# Circuit and System Analysis
EHB 232E

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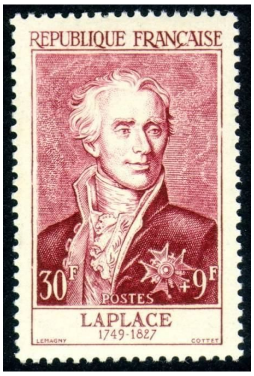
# Outline I

1. Laplace Transform in Circuit Analysis
    - Laplace Transform
    - Inverse Laplace Transform
    - Analysis of state space equation

# Why Laplace Transform




Laplace transform ($F(s)$) of $f(t)$ function is given by

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t) dt.$$

The Laplace transform converts linear differential equations into algebraic equations. These are linear equations with polynomial coefficients. The solution of these linear equations therefore leads to rational function expressions for the variables involved.

Initial Conditions, Generalized Functions, and the Laplace Transform, by Kent H. Lundberg; Haynes R. Miller ; David L. Trumper, IEEE Control Systems Link

# Laplace Transform

| Signal | Laplace Transform |
|---|---|
| $\delta(t)$ | $1$ |
| $u(t)$ | $\frac{1}{s}$ |
| $tu(t)$ | $\frac{1}{s^2}$ |
| $\frac{t^n}{n!}u(t)$ | $\frac{1}{s^{n+1}}$ |
| $e^{-at}u(t)$ | $\frac{1}{s+a}$ |
| $\cos(at)$ | $\frac{s}{s^2+a}$ |
| $\sin(at)$ | $\frac{a}{s^2+a}$ |
| $2\|k\|e^{-\alpha t}\cos(\beta t+\theta)$ | $\frac{k}{(s+\alpha-j\beta)}+\frac{\hat{k}}{(s+\alpha+j\beta)}$ |

# Properties of the Laplace Transform

- $$\mathcal{L}\left\{k_1 f_1(t) + k_2 f_2(t)\right\} = k_1 F_1(s) + k_2 F_2(s)$$

- $$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$$

  and

  $$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f(0) - \ldots - f(0)$$

- $$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{F(s)}{s}$$

- $$\mathcal{L}\left\{f(t - t_0) u(t - t_0)\right\} = F(s) e^{-t_0 s}$$

# $\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$

## Distinguishing $0^-$, 0, and $0^+$

While the unilateral Laplace transform is frequently associated with the solution of differential equations, the need to clearly distinguish $0^-$, 0, and $0^+$ is independent of any dynamic-systems context. For a discontinuous function $f(t)$, the derivative $f'(t)$ must be interpreted as the generalized derivative of $f(t)$, which includes the singularity function

$$(f(t_0^+) - f(t_0^-))\delta(t - t_0)$$

at every point $t_0$ at which $f(t)$ is discontinuous. In particular, if $f(0^-) \neq f(0^+)$, then the derivative includes a delta function at the origin.

In the following example, adapted from Problem 11.17 of [29], we apply the unilateral transform to three signals and their derivatives. This example clarifies that the need for using the Laplace transform (3) and properties (4) and (5) is really a matter of properly defining signals and their transforms, and is not fundamentally connected to the solution of differential equations.

Consider the three signals $f(t)$, $g(t)$, and $h(t)$ defined for all time

$$f(t) = e^{-at},$$
$$g(t) = e^{-at}u(t),$$
$$h(t) = e^{-at}u(t) - u(-t),$$

which are plotted for the value $a = 1$ in Figure S1. Since all three functions are nonsingular and agree for positive time they have the same Laplace transform by means of (3). However, their derivatives include different impulse values at $t = 0$, and thus the Laplace transforms of their derivatives must differ. Our choice of Laplace transform properties should give consistent and correct results when operating on these signals and their derivatives. The associated transforms are calculated below to show that this consistency is found. We also demonstrate the initial-value theorem in the context of these signals.

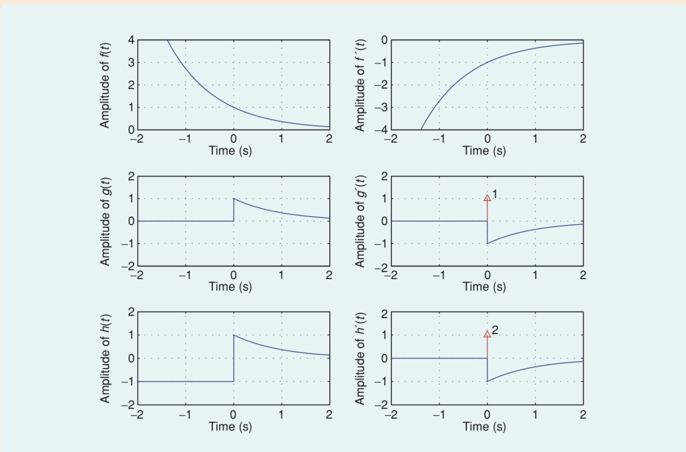



FIGURE S1 Three functions $f(t) = e^{-at}$, $g(t) = e^{-at}u(t)$, $h(t) = e^{-at}u(t) - u(-t)$, and their derivatives, plotted for $a = 1$ and defined for all time. Impulses are represented by the red arrows, with the impulse area denoted by the number next to the arrowhead. Since all three functions are identical for positive time, they have identical unilateral Laplace transforms. However, their derivatives differ at the origin. Therefore, the Laplace transforms of their derivatives also differ.

