Circuit and System Analysis
EEF 232E

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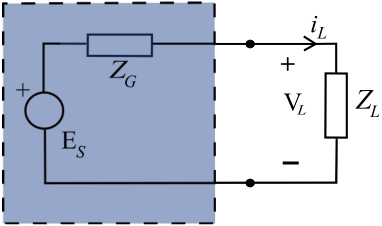
Outline I

Maximum Power Transfer

Maximum amount of power from the source to the load.



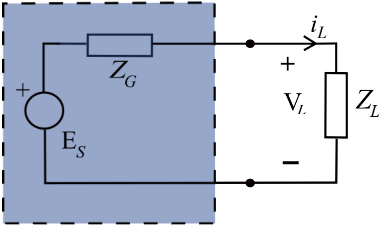
We must determine the load impedance $Z_L = R_L + X_L j$ that results in the delivery of maximum average power to its terminal.

$$-P_L = P_S + P_G = \frac{1}{2}|E_S||I_S|\cos(\theta_S) + P_G$$

$I_S = -I_L = |I_L|e^{j(\theta_L+\pi)}$ and $Z_G = R_G + X_G j$

$$-P_L = P_S + P_G = \frac{1}{2}|E_S||I_L|\cos(\theta_L + \pi) + \frac{1}{2}\underbrace{R_G|I_L|e^{\theta_L}}_{V_{R_G}}|I_L|e^{-\theta_L}$$

Maximum Power Transfer



$$-P_L = \frac{1}{2}|E_S||I_L|\cos(\theta_L + \pi) + \frac{1}{2}\underbrace{R_G|I_L|e^{\theta_L}}_{V_{R_G}}|I_L|e^{-\theta_L}$$

$$= -\frac{1}{2}|E_S||I_L|\cos(\theta_L) + \frac{1}{2}R_G|I_L|^2$$

$$P_L = \frac{1}{2}E_G|I_L|\cos(\theta_L) - \frac{1}{2}R_G|I_L|^2$$

When P_L is maximum ?

First let $\cos(\theta_L) = 1$ to maximize P_L. Meaning $\theta_L = \theta_V - \theta_i = 0$!

$$P_L = \frac{1}{2}E_S|I_L| - \frac{1}{2}R_G|I_L|^2$$

Then we must find the values of Z_L where $\frac{dP_L}{d|I_L|} = 0$.

$$\frac{dP_L}{d|I_L|} = \frac{1}{2}E_S - R_G|I_L|$$

then

$$|I_L| = \frac{E_S}{2R_G}$$

For the maximum power transfer we must meet the conditions :

- $\cos(\theta_L) = 1$
- $|I_L| = \frac{E_S}{2R_G}$.

$\cos(\theta_L) = 1$ meaning $\theta_L = 0$. From circuit

$$I_L = \frac{E_S}{Z_L + Z_G} = |I_L|e^{\theta_L j} = |I_L|e^{0j} = |I_L|$$

Hence $Z_L + Z_G$ must be resistive. Therefore $X_L = -X_G$.

$$|I_L| = \frac{E_S}{Z_L + Z_G} = \frac{E_S}{R_L + R_G}$$

Use the second condition

$$|I_L| = \frac{E_S}{Z_L + Z_G} = \frac{E_S}{R_L + R_G} = \frac{E_S}{2R_G}$$

then

$$R_L = R_G$$

For maximum average power transfer

$$Z_L = \bar{Z}_G$$

Find the load impedance that transfers maximum power to the load and determine the maximum power delivered to the load for the circuit including serial connected $Z_G = (5 - 6j)\Omega$ and $V_S = 10\angle 0^\circ$.

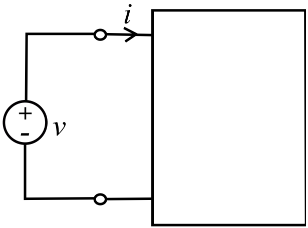
$$Z_L = (5 + 6j)\Omega$$

$$I_L = \frac{10\angle 0^\circ}{5 + 5} = 1\angle 0^\circ A$$

the average power transferred to the load is

$$P_L = \frac{|I|^2}{2} R_L = 2.5W$$

Average Power Due to Several Sinusoidal Inputs



We drive a linear time-invariant one-port by a voltage source given by

$$v(t) = V_1 \cos\left(w_1 t + \theta_{v1}\right) + V_2 \cos\left(w_2 t + \theta_{v2}\right)$$

and the port current

$$i(t) = I_1 \cos\left(w_1 t + \theta_{i1}\right) + I_2 \cos\left(w_2 t + \theta_{i2}\right)$$

Using standard formulas, The (instantaneous) power delivered by the voltage source to the one-port

$$
\begin{aligned}
P &= v(t)i(t) \\
&= \frac{1}{2}V_1 I_1\{\cos(\theta_{v1}-\theta_{i1}) + \frac{1}{2}V_2 I_2\{\cos(\theta_{v2}-\theta_{i2}) \\
&+ \frac{1}{2}V_1 I_1 \cos(2wt+\theta_{v1}+\theta_{i1})\} + \frac{1}{2}V_2 I_2 \cos(2wt+\theta_{v2}+\theta_{i2})\} \\
&+ \frac{1}{2}V_1 I_2 \cos((w_1+w_2)t+\theta_{v1}+\theta_{i2})\} + \frac{1}{2}V_2 I_1 \cos((w_1+w_2)t+\theta_{v2}+\theta_{i1} \\
&+ \frac{1}{2}V_1 I_2 \cos((w_1-w_2)t+\theta_{v1}-\theta_{i2})\} + \frac{1}{2}V_2 I_1 \cos((w_1-w_2)t+\theta_{i1}-\theta_{v2}
\end{aligned}
$$

The average power over $T_c = n_1 T_1 = n_2 T_2$

$$
\begin{aligned}
P_{\mathrm{ort}} &= \frac{1}{T_c}\int_0^{T_c} P(t)dt \\
&= \frac{1}{2}V_1 I_1\{\cos(\theta_{v1}-\theta_{i1}) + \frac{1}{2}V_2 I_2\{\cos(\theta_{v2}-\theta_{i2})
\end{aligned}
$$

The superposition of average power

The average power delivered to a circuit by several sinusoidal sources, acting together, is equal to the sum of the average power delivered to the circuit by each source acting alone, if, and only if, no two of the sources have the same frequency.

! if $w_1 = w_2$, it does not hold !

If two or more sources are operating at the same frequency, the principle of power superposition is not valid, but the principle of superposition remains valid.

Mesh-Current Method in Frequency Domain

The number of equations to be solved are equal to the number of independent loops $(n_e - n_d + 1)$. There exists a tree such that the meshes are Fundamental loops*.

$$B_1 \mathbf{R} B_1^T i_c + B_2 v_k = 0$$

where v_k and v_R voltages of independent voltage sources and resistors. Instead of $V_e = \mathbf{R} I_e$ using

$$V_e = \mathbf{Z} I_e$$

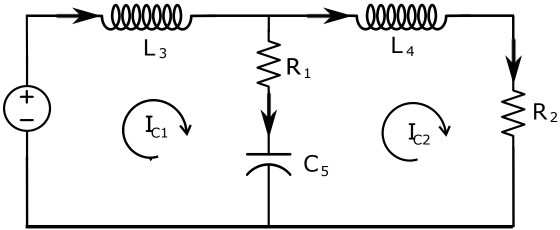
we have

$$B_1 \mathbf{Z} B_1^T I_c + B_2 V_k = 0$$

where $B_1 \mathbf{Z} B_1^T$ mesh impedance matrix.

See :EHB211 E

Example



$$M1 \quad L_3 jwI_{c1} + (\frac{1}{C_5 jw} + R)(I_{c1} - I_{c2}) - V_G = 0$$
$$M2 \quad L_4 jwI_{c2} + R_2 I_{c2} + (\frac{1}{C_5 jw} + R)(I_{c2} - I_{c1}) = 0$$

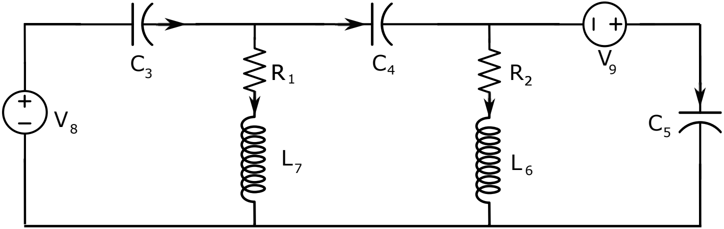
In matrix form

$$\begin{bmatrix} R_1 + \frac{1}{C_5 jw} + L_3 jw & -\frac{1}{C_5 jw} - R_1 \\ -\frac{1}{C_5 jw} - R_1 & L_4 jw + R_2 + R_1 + \frac{1}{C_5 jw} \end{bmatrix} \begin{bmatrix} I_{c1} \\ I_{c2} \end{bmatrix} = \begin{bmatrix} V_G \\ 0 \end{bmatrix}$$

$$V_{L_3} = L_3 jw[1\ \ 0] \begin{bmatrix} \cdot & -\frac{1}{Cjw} - R_1 \\ -\frac{1}{Cjw} - R_1 & \cdot \end{bmatrix}^{-1} \begin{bmatrix} V_G \\ 0 \end{bmatrix}$$

$v_G(t) = 4\cos(2\pi 60 t + \frac{\pi}{3}),\ L_3 = L_4 = 3mH,\ C = 4\mu F,\ R_1 = R_2 = 2k\Omega$

$$V_{L_3} = 310^{-3} j2\pi 60[1\ \ 0] \begin{bmatrix} \cdot & -\frac{10^6}{4j2\pi 60} - 2k \\ -\frac{10^6}{4j2\pi 60} - 2k & \cdot \end{bmatrix}^{-1} \begin{bmatrix} 4e^{\frac{\pi}{3}j} \\ 0 \end{bmatrix}$$



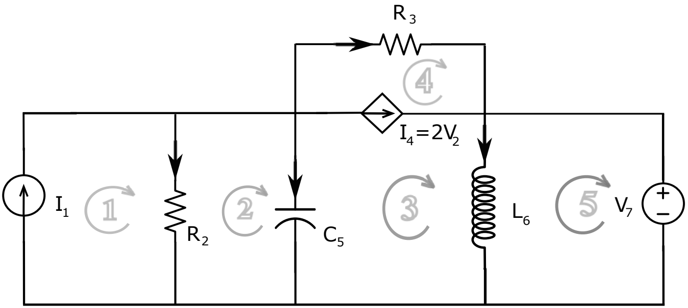
$$Z\begin{bmatrix} I_{c1} \\ I_{c2} \\ I_{c3} \end{bmatrix} = \begin{bmatrix} V_8 \\ 0 \\ V_9 \end{bmatrix}$$

where

$$Z =$$

$$\begin{bmatrix} \frac{1}{C_3 jw} + R_1 + L_7 jw & -R_1 - L_7 jw & 0 \\ -R_1 - L_7 jw & R_1 + (L_7 + L_6) jw + \frac{1}{C_4 jw} + R_2 & -R_2 - L_6 jw \\ 0 & -R_2 - L_6 jw & R_2 + L_6 jw + \frac{1}{C_5 jw} \end{bmatrix}$$

$I_4 = 2V_2$



$$\begin{bmatrix} R_2 & -R_2 & 0 & 0 & 0 \\ -R_2 & R_2 + \frac{1}{C_5 jw} & -\frac{1}{C_5 jw} & 0 & 0 \\ 0 & -\frac{1}{C_5 jw} & \frac{1}{C_5 jw} + L_6 jw & 0 & -L_6 jw \\ 0 & 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & -L_6 jw & L_6 jw \end{bmatrix} \begin{bmatrix} I_{c1} \\ I_{c2} \\ I_{c3} \\ I_{c4} \\ I_{c5} \end{bmatrix} = \begin{bmatrix} -V_1 \\ 0 \\ -V_4 \\ V_4 \\ -V_7 \end{bmatrix}$$

$$
\begin{aligned}
I_1 &= I_{c1} \\
I_4 &= 2V_2 \\
I_{c3} - I_{c4} &= -2V_1
\end{aligned}
$$

with above equ.s

$$
\begin{bmatrix}
R_2 & -R_2 & 0 & 0 & 0 & 1 & 0 \\
-R_2 & R_2 + \frac{1}{C_5 jw} & -\frac{1}{C_5 jw} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{C_5 jw} & \frac{1}{C_5 jw} + L_6 jw & 0 & -L_6 jw & 0 & 1 \\
0 & 0 & 0 & R_3 & 0 & 0 & -1 \\
0 & 0 & -L_6 jw & 0 & L_6 jw & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
I_{c1} \\ I_{c2} \\ I_{c3} \\ I_{c4} \\ I_{c5} \\ V_1 \\ V_4
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ -V \\ I_1 \\ 0
\end{bmatrix}
$$

Node-voltage Method

The fundamental cut-set equations for the nodes(which do not correspond to node of a voltage sources)

$$Ai = 0$$

current sources i_k and currents of one ports i_e

$$A_1 i_e + A_2 i_k = 0$$

in Sinusoidal steady-state $I_e = YV_e$

$$A_1 YV_e + A_2 I_k = 0$$

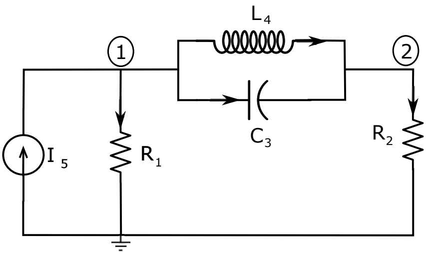
using $V_e = A_1^T V_d$ we have

$$A_1 YA_1^T V_d + A_2 i_k = 0$$

where V_d is phasor of the node voltage.

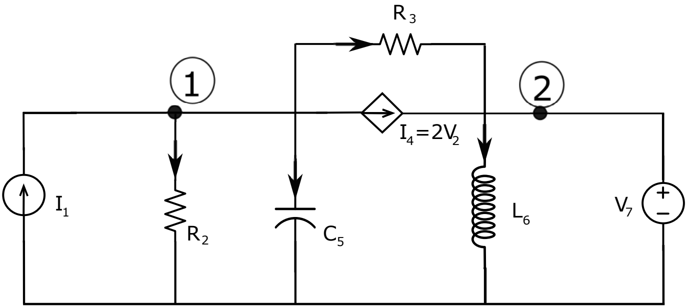
See :EHB211 E

Example



$$\begin{bmatrix} G_1 + C_3jw + \frac{1}{L_3jw} & -C_3jw - \frac{1}{L_3jw} \\ -C_3jw - \frac{1}{L_3jw} & G_2 + C_3jw + \frac{1}{L_3jw} \end{bmatrix} \begin{bmatrix} V_{d1} \\ V_{d2} \end{bmatrix} = \begin{bmatrix} I_1 \\ 0 \end{bmatrix}$$

Example



$$\begin{bmatrix} G_2 + G_3 + C_5 jw & -G_3 \\ -G_3 & G_3 + \frac{1}{L_6 jw} \end{bmatrix} \begin{bmatrix} V_{d1} \\ V_{d2} \end{bmatrix} = \begin{bmatrix} I_1 - I_4 \\ I_4 - I_7 \end{bmatrix}$$

Example

$$V_7 = V_{d1}$$

$$I_4 = 2V_2 = 2V_{d1}$$

$$\begin{bmatrix} G_2 + G_3 + C_5 jw & -G_3 & 0 \\ -G_3 & G_3 + \frac{1}{L_6 jw} & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{d1} \\ V_{d2} \\ I_7 \end{bmatrix} = \begin{bmatrix} I_1 \\ 0 \\ V_7 \end{bmatrix}$$

$v_7(t) = 2\cos(2\pi 60 t + \frac{2\pi}{3}),\ i_1(t) = 3\cos(2\pi 60 t + \frac{7\pi}{3}),\ L_6 = 2H,\ C = 2F,$
$R_2 = R_3 = 1\Omega$

$$\begin{bmatrix} 1 + 1 + 2j2\pi 60 & -1 & 0 \\ -1 & 1 + \frac{1}{2j2\pi 60} & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{d1} \\ V_{d2} \\ I_7 \end{bmatrix} = \begin{bmatrix} 3e^{\frac{7\pi}{3}j} \\ 0 \\ 2e^{\frac{2\pi}{3}j} \end{bmatrix}$$

Network functions

Network functions :

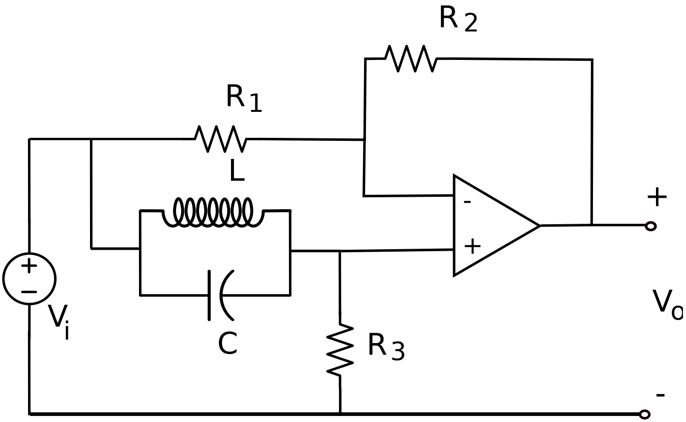
(a) Voltage transfer functions $\frac{V_o}{V_i}$, (b) Transfer admittances $\frac{I_o}{V_i}$, (c) Current transfer function $\frac{I_o}{V_i}$, (d) Transfer impedance $\frac{V_o}{I_i}$.

Consider a general linear time-invariant circuit N. Assume that N is driven by one independent source, say, the sinusoidal current source I_i represented by the phasor.
Suppose we want to calculate the node voltage E, and consider the dependence of the phasor E on w.

$$\frac{E(jw)}{I_i}$$

is a function of jw which depends only on the circuit N and not on I_i. it is called the transfer impedance from I_i, to E.

Find voltage transfer functions from V_i to V_0.



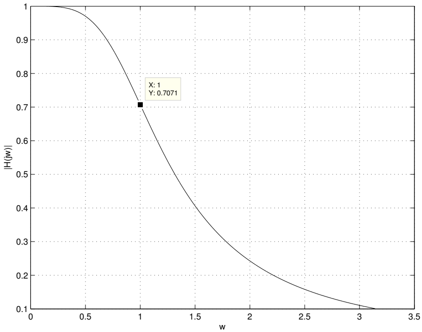
Example : Chua's book, Page 526, Examples 2 and 3.

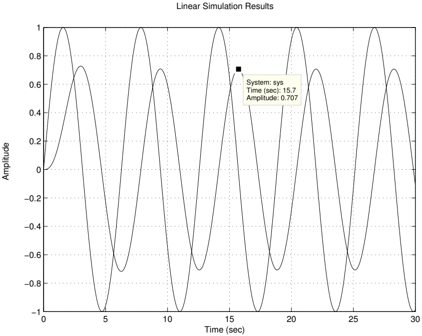
A linear time-invariant circuit N in the sinusoidal steady state of frequency w. $H(jw) = |H(jw)|e^{j\angle H(jw)}$ is the voltage transfer function from $V_s = |V_s|e^{j\angle V_s}$ to $V_k = |V_k|e^{j\angle V_k}$.

$$V_k = |V_k|e^{j\angle V_k} = |H(jw)|e^{j\angle H(jw)}|V_s|e^{j\angle V_s}$$

$$= |H(jw)||V_s|e^{j(\angle H(jw)+\angle V_s)}$$

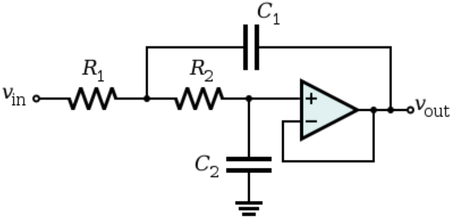
$$v_k(t) = |H(jw)||V_s|\cos(wt + \angle H(jw) + \angle V_s)$$





