1} + e^{-1.414 \cdot T} z^{-2}]}$$

$$H(z)\big|_{T=1.0} = \frac{0.94546 \cdot z^{-1} - 0.45205 \cdot z^{-2}}{1 - 0.74971 \cdot z^{-1} + 0.24312 \cdot z^{-2}}$$

$$H(z)\big|_{T=0.1} = \frac{0.13643 \cdot z^{-1} - 0.12711 \cdot z^{-2}}{1 - 1.8588 \cdot z^{-1} + 0.86812 \cdot z^{-2}}$$

Note that the pole locations are the same as for the impulse invariant design but that the zero locations have changed. We can compare the frequency responses of these two discrete-time filters as shown in Figure 6.2.
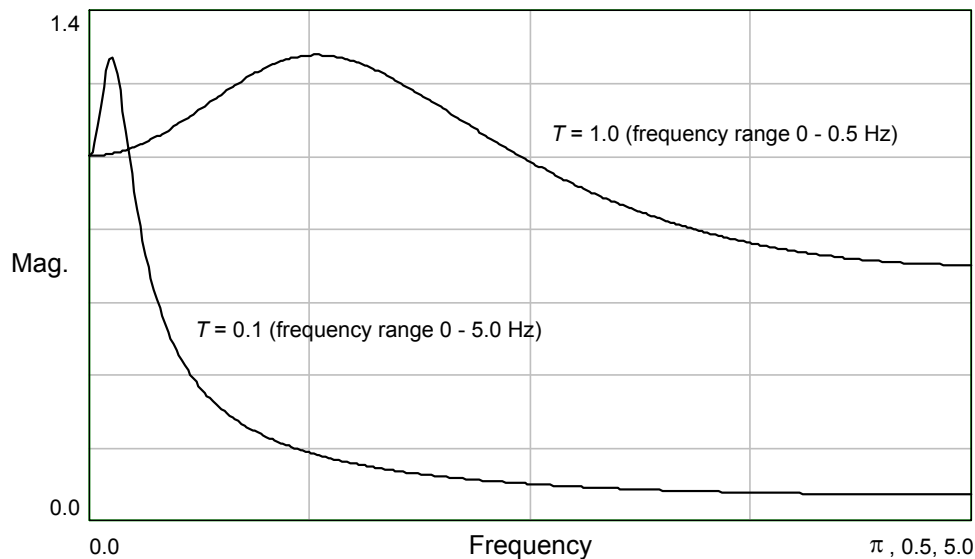

Figure 6.2 Frequency responses for Example 6.4.

As we can see, there is significant difference between the two implementations, but the differences are again a result of the frequency axis having two different scales. In the step invariant design method, there is no need