STATICS

Assist. Prof. Dr. Cenk Üstündağ

7 Center of Gravity and Centroid

Chapter Objectives

- Concept of the center of gravity, center of mass, and the centroid
- Determine the location of the center of gravity and centroid for a system of discrete particles and a body of arbitrary shape
- Theorems of Pappus and Guldinus
- Method for finding the resultant of a general distributed loading

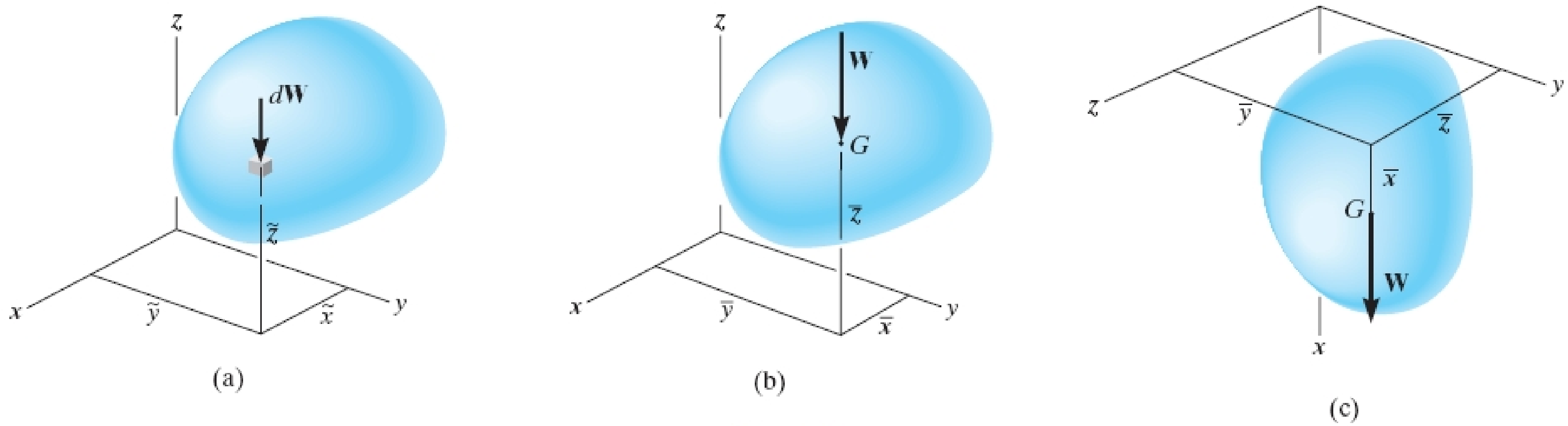
Chapter Outline

1. Center of Gravity and Center of Mass for a System of Particles
2. Composite Bodies
3. Theorems of Pappus and Guldinus

Center of Gravity and Center of Mass for a System of Particles

Center of Gravity

- Locates the resultