

Circuit and System Analysis

EEF 232E

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Why Laplace Transform



Laplace transform ($F(s)$) of $f(t)$ function is given by

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t) dt.$$

The Laplace transform converts linear differential equations into algebraic equations. These are linear equations with polynomial coefficients. The solution of these linear equations therefore leads to rational function expressions for the variables involved.

Initial Conditions, Generalized Functions, and the Laplace Transform, by Kent H. Lundberg; Haynes R. Miller ; David L. Trumper, IEEE Control Systems [▶ Link](#)

Laplace Transform

Signal	Laplace Transform
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$\frac{t^n}{n!} u(t)$	$\frac{1}{s^{n+1}}$
$e^{-at} u(t)$	$\frac{1}{s+a}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$2 k e^{-\alpha t} \cos(\beta t + \theta)$	$\frac{k}{(s+\alpha-j\beta)} + \frac{\hat{k}}{(s+\alpha+j\beta)}$

Properties of the Laplace Transform

- $$\mathcal{L}\{k_1 f_1(t) + k_2 f_2(t)\} = k_1 F_1(s) + k_2 F_2(s)$$

- $$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$$

and

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

- $$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{F(s)}{s}$$

- $$\mathcal{L}\{f(t - t_0)u(t - t_0)\} = F(s)e^{-t_0 s}$$

$$\mathcal{L} \left\{ \frac{df(t)}{dt} \right\} = sF(s) - f(0^-)$$

Distinguishing 0^- , 0 , and 0^+

While the unilateral Laplace transform is frequently associated with the solution of differential equations, the need to clearly distinguish 0^- , 0 , and 0^+ is independent of any dynamic-systems context. For a discontinuous function $f(t)$, the derivative $f'(t)$ must be interpreted as the generalized derivative of $f(t)$, which includes the singularity function

$$(f(t_0^+) - f(t_0^-))\delta(t - t_0)$$

at every point t_0 at which $f(t)$ is discontinuous. In particular, if $f(0^-) \neq f(0^+)$, then the derivative includes a delta function at the origin.

In the following example, adapted from Problem 11.17 of [29], we apply the unilateral transform to three signals and their derivatives. This example clarifies that the need for using the Laplace transform (3) and properties (4) and (5) is

really a matter of properly defining signals and their transforms, and is not fundamentally connected to the solution of differential equations.

Consider the three signals $f(t)$, $g(t)$, and $h(t)$ defined for all time

$$\begin{aligned} f(t) &= e^{-at}, \\ g(t) &= e^{-at}u(t), \\ h(t) &= e^{-at}u(t) - u(-t), \end{aligned}$$

which are plotted for the value $a = 1$ in Figure S1. Since all three functions are nonsingular and agree for positive time they have the same Laplace transform by means of (3). However, their derivatives include different impulse values at $t = 0$, and thus the Laplace transforms of their derivatives must differ. Our choice of Laplace transform properties should give consistent

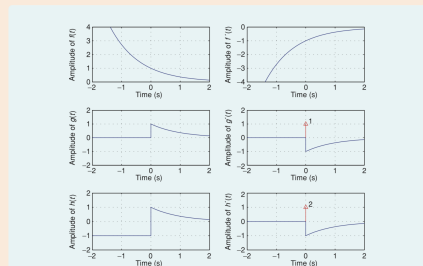


FIGURE S1 Three functions $f(t) = e^{-at}$, $g(t) = e^{-at}u(t)$, $h(t) = e^{-at}u(t) - u(-t)$, and their derivatives, plotted for $a = 1$ and defined for all time. Impulses are represented by the red arrows, with the impulse area denoted by the number next to the arrowhead. Since all three functions are identical for positive time, they have identical unilateral Laplace transforms. However, their derivatives differ at the origin. Therefore, the Laplace transforms of their derivatives also differ.

Properties of the Laplace Transform

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$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

-

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$$

Inverse Laplace Transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \int_{c-j\infty}^{c+j\infty} e^{st} F(s) ds$$

Partial-Fraction Expansion Method

s-transform of a linear time invariant system is often of the form ($n > m$)

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \dots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}$$

which is ratio of two polynomials. The value(s) for s where $P(s) = 0$ are called *zeros*. The value(s) for s where $Q(s) = 0$ are called *poles*.

If $s_{pi} \neq s_{pj}$, poles distinct.

if $\lim_{s \rightarrow \infty} F(s)(s - s_{pi}) = \infty$ and $\lim_{s \rightarrow \infty} F(s)(s - s_{pi})^k$ is constant then $s = s_{pi}$ is a k -multiple pole.