**PROPERTIES OF $f(t)$**

Consider the function $f(t) = e^{-at}$ with the pre-initial value $f(0^-) = 1$. The Laplace transform of $f(t)$ is

$$F(s) = \mathcal{L}_-\{e^{-at}\} = \frac{1}{s+a}.$$

The time derivative of $f(t)$ is

$$f'(t) = -ae^{-at},$$

and the Laplace transform of the time derivative is

$$\mathcal{L}_-\{-ae^{-at}\} = \frac{-a}{s+a}. \quad \text{(S1)}$$

Using the derivative rule (4)

$$\mathcal{L}_-\{f'(t)\} = sF(s) - f(0^-) = \frac{s}{s+a} - 1 = \frac{-a}{s+a}$$

produces the same result. The results from the initial-value theorem are

$$f(0^+) = \lim_{s\to\infty\cdot 1} \frac{s}{s+a} = 1$$

and

$$f'(0^+) = \lim_{s\to\infty\cdot 1} \frac{-sa}{s+a} = -a.$$

The time-domain and Laplace-domain calculations are thus consistent.

**PROPERTIES OF $g(t)$**

The function $g(t) = e^{-at}u(t)$ has an associated pre-initial value $g(0^-) = 0$. The Laplace transform of $g(t)$ is the same as for $f(t)$, namely,

$$G(s) = \mathcal{L}_-\{e^{-at}u(t)\} = \frac{1}{s+a}.$$

However, the time derivative now includes an impulse at the origin

$$g'(t) = \delta(t) - ae^{-at}u(t).$$

The Laplace transform of this time derivative is

$$\mathcal{L}_-\{g'(t)\} = 1 - \frac{a}{s+a} = \frac{s}{s+a},$$

which is different from the result (S1). Using the derivative rule (4),

$$\mathcal{L}_-\{g'(t)\} = sG(s) - g(0^-) = \frac{s}{s+a} - 0 = \frac{s}{s+a}, \quad \text{(S2)}$$

we obtain the correct result. The initial-value theorem gives

$$g(0^+) = \lim_{s\to\infty\cdot 1} \frac{s}{s+a} = 1,$$

producing the value at $t = 0^+$. We can also apply the more general initial-value theorem to the transform of the derivative as follows. Expanding out the nonsingular part of the transform gives

$$G(s) = 1 - \frac{a}{s+a} \equiv 1 + \tilde{G}(s),$$

and thus

$$g'(0^+) = \lim_{s\to\infty\cdot 1} s\tilde{G}(s) = -a,$$

which is the correct value.

**PROPERTIES OF $h(t)$**

Finally consider the function

$$h(t) = e^{-at}u(t) - u(-t) = \begin{cases} -1, & t < 0, \\ e^{-at}, & t > 0, \end{cases}$$

which has an associated pre-initial value $h(0^-) = -1$. The Laplace transform of this signal is the same as the Laplace transforms of $f$ and $g$. Computing the time derivative gives

$$h'(t) = 2\delta(t) - ae^{-at}u(t).$$

The Laplace transform of this time derivative is

$$\mathcal{L}_-\{h'(t)\} = 2 - \frac{a}{s+a} = \frac{2s+a}{s+a},$$

which is different from the results (S1) and (S2) above. Using the derivative rule (4),

$$\mathcal{L}_-\{h'(t)\} = sH(s) - h(0^-) = \frac{s}{s+a} + 1 = \frac{2s+a}{s+a}$$

gives the correct result. Finally, the initial-value theorem gives a correct result for both $h$ and its derivative, $h(0^+) = 1$ and $h'0^+) = -a$, although we don't show the details here.

The formulas (3), (4), and (5) give consistent results. We hope that the signal examples presented above help to clarify the need for and application of these formulas.

## Properties of the Laplace Transform

- $$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a)$$
- $$\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n F(s)}{ds^n}$$

Inverse Laplace Transform

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = \int_{c-j\infty}^{c+j\infty} e^{st}F(s)ds$$

# Partial-Fraction Expansion Method

s-transform of a linear time invariant system is often of the form $(n > m)$

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0}$$

which is ratio of two polynomials. The value(s) for $s$ where $P(s) = 0$ are called *zeros*. The value(s) for $s$ where $Q(s) = 0$ are called *poles*.

If $s_{pi} \neq s_{pj}$, poles distinct.

if $\lim_{s \to \infty} F(s)(s - s_{pi}) = \infty$ and $\lim_{s \to \infty} F(s)(s - s_{pi})^k$ is constant then $s = s_{pi}$ is a k-multiple pole.

Lets assume that poles distinct

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{k_1}{(s-s_{p1})} + \frac{k_2}{(s-s_{p2})} \dots + \frac{k_n}{(s-s_{pn})}\right\}$$

$k_i$ is the residue located at the corresponding pole $s_{pi}$ which is

$$k_i = F(s)\,(s-s_{pi})\big|_{s=s_{pi}}$$

$$\mathcal{L}^{-1}\{F(s)\} = k_1 e^{s_{p1}t}u(t) + k_2 e^{s_{p2}t}u(t) + \dots + k_n e^{s_{pn}t}u(t)$$

$$\mathcal{L}^{-1}\{k_0 + F(s)\} = k_0\delta(t) + k_1 e^{s_{p1}t}u(t) + k_2 e^{s_{p2}t}u(t) + \dots + k_n e^{s_{pn}t}u(t)$$

$$Y(s) = \frac{-3s^2 + 23s - 38}{(s-1)(s-2)(s-3)} = \frac{-9}{s-1} + \frac{2}{s-2} + \frac{2}{s-3}$$

$$y(t) = -9e^t + 4e^{2t} + 2e^{3t} \;\; for \;\; t > 0$$

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$$Y(s) = \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} = \frac{1/6j}{s+1-j} - \frac{1/6j}{s+1+j} + \frac{1/6j}{s+1-2j} - \frac{}{s +}$$

$$y(t) = \frac{1}{3}e^{-t}\sin(t) + \frac{1}{3}e^{-t}\sin(2t) \;\; for \;\; t > 0$$

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$$\dot{x} = 2x - 3y, \quad x(0) = 8$$

$$\dot{y} = -2x + y, \quad y(0) = 3$$

Using Laplace transform

$$sX - 8 = 2X - 3Y \;\; \rightarrow \;\; (s-2)X + 3Y = 8$$

$$sY - 3 = -2x + y \;\; \rightarrow \;\; 2X + (s-1)Y = 3$$

$$X = \frac{8s - 17}{s^2 - 3s - 4} = \frac{5}{s+1} + \frac{3}{s-4} \longrightarrow x(t) = 5e^{-t} + 3e^{4t}$$

$Q(s)$ has a multiple pole.

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \mathcal{L}^{-1}\left\{\frac{k_{i1}}{(s-s_{pi})} + \frac{k_{i2}}{(s-s_{pi})^2} \dots + \frac{k_{ik}}{(s-s_{pi})^k} + \frac{P_1(s)}{Q_1(s)}\right\}$$

where

$$k_{ik} = F(s)\,(s-s_{pi})^k\Big|_{s=s_{pi}}$$

and

$$k_{il} = \frac{1}{(k-l)!}\,\frac{d^{k-l}F(s)(s-s_{pi})^k}{ds^{k-l}}\Bigg|_{s=s_{pi}}$$

for $l = 1, 2, \dots k-1$.

$$Y(s) = \frac{2s^2 - 25s - 33}{(s+1)^2(s-5)} = \frac{k_1}{s+1} + \frac{k_2}{(s+1)^2} + \frac{k}{s-5}$$

