Basic of Electrical Circuits
EHB 211E

Prof. Dr. Müştak E. Yalçın

Istanbul Technical University
Faculty of Electrical and Electronic Engineering

mustak.yalcin@itu.edu.tr

Lecture 14

Contents I

Solution of State Equations

A first-order differential equation, which may be written in a standard form as

$$\dot{x} = ax + be$$

where $a, b \in R$ and $e \in R$ is independent source.
Given initial condition $x(t_0) = x_0$ at t_0 DE has a unique solution.
In order to obtain x(t), lets multiply the eqn. by e^{-at}

$$\begin{aligned}
e^{-at}\dot{x} &= e^{-at}(ax + be) \\
e^{-at}\dot{x} - e^{-at}ax &= e^{-at}be \\
\tfrac{d}{dt}(e^{-at}x) &= e^{-at}be
\end{aligned}$$

Then integrate the eqn.

$$\int_{t_0}^{t} \frac{d}{dt}(e^{-at}x) = \int_{t_0}^{t} e^{-at}be(\tau)d\tau$$

Solution of State Equations

The solution of the 1st order differential equation

$$x(t) = e^{a(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{a(t-\tau)} b e(\tau) d\tau$$

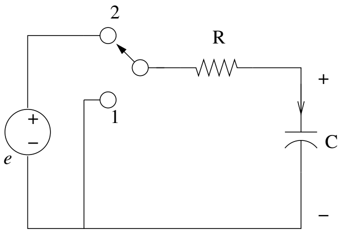
Zero-input response;

$$x_{\mathrm{zi}}(t) = e^{a(t-t_0)}x(t_0)$$

Zero-state response;

$$x_{\mathrm{zs}}(t) = \int_{t_0}^{t} e^{a(t-\tau)} b e(\tau) d\tau$$

Example



The switch is in 1, the current of capacitor

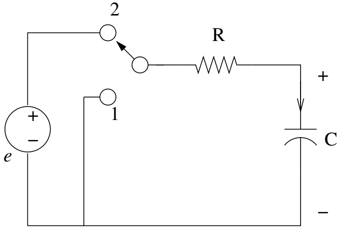
$$C\frac{dV_C}{dt} = -G(V_C)$$

The state equation;

$$\frac{dV_C}{dt} = -\frac{1}{RC}V_C$$

Zero-input response

$$V_C(t) = e^{-\frac{1}{RC}(t-t_0)}V_C(0)$$



The switch is in 2, the state equation of the circuit

$$C\frac{dV_C}{dt} = G(e - V_C)$$

in standart form

$$\frac{dV_C}{dt} = -\frac{1}{RC}V_C + \frac{1}{RC}e$$

The solution of $V_C(t)$;

$$V_C(t) = e^{-\frac{1}{RC}(t-t_0)}V_C(0) + \int_0^t e^{-\frac{1}{RC}(t-\tau)}\frac{1}{RC}e(\tau)d\tau$$

Homogeneous Solution

The homogeneous solution is also called the natural response is the general solution of DE when the input is set to zero;

$$\dot{x} = ax.$$

The homogeneous solution has the form

$$x_h(t) = Ke^{at}$$

Particular Solution (Forced response) $x_p(t)$ is depend on the source e and it will be picked up from the Table. Substituting $x_p(t)$ into DE eqn. we will obtain the parameters of the $x_p(t)$.

$$\dot{x_p} = ax_p + be$$

Particular Solutions

SOURCE	PARTICULAR SOLUTION
E	K
$Ee^{\alpha t}$	$Ke^{\alpha t}$
Ee^{at}	$K_1 e^{at} + K_2 te^{at}$
Et	$K_1 + K_2 t$
$E\cos(wt)$	$K_1 cos(wt) + K_2 \sin(wt)$
$E\sin(wt)$	$K_1 cos(wt) + K_2 \sin(wt)$
$E_1 \sin(w_1 t) + E_2 \cos(w_2 t)$	$K_1 cos(w_1 t) + K_2 \sin(w_2 t)$ $+K_3 cos(w_2 t) + K_4 \sin(w_2 t)$