Lets assume that poles distinct

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{k_1}{(s-s_{p1})} + \frac{k_2}{(s-s_{p2})} \cdots + \frac{k_n}{(s-s_{pn})}\right\}\end{aligned}$$

k_i is the residue located at the corresponding pole s_{pi} which is

$$k_i = F(s) (s - s_{pi})|_{s=s_{pi}}$$

$$\mathcal{L}^{-1}\{F(s)\} = k_1 e^{s_{p1}t} u(t) + k_2 e^{s_{p2}t} u(t) + \dots + k_n e^{s_{pn}t} u(t)$$

$$\mathcal{L}^{-1}\{k_0 + F(s)\} = k_0 \delta(t) + k_1 e^{s_{p1}t} u(t) + k_2 e^{s_{p2}t} u(t) + \dots + k_n e^{s_{pn}t} u(t)$$

$$Y(s) = \frac{-3s^2 + 23s - 38}{(s-1)(s-2)(s-3)} = \frac{-9}{s-1} + \frac{2}{s-2} + \frac{2}{s-3}$$

$$y(t) = -9e^t + 4e^{2t} + 2e^{3t} \text{ for } t > 0$$

$$Y(s) = \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} = \frac{1/6j}{s+1-j} - \frac{1/6j}{s+1+j} + \frac{1/6j}{s+1-2j} - \frac{1/6j}{s+1+2j}$$

$$y(t) = \frac{1}{3}e^{-t} \sin(t) + \frac{1}{3}e^{-t} \sin(2t) \text{ for } t > 0$$

$$\dot{x} = 2x - 3y, \quad x(0) = 8$$

$$\dot{y} = -2x + y, \quad y(0) = 3$$

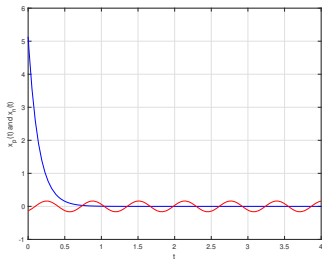
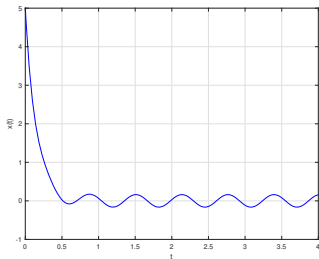
Using Laplace transform

$$sX - 8 = 2X - 3Y \rightarrow (s-2)X + 3Y = 8$$

$$sY - 3 = -2x + y \rightarrow 2X + (s-1)Y = 3$$

$$X = \frac{8s - 17}{s^2 - 3s - 4} = \frac{5}{s+1} + \frac{3}{s-4} \rightarrow x(t) = 5e^{-t} + 3e^{4t}$$

Example: $\dot{x} + 7x = 2 \sin 10t$ and $x(0) = 5$



$$x(t) = \underbrace{5e^{-7t}}_{\text{zero-input}} + \underbrace{\frac{20}{149}e^{-7t} + \frac{14}{149}\sin(10t) - \frac{20}{149}\sin(10t + \frac{\pi}{2})}_{\text{zero-state}}$$

Using Laplace Transform

$$sX(s) - 5 + 7X(s) = \frac{20}{s^2 + 10}$$

$$(s + 7)X(s) = 5 + \frac{20}{s^2 + 10}$$

$$X(s) = \underbrace{\frac{5}{(s + 7)}}_{\text{zero-input}} + \underbrace{\frac{20}{(s^2 + 10)(s + 7)}}_{\text{zero-state}} \quad \text{▶ On line partial fraction}$$

$$X(s) = \frac{5}{(s + 7)} + \frac{20}{149(s + 7)} + \frac{-20s + 140}{149(s^2 + 10)}$$

$$X(s) = \underbrace{\frac{5}{(s + 7)}}_{\text{zero-input}} + \underbrace{\frac{20}{149(s + 7)} + \frac{-20s}{149(s^2 + 10)}}_{\text{zero-state}} + \frac{140}{149(s^2 + 10)}$$

$$x(t) = 5e^{-7t} + \frac{20}{149}e^{-7t} - \frac{20}{149}\cos(10t) + \frac{14}{149}\sin(10t)$$

$Q(s)$ has a multiple pole.