Example: Low Pass Filter

YouTube Video: RC Filter



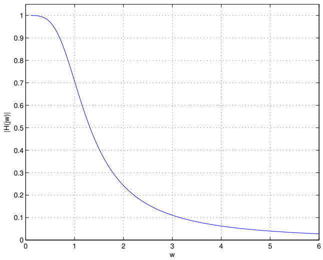
Verify

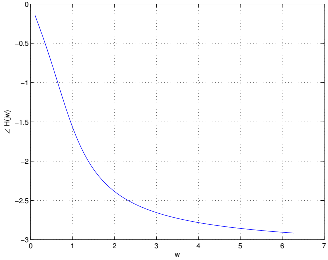
$$H(j\omega) = \frac{V_o}{V_i} = \frac{\omega_0^2}{\omega_0^2 - \omega + 2\alpha j\omega}$$

where $\omega_o^2 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$ and $2\alpha = \frac{(R_1+R_2)}{(R_1 R_2)} \frac{1}{C_1}$ (Note that $Q = \frac{\omega_0}{2\alpha}$)

Example: Low Pass Filter

$\omega_o = 1$ and $2\alpha = \sqrt{2}$



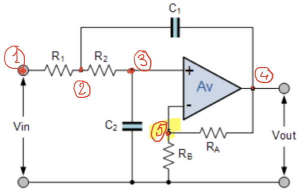


$H(j1) = \frac{1}{\sqrt{2}j} = \frac{1}{\sqrt{2}} e^{-\frac{\pi}{2}}$

$|H(j\omega)|\,|_{\text{in decibels}} = 20log(|H(j\omega)|)$

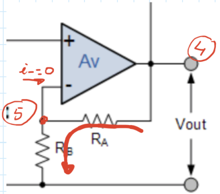
$|H(j0)| = 0dB, |H(j1)| = \frac{1}{\sqrt{2}} = -3dB$

Second Order Low Pass Filter



Transfer function: $\frac{V_o}{V_i} = ?$

NODE 5

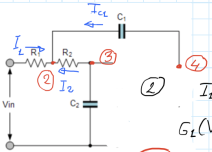


for OPAMP $V_{n3} = V_{n5}$

$V_{n5} = \frac{R_B}{R_A + R_B} \cdot V_{n4}$ Note $V_{n4} = V_o$

$= k V_{n4}$

NODE 2

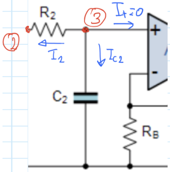


(2) $I_1 + I_2 + I_{C1} = 0$

$G_1(V_{n1} - V_{n2}) + G_2(V_{n3} - V_{n2}) + C_1 j\omega (V_{n4} - V_{n2}) = 0$

Equ 1: $G_1 V_{n1} - (G_1 + G_2 + C_1 j\omega) V_{n2} + G_2 V_{n3} + C_1 j\omega V_{n4} = 0$ Note $V_{n1} = V_i$

$V_{n3} = V_{n5} = k V_{n4}$



$V_{n2} = ?$

$I_2 + I_{c2} = 0$ KCL

$G_2(V_{n3} - V_{n2}) + V_{n3} C_2 j\omega = 0$

$(G_2 + C_2 j\omega) V_{n3} = G_2 V_{n2} \Rightarrow V_{n2} = \frac{(G_2 + C_2 j\omega)}{G_2} \cdot V_{n3} = \frac{(G_2 + C_2 j\omega)}{G_2} \cdot k V_{n4}$

Equ.2

$$G_1 V_{n1} - (G_1 + G_2 + C_1 j\omega) \cdot \frac{(G_2 + C_2 j\omega)}{G_2} \cdot k V_{n4} + G_2 \cdot k V_{n4} + C_1 j\omega V_{n4} = 0$$

$V_{n1} \to V_i$, $V_{n4} \to V_o$

$$G_1 V_{n1} + \left(C_1 j\omega + G_2 k - (G_1 + G_2 + C_1 j\omega) \frac{(G_2 + C_2 j\omega)}{G_2} \cdot k \right) V_{n4} = 0$$

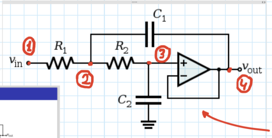
$$\frac{V_o}{V_i} = \frac{G_1}{(G_1 + G_2 + C_1 j\omega) \frac{(G_2 + C_2 j\omega)}{G_2} \cdot k - C_1 j\omega - G_2 k}$$