$$k = \frac{2s^2 - 25s - 33}{(s+1)^2} |_{s=5} = -3$$

$$k_1 = \frac{1}{1!}\frac{d}{ds}\frac{2s^2 - 25s - 33}{(s-5)} |_{s=-1} = 5$$

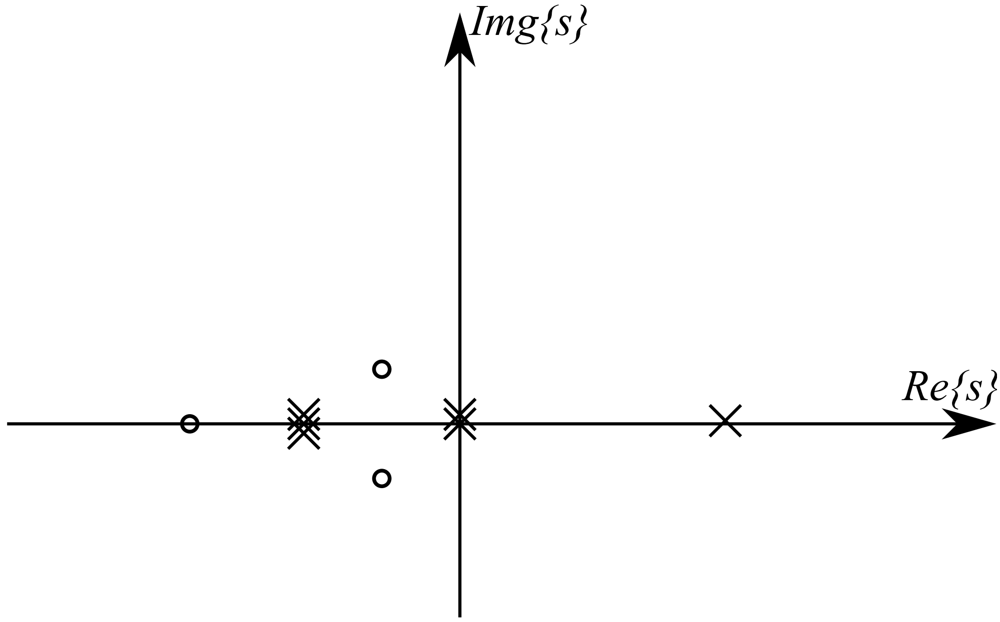
$$k_2 = \frac{2s^2 - 25s - 33}{(s-5)} |_{s=-1} = 1$$

$$y(t) = 5e^{-t} + te^{-t} - 3e^{5t}$$

# s-plane

$$F(s) = \frac{(s+3)(s^2+2s+2)}{s^2(s-4)(s+2)^2}$$

zeros "o" and poles "x" on s-plane:



# The Convolution Theorem

**The convolution operation:** Let $f_1(t)$ and $f_2(t)$ be functions defined on $[0, \infty)$ , and let us take them to be equal to zero for $t < 0$: The convolution of the time functions $f_1$ and $f_2$ is a new time function denoted by $(f_1 \star f_2)(t)$ and defined for all $t$ by

$$f(t) = f_1(t) \star f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau = \int_0^t f_2(t-\tau) f_2(\tau) d\tau$$

## The Convolution Theorem

Let $f_1(t)$ and $f_2(t)$ have $F_1(s)$ and $F_2(s)$ as Laplace transforms. We assume that for $i = 1, 2$, $f(t) = 0$ for $t < 0$. Laplace transform of the convolution of $f_1$ and $f_2$ is given by

$$\mathcal{L}\{f_1(t) \star f_2(t)\} = F_1(s) F_2(s)$$

Thus, the operation of convolution in the time domain is equivalent to multiplication in the frequency domain.

# Example

Using The Convolution Theorem, lets find inverse Laplace transform of $F(s) = \frac{1}{s^2(s+2)}$

$$\frac{1}{s^2(s+2)} = \frac{1}{s^2}\frac{1}{(s+2)} = \mathcal{L}\left\{tu(t) \star e^{-2t}u(t)\right\}$$

$$tu(t) \star e^{-2t}u(t) = \int_0^t \tau e^{-2(t-\tau)} d\tau$$

# Analysis of state space equation

Linear time invariant system

$$\begin{aligned} \dot{x} &= Ax + Be \\ y &= Cx + De \end{aligned}$$

where $x$ state variable and $y$ is output, $u$ is input vectors. Using Laplace transform

$$\begin{aligned} sX(s) - x(0) &= AX(s) + BE(s) \\ Y(s) &= CX(s) + DE(s) \end{aligned}$$

we have Laplace transform state variable

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BE(s)$$

Output

$$Y(s) = C(sI - A)^{-1}x(0) + (C(sI - A)^{-1}B + D)E(s)$$

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BE(s) = Q(s)x(0) + Q(s)BE(s)$$

where $Q(s) = (sI - A)^{-1}$ and $q(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$.
Using The Convolution Theorem, we have

$$Q(s)x(0) + Q(s)BE(s) = \mathcal{L}\left\{q(t)x(0) + \int_0^t q(t-\tau)Be(\tau)d\tau\right\}$$

the state variable in time domain

$$x(t) = q(t)x(0) + \int_0^t q(t-\tau)Be(\tau)d\tau$$

we know that

$$x(t) = \underbrace{\Phi(t)x(0)}_{\text{zero-input response}} + \underbrace{\int_0^t \Phi(t-\tau)Be(\tau)d\tau}_{\text{zero-state response}}$$

In this case

$$\Phi(s) = (sI - A)^{-1}$$

and

$$\Phi(t) = \mathcal{L}\{(sI - A)^{-1}\}$$

$$X(s) = \Phi(s)x(0) + \Phi(s)BE(s)$$
$$Y(s) = C\Phi(s)x(0) + (C\Phi(s)B + D)E(s)$$