$$\mathcal{L}^{-1} \left\{ \frac{P(s)}{Q(s)} \right\} = \mathcal{L}^{-1} \left\{ \frac{k_{i1}}{(s - s_{pi})} + \frac{k_{i2}}{(s - s_{pi})^2} \cdots + \frac{k_{ik}}{(s - s_{pi})^k} + \frac{P_1(s)}{Q_1(s)} \right\}$$

where

$$k_{ik} = F(s) (s - s_{pi})^k \Big|_{s=s_{pi}}$$

and

$$k_{il} = \frac{1}{(k-l)!} \frac{d^{k-l} F(s) (s - s_{pi})^k}{ds^{k-l}} \Big|_{s=s_{pi}}$$

for $l = 1, 2, \dots, k - 1$.

$$Y(s) = \frac{2s^2 - 25s - 33}{(s+1)^2(s-5)} = \frac{k_1}{s+1} + \frac{k_2}{(s+1)^2} + \frac{k}{s-5}$$

$$k = \frac{2s^2 - 25s - 33}{(s+1)^2} \Big|_{s=5} = -3$$

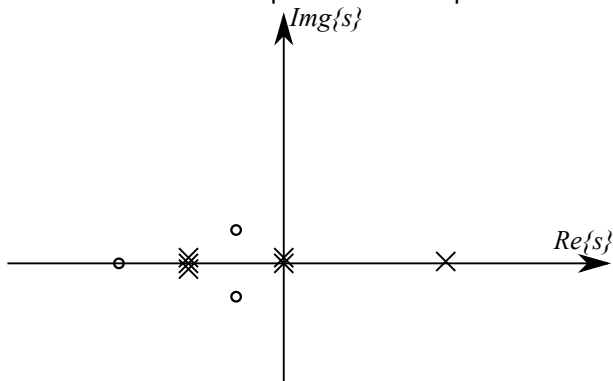
$$k_1 = \frac{1}{1!} \frac{d}{ds} \frac{2s^2 - 25s - 33}{(s-5)} \Big|_{s=-1} = 5$$

$$k_2 = \frac{2s^2 - 25s - 33}{(s-5)} \Big|_{s=-1} = 1$$

$$y(t) = 5e^{-t} + te^{-t} - 3e^{5t}$$

$$F(s) = \frac{(s + 3)(s^2 + 2s + 2)}{s^2(s - 4)(s + 2)^2}$$

zeros "o" and poles "x" on s-plane:



The Convolution Theorem

The convolution operation: Let $f_1(t)$ and $f_2(t)$ be functions defined on $[0, \infty)$, and let us take them to be equal to zero for $t < 0$: The convolution of the time functions f_1 and f_2 is a new time function denoted by $(f_1 \star f_2)(t)$ and defined for all t by

$$f(t) = f_1(t) \star f_2(t) = \int_0^t f_1(\tau) f_2(t - \tau) d\tau = \int_0^t f_2(t - \tau) f_1(\tau) d\tau$$

The Convolution Theorem

Let $f_1(t)$ and $f_2(t)$ have $F_1(s)$ and $F_2(s)$ as Laplace transforms. We assume that for $i = 1, 2$, $f_i(t) = 0$ for $t < 0$. Laplace transform of the convolution of f_1 and f_2 is given by

$$\mathcal{L}\{f_1(t) \star f_2(t)\} = F_1(s)F_2(s)$$

Thus, the operation of convolution in the time domain is equivalent to multiplication in the frequency domain.

Example

Using The Convolution Theorem, lets find inverse Laplace transform of

$$F(s) = \frac{1}{s^2(s+2)}$$

$$\frac{1}{s^2(s+2)} = \frac{1}{s^2} \frac{1}{(s+2)} = \mathcal{L} \{ tu(t) \star e^{-2t} u(t) \}$$

$$tu(t) \star e^{-2t} u(t) = \int_0^t \tau e^{-2(t-\tau)} d\tau$$

Analysis of state space equation

Linear time invariant system

$$\begin{aligned}\dot{x} &= Ax + Be \\ y &= Cx + De\end{aligned}$$

where x state variable and y is output, u is input vectors. Using Laplace transform

$$\begin{aligned}sX(s) - x(0) &= AX(s) + BE(s) \\ Y(s) &= CX(s) + DE(s)\end{aligned}$$

we have Laplace transform state variable

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BE(s)$$

Output

$$Y(s) = C(sl - A)^{-1}x(0) + (C(sl - A)^{-1}B + D)E(s)$$

$$X(s) = (sl - A)^{-1}x(0) + (sl - A)^{-1}BE(s) = Q(s)x(0) + Q(s)BE(s)$$

where $Q(s) = (sl - A)^{-1}$ and $q(t) = \mathcal{L}^{-1}\{(sl - A)^{-1}\}$.