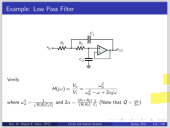
$$= \frac{G_1 \cdot G_2}{-k C_1 \cdot C_2 \omega^2 + k((G_1 + G_2) C_2 + G_2 C_1) j\omega + (G_1 + G_2) k G_2 - G_2 C_1 j\omega - G_2^2 k}$$

$$= \frac{G_1 \cdot G_2}{-k C_1 \cdot C_2 \omega^2 + k (G_1 + G_2) C_2 j\omega + G_1 G_2 k} \qquad G = \frac{1}{R}$$

$$= \left(\frac{1}{k}\right) \frac{1}{-C_1 C_2 R_1 R_2 \omega^2 + (R_1 + R_2) C_2 j\omega + 1}$$

$$k = \frac{R_B}{R_B + R_A} \Rightarrow \frac{1}{k} = 1 + \frac{R_A}{R_B} \geq 1$$





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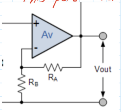


$$= \frac{1}{-C_1C_2R_1R_2\omega^2 + (R_1+R_2)C_2 j\omega + 1}$$

without k

$$H(j\omega) = \frac{V_o}{V_i} = \frac{\frac{1}{R_1R_2C_1C_2}}{-\omega^2 + \frac{(R_1+R_2)C_2}{R_1R_2C_1C_2} j\omega + \frac{1}{R_1R_2C_1C_2}}$$

$\omega = 0 \qquad H(j0) = 1$

$$\omega^2 = \frac{1}{R_1R_2C_1C_2} \qquad H(j\omega) = \frac{\frac{1}{R_1R_2C_1C_2}}{-\frac{1}{R_1R_2C_1C_2} + \frac{(R_1+R_2)C_2}{R_1R_2C_1C_2} \cdot \frac{1}{\sqrt{R_1R_2C_1C_2}} j + \frac{1}{R_1R_2C_1C_2}}$$

$$= -j \frac{\sqrt{R_1R_2C_1C_2}}{(R_1+R_2)C_2}$$

Let's $R_1 = R_2 = R$ and $C_1 = 2C_2$

$$= -j \frac{R}{2R} \sqrt{\frac{C_1C_2}{C_2^2}}$$

$$= -j \frac{1}{2} \sqrt{2}$$

$$= -j \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} e^{-\frac{\pi}{4} j}$$

$$\omega = \hat{\omega} = \frac{1}{\sqrt{R_1R_2C_1C_2}} \qquad H(j\hat{\omega}) = \frac{1}{\sqrt{2}} e^{-j\pi/4}$$

$\omega = \infty \qquad H(j\omega) = 0$

SPICE PROG.

OPAMP →

OPAMP MODEL

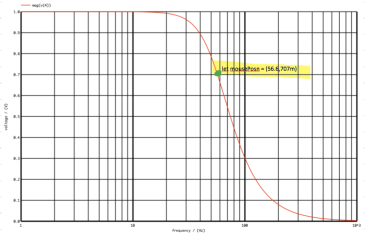
```
R1 1 2 1k
R2 2 3 1k
C2 3 0 2U
C1 2 4 4U
XOP 3 4 4 OPAMP1
.SUBCKT OPAMP1     1   2   6
* INPUT IMPEDANCE
RIN      1         2         10MEG
* DC GAIN (100K) AND POLE 1 (10HZ)
EGAIN   3 0        1 2       100K
RP1      3         4         1K
CP1      4         0         15.915UF
* OUTPUT BUFFER AND RESISTANCE*
EBUFFER           5 0       4 0      1
ROUT    5          6         10
.ENDS
Vin 1 0 DC 0.0 AC 1.0 0.0
.ac dec 100 1Hz 1000Hz
.END
```

⟹ plot mag(v(4))

$$\omega = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$$

$$\omega = 353{,}55$$

$$f = 56{,}2\,Hz$$



Superposition of Sinusoidal Steady States

Let N be a linear time-invariant circuit which is driven by two sinusoidal independent sources operating at two different frequencies.

Voltage source is specified by phasor E, and operates at frequency w_1. The current source is specified by phasor I, and operates at frequency w_2.