The complete response

The complete response is the sum of natural response and forced response.

$$x(t) = x_h(t) + x_p(t)$$

Using initial condition $x(t_0)$

$$x_0 = Ke^{at_0} + x_p(t_0)$$

Parameter for the natural response can be obtained $K = \frac{x_0 - x_p(t_0)}{e^{at_0}}$.
The complete response

$$x(t) = \underbrace{(x_0 - x_p(t_0))e^{a(t-t_0)}}_{\text{Natural response}} + \underbrace{x_p(t)}_{\text{Particular solution}}$$

Zero-input and zero-state (forced response) responses express in term of natural response and Particular solution.

$$x(t) = \underbrace{x_0 e^{a(t-t_0)}}_{\text{zero-input response}} + \underbrace{x_p(t) - x_p(t_0)e^{a(t-t_0)}}_{\text{zero-state response}}$$

A first-order differential equation is given by

$$x = -2\dot{x} + e(t)$$

where $e(t) = e^{-t}u(t)$ and $x(0) = 2$. Find $x(t)$ for $t > 0$. The homogeneous solution is

$$x_h(t) = Ke^{-2t}$$

and the particular solution is

$$x_p(t) = Ee^{-t}$$

Substituting the particular solution into the DE

$$-Ee^{-t} = -2Ee^{-t} + e^{-t}$$

then $E = 1$, we obtain the complete the solution

$$x(t) = Ke^{-2t} + e^{-t}$$

Applying the initial condition to the above equation, we obtain $K = 1$. Then

$$x(t) = (e^{-2t} + e^{-t})u(t)$$

The zero-input response is

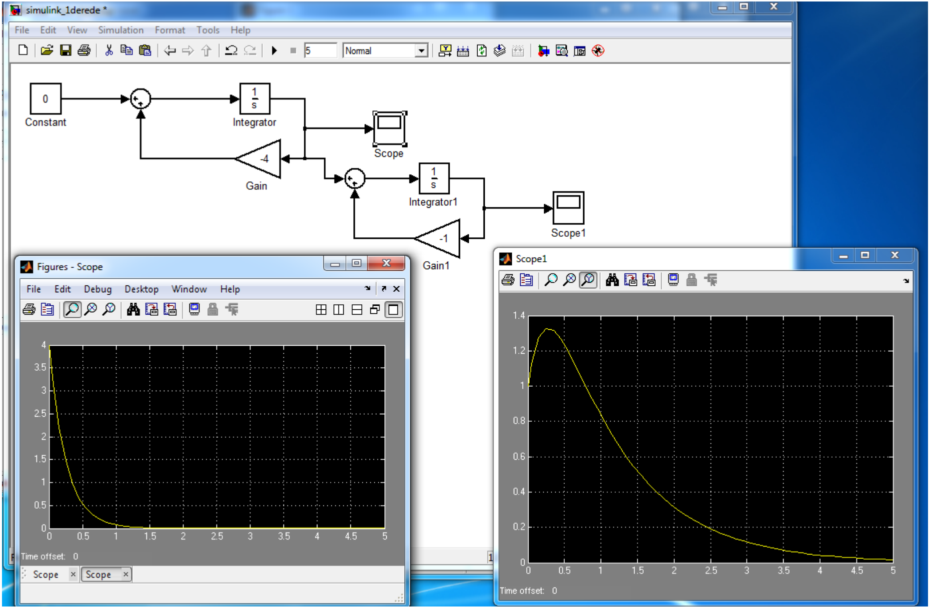
$$x_{zi}(t) = 2e^{-2t}$$

and the zero-state response is

$$\begin{aligned} x_{zs}(t) &= \int_0^t e^{-2(t-\tau)} e^{-\tau} d\tau \\ &= e^{-2t} \int_0^t e^{\tau)} d\tau \\ &= e^{-2t}(e^t - 1) \end{aligned}$$

The complete solution

$$x(t) = 2e^{-2t}u(t) + e^{-t} - e^{-2t}$$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Scope
Scope
Scope1
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A first-order differential equation is given by $\dot{x} = -4x$ where $x(0) = 4$. Find $x(t)$ for $t > 0$. The homogeneous solution is $x(t) = Ke^{-4t}$ Applying the initial condition to the above equation, we obtain $K = 4$. Then
$x(t) = 4e^{-4t}u(t)$
It's connected to a second first order system which is

$$\dot{z} = -z + x(t)$$

where $z(0) = 1$. Find $z(t)$ for $t > 0$.
The homogeneous and the particular solutions are

$$z_h(t) = Ke^{-t} \quad z_p(t) = Ee^{-4t}$$

Substituting the particular solution into the DE

$$-4Ee^{-4t} = -Ee^{-4t} + 4e^{-4t}$$

then $E = -4/3$, we obtain the complete the solution
$z(t) = Ke^{-t} - 4/3e^{-4t}$ Applying the initial condition to the above equation, we obtain $K = 7/3$. Then

$$z(t) = (7/3e^{-t} - 4/3e^{-4t})u(t)$$

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>> t=0:0.1:5;
>> plot(t,7*exp(-t)/3-4*exp(-4*t)/3);
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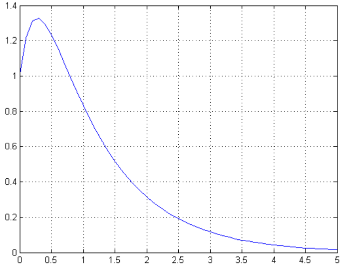

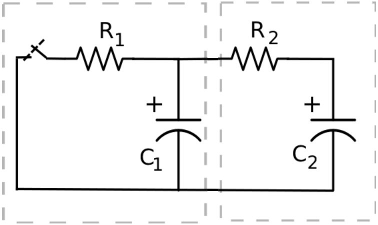
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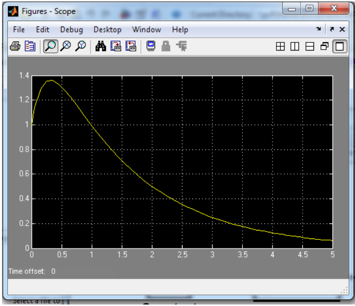
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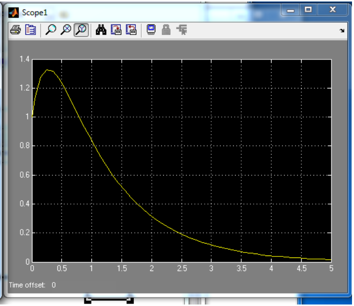
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Solution of Second Order State Equations

A second order state equation is given by standard form such as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e$$

or

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + b_1 e \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + b_2 e \end{aligned}$$

The output of this system is given by

$$y(t) = c_1 x_1(t) + c_2 x_2(t) + de(t)$$

In order to have differential equation in the output variable

$$\frac{d^2x_1}{dt^2} - (a_{11} + a_{22})\frac{dx_1}{dt} + (a_{11}a_{22} - a_{12}a_{21})x_1 = b_1\frac{du}{dt} + (a_{12}b_2 - a_{22}b_1)e$$

$$\frac{d^2x_2}{dt^2} - (a_{11} + a_{22})\frac{dx_2}{dt} + (a_{11}a_{22} - a_{12}a_{21})x_2 = b_2\frac{du}{dt} + (a_{21}b_1 - a_{11}b_2)e.$$

Undamped natural frequency

$$w = \sqrt{a_{11}a_{22} - a_{12}a_{21}}$$

Damping ratio

$$Q = -\frac{1}{2w}(a_{11} + a_{22})$$

With these new coefficients

$$\frac{d^2x_1}{dt^2} + 2Qw\frac{dx_1}{dt} + w^2x_1 = b_1\frac{du}{dt} + (a_{12}b_2 - a_{22}b_1)e$$

$$\frac{d^2x_2}{dt^2} + 2Qw\frac{dx_2}{dt} + w^2x_2 = b_2\frac{du}{dt} + (a_{21}b_1 - a_{11}b_2)e.$$

Defining

$$\begin{aligned}
q_0 &= c_1(-b_1a_{22} + a_{12}b_2) + c_2(-b_2a_{11} + a_{21}b_1) + d(a_{11}a_{22} - a_{12}a_{21}) \\
q_1 &= c_1b_1 + c_2b_2 - d(a_{11} + a_{22}) \\
q_2 &= d
\end{aligned}$$

The second order differential equation

$$\frac{d^2y}{dt^2} - (a_{11} + a_{22})\frac{dy}{dt} + (a_{11}a_{22} - a_{12}a_{21})y = q_2\frac{d^2e}{dt^2} + q_1\frac{de}{dt} + q_0e.$$

and it is in the term of standard parameter

$$\frac{d^2y}{dt^2} + 2Qw\frac{dy}{dt} + w^2y = q_2\frac{d^2e}{dt^2}q_1\frac{de}{dt} + q_0e.$$

Solution of the Homogeneous Second-Order Equation

For $e(t) = 0$, 2nd order equation

$$\frac{d^2y}{dt^2} + 2Qw\frac{dy}{dt} + w^2y = 0$$

$$y_h(t) = K_1e^{\lambda_1 t} + K_2e^{\lambda_2 t}$$

where K_1 and K_2 are constants defined by the initial conditions, and the eigenvalues λ_1 and λ_2 are the roots of the characteristic equation. Or λ_1 and λ_2 are found from

$$\lambda_i^2 + 2Qw\lambda_i + w^2 = 0$$

he eigenvalues

$$\lambda_1, \lambda_2 = -Qw \mp w\sqrt{Q^2 - 1}.$$

K_1 and K_2 are obtain

$$y(0) = c_1x(0) + c_2x_2(0)$$

using initial conditions

$$y(0) = K_1 + K_2$$

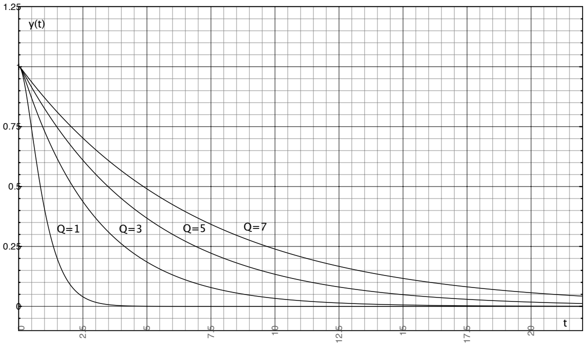
and

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = c_1(a_{11}x_1(0)+a_{12}x_2(0))+c_2(a_{21}x_1(0)+a_{22}x_2(0)) = K_1\lambda_1+K_2\lambda_2$$

The response of the system for $e = 0$ is depend on

$$\lambda_1, \lambda_2 = w(-Q \mp \sqrt{Q^2 - 1}).$$

- $Q > 1$: $\lambda < 0$ and real. In this case, the response is said to be overdamped.



- $Q = 1$: The response is said to be critically damped.

$$\lambda_1, \lambda_2 = -w.$$

Homogenous solution is

$$y_h(t) = K_1 e^{-wt} + K_2 t e^{-wt}$$

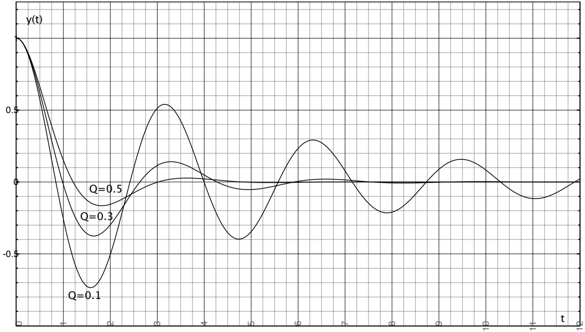
- $0 \leq Q < 1$: The response is said to be underdamped.