Using The Convolution Theorem, we have

$$Q(s)x(0) + Q(s)BE(s) = \mathcal{L} \left\{ q(t)x(0) + \int_0^t q(t - \tau)Be(\tau)d\tau \right\}$$

the state variable in time domain

$$x(t) = q(t)x(0) + \int_0^t q(t - \tau)Be(\tau)d\tau$$

we know that

$$x(t) = \underbrace{\Phi(t)x(0)}_{\text{zero-input response}} + \underbrace{\int_0^t \Phi(t-\tau)Be(\tau)d\tau}_{\text{zero-state response}}$$

In this case

$$\Phi(s) = (sI - A)^{-1}$$

and

$$\Phi(t) = \mathcal{L}\{(sI - A)^{-1}\}$$

$$X(s) = \Phi(s)x(0) + \Phi(s)BE(s)$$

$$Y(s) = C\Phi(s)x(0) + (C\Phi(s)B + D)E(s)$$

Example: $\dot{x} + 7x = 2 \sin 10t$ and $x(0) = 5$

$$sX(s) - 5 + 7X(s) = \frac{20}{s^2 + 10}$$
$$X(s) = \underbrace{\frac{5}{(s+7)}}_{\text{zero-input}} + \underbrace{\frac{20}{(s+7)(s^2+100)}}_{\text{zero-state}}$$

$$\Phi(s) = \frac{1}{(s+7)}$$

$$X(s) = \underbrace{\Phi(s)5}_{\text{zero-input}} + \underbrace{\Phi(s)\frac{20}{(s^2+100)}}_{\text{zero-state}}$$

Network Functions

$$\text{Network Function} = \frac{\mathcal{L}\{\text{zero - state response}\}}{\mathcal{L}\{\text{input}\}}$$

$$\frac{Y(s)}{E(s)} = (C\Phi(s)B + D) = H(s)$$

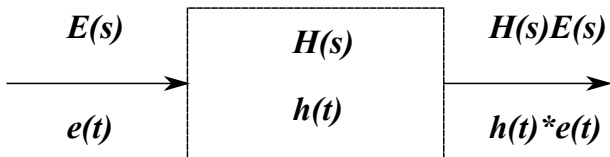
Transfer (Network) functions : voltage transfer functions, transfer admittances, current transfer function, transfer impedance.

$$y(t) = \mathcal{L}^{-1}\{H(s)E(s)\} = h(t) \star e(t) = \int_0^t h(t - \tau)e(\tau)d\tau$$

if we chose $e(t) = \delta(t)$

$$y(t) = \mathcal{L}^{-1}\{H(s) 1\} = \int_0^t h(t - \tau)\delta(\tau)d\tau = h(t)$$

NETWORK FUNCTION = $\mathcal{L}\{\text{IMPULS RESPONSE}\}$



$$\begin{aligned} \text{impuls response} &= h(t) = \mathcal{L}^{-1}\{H(s)\} \\ \text{step response} &= h(t)u(t) = \mathcal{L}^{-1}\left\{H(s)\frac{1}{s}\right\} \\ &\quad ! \text{ zero-input response} = 0 ! \end{aligned}$$

what happen if zero-input response is not zero (or system is not stable)?

Characteristic polynomial

$$H(s) = \frac{P(s)}{Q(s)} = \frac{Y(s)}{E(s)} = (C\Phi(s)B + D)$$

The characteristic polynomial of A is defined by

$$Q(s) = \det\{sI - A\} = 0$$

The solutions of the characteristic equation are precisely the eigenvalues of the matrix A . The roots of the characteristic equation is $s_{pi} = \sigma_i + j\omega_i$

- if $\sigma_i > 0$ unstable
- if $\sigma_i < 0$ stable
- $\sigma_i = 0$ check the repeated eigenvalue !

Example : The transfer function for a linear time-invariant circuit is $H(s) = \frac{1}{s+1}$. If $E(t) = 3 \cos t$ what is the steady-state expression of the output?

$$Y(s) = \frac{3s}{s^2 + 1} \frac{1}{s + 1}$$

and in time domain

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{3}{2} \frac{s+1}{s^2+1} - \frac{3}{2} \frac{1}{s+1} \right\} \\ &= \frac{3}{2} (\cos t + \sin t - e^{-t}) \end{aligned}$$

Using convolution

$$\begin{aligned} h(t) &= \mathcal{L} \left\{ \frac{1}{s+1} \right\} = e^{-t} \\ y(t) &= \int_0^t e^{-t+\tau} 3 \cos(\tau) d\tau \\ &= 3e^{-t} \left[\frac{e^\tau}{2} (\cos(\tau) + \sin(\tau)) \right]_0^t \\ &= \frac{3}{2} (\cos t + \sin t - e^{-t}) \end{aligned}$$