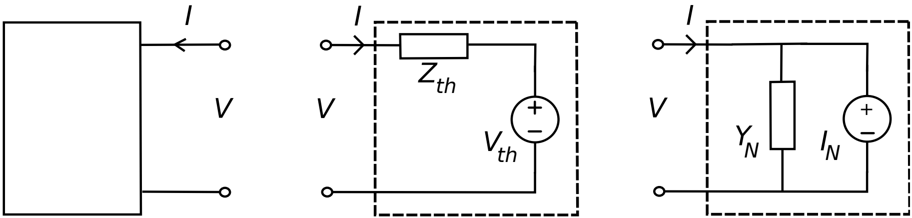
$H_1(jw)$ be the voltage transfer function of from E to V_k and $H_2(jw)$ be the transfer impedance from I to V_k.

By the superposition theorem, the resulting steady state is the superposition of two sinusoids

$$v_k(t) = |H_1(jw)||E|\cos(w_1 t + \angle H_1(jw) + \angle E) + |H_2(jw)||I| \cos w_2 t + ...$$

$w_1 = rw_2$ if r is a rational number then periodic if r is a irrational number then almost periodic

Thevenin - Norton Equivalent Circuits



The techniques for finding the Thevenin equivalent voltage (V_{th}) and impedance ($Z_{\mathrm{th}}(jw)$) are identical to those used for resistive circuits, except that the frequency domain equivalent circuit involves the manipulation of complex quantities.

▸ More detail EHB211 E: Slayt 192

Thevenin - Norton Equivalent Circuits

Driving-point characteristic of Thevenin equivalent circuit is defined by

$$V = Z_{\mathrm{th}}(jw)I + V_{\mathrm{th}}$$

and

$$V = V_{\mathrm{th}}\big|_{I=0} \text{ and } Z_{\mathrm{th}} = \frac{V}{I}\bigg|_{V_{\mathrm{th}}=0}.$$

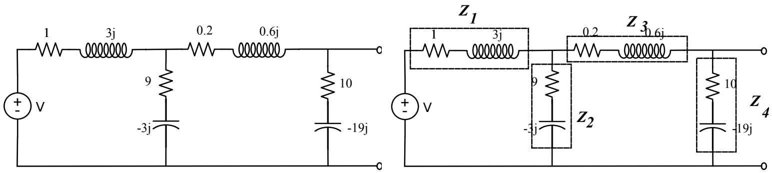
Driving-point characteristic of Norton equivalent circuit is defined by

$$I = Y_{\mathrm{N}}(jw)V + I_{\mathrm{N}}$$

and

$$I = I_{\mathrm{N}}\big|_{V=0} \text{ and } Y_{\mathrm{N}} = \frac{I}{V}\bigg|_{I_{\mathrm{N}}=0}.$$

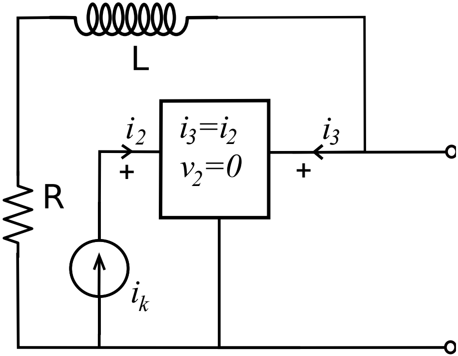
Example



$$Z_{\mathrm{th}} = (((Z_1//Z_2) + Z_3)//Z_4)$$

$$V_1 = \frac{(Z_3 + Z_4)//Z_2}{(Z_3 + Z_4)//Z_2 + Z_1} V, \;\; V_{\mathrm{th}} = \frac{Z_4}{Z_3 + Z_4} \; V_1$$

Example



Example

$Y = j$, $i_1 = 2v_2$ and $i_2 = -2v_1$ In steady state $I_{c1} = 1 - j$. Find complex power of two-port.

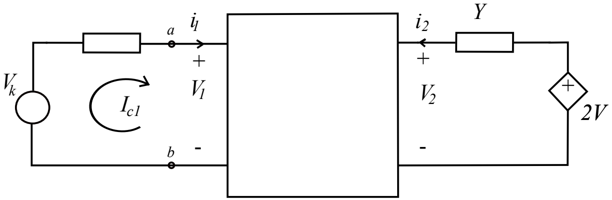


Figure: ($i_1 = 2v_2$, $i_2 = -2v_1$ ve $Y = j$ and $I_{c1} = 1 - j$).