$$\lambda_1, \lambda_2 = -Qw \mp jw\sqrt{1 - Q^2}.$$

The output

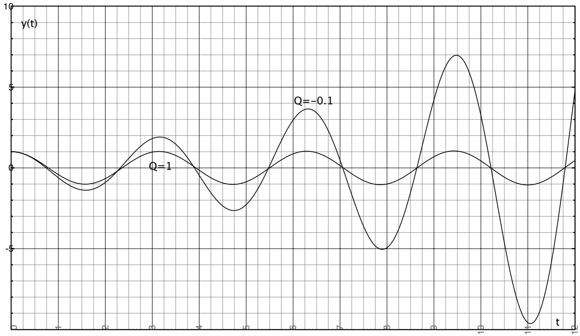
$$y_h(t) = y_0 \frac{e^{-Qwt}}{\sqrt{1 - Q^2}} \cos\left(w_d t - \phi\right)$$

where $w_d = w\sqrt{1 - Q^2}$ and $\phi = \tan -1 \frac{Q}{\sqrt{1-Q^2}}$.

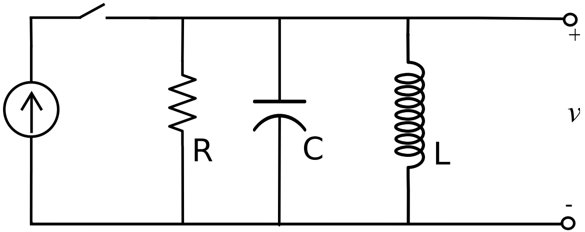


- $Q < 0$

$$\text{Real}\{\lambda\} > 0$$



Natural Response of a Parallel RLC Circuit



$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$w = \frac{1}{\sqrt{LC}}$, $Q = \frac{\sqrt{LC}}{2RC}$,
Electric Circuits, James W. Nilsson and Susan A. Riedel, pp. 286-301

We will find the output for $e(t) = u(t)$ which is given by

$$y(t) = y_h(t) + y_{\ddot{o}}(t)$$

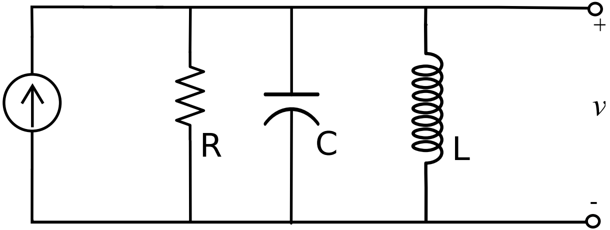
The particular solution $y_{\ddot{o}}(t) = K$. Substituting the particular solution into the eqn. we will have $K = \frac{1}{w_n^2}$. Step response is obtain such as

$$y(t) = K_1 e^{\lambda_1 t} + K_1 e^{\lambda_2 t} + \frac{1}{w_n^2}$$

K_1 and K_2 are obtained form the initial conditions with

$$y(0) = K_1 + K_2 + \frac{1}{w_n^2}$$

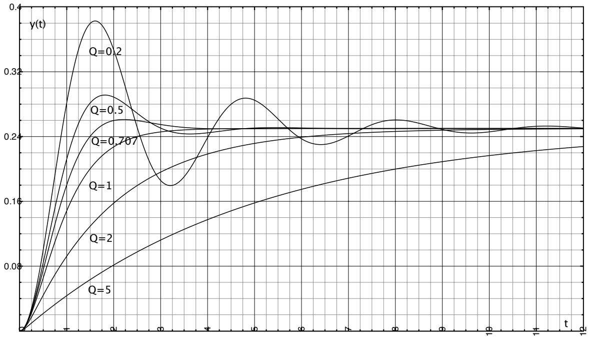
$$\left.\frac{dy(t)}{dt}\right|_{t=0} = K_1\lambda_1 + K_2\lambda_2$$



$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}i_k$$

Electric Circuits, James W. Nilsson and Susan A. Riedel, pp. 301-307

Step response



Example

$$\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e$$

where $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the initial condition and the output is $y(t) = x_1(t)$ Let find the output for $e(t) = 0$, e(t)=u(t) and $e(t) = 10\cos(t)$.
We have

$$\begin{aligned} \dot{x_1} &= 2x_2 \\ \dot{x_2} &= -x_1 - 3x_2 + e \end{aligned}$$

The output is obtain such as

$$\begin{aligned} \ddot{x_1} &= 2\ddot{x_2} \\ \dot{x_1} &= 2(-x_1 - 3x_2 + e) \\ &= -2x_1 - 6x_2 + 2e \\ &= -3\dot{x_1} - 2x_1 + 2e \end{aligned}$$

For the homogeneous solution, we must find eigenvalues

$$\det\left\{\lambda I - \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}\right\} = \lambda^2 + 3\lambda + 2 = 0$$

$\lambda = -1$ ve $\lambda = -2$. The homogeneous solution

$$y(t) = K_1 e^{-t} + K_2 e^{-2t}$$

Using initial condition

$$\begin{aligned} y(0) &= K_1 + K_2 = 1 \\ \dot{y}(0) &= -K_1 - 2K_2 = 0 \end{aligned}$$

$K_1 = 2$ and $K_2 = -1$ are obtained. The solution is

$$y(t) = 2e^{-t} - e^{-2t}.$$

For $e(t) = u(t)$, the particular solution is chosen from table which is

$$y_{ö}(t) = E$$

Substituting the particular solution into the DE equ. We have $E = 1$.

$$y(t) = K_1 e^{-t} + K_2 e^{-2t} + 1$$

Using the initial conditions

$$\begin{aligned} y(0) &= K_1 + K_2 + 1 = 1 \\ \dot{y}(0) &= -K_1 - 2K_2 + 1 = 0 \end{aligned}$$

we will have $K_1 = -1$ and $K_2 = 1$. The complete solution is given by

$$y(t) = -e^{-t} + e^{-2t} + u(t)$$

For $e(t) = 10\cos(t)$, the particular solution is chosen from table which is

$$y_{ö}(t) = E_1 \cos t + E_2 \sin t$$

Substituting the particular solution into the DE equ.

$$-E_1 \cos t - E_2 \sin t = 3E_1 \sin t - 3E_2 \cos t - 2E_1 \cos t - 2E_2 \sin t + 20 \cos t$$

we have $E_1 = 2$ ve $E_2 = 6$ then

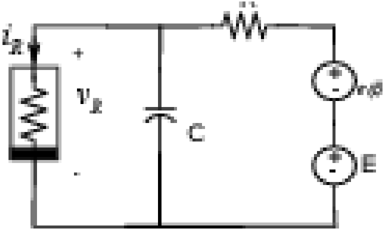
$$y(t) = K_1 e^{-t} + K_2 e^{-2t} + 2\cos t + 6\sin t$$

Using the initial conditions

$$\begin{aligned} y(0) &= K_1 + K_2 + 2 = 1 \\ \dot{y}(0) &= -K_1 - 2K_2 + 6 = 0 \end{aligned}$$

we will have $K_1 = -8$ and $K_2 = 7$. The complete solution is given by

$$y(t) = -8e^{-t} + 7e^{-2t} + 2\cos t + 6\sin t$$



$E = -2V,\ R = 2\Omega,\ e(t) = 0.2\sin(wt),\ C = 1F,\ v_R = i_R^2$

DC Analysis:

$$C\frac{dV_c}{dt} = i_C = 0$$

$$e = iR + v_R = -4 = 2i_R + i_R^2$$

$$i_R = -2Amps$$

AC Analysis: $R_Q = -4\Omega$

$$CC\frac{dV_c}{dt} = \frac{(e - v_C)}{R} - \frac{v_C}{R_Q}$$

$$\frac{dv_C}{dt} = -\frac{v_C}{4} + 0.1\sin(wt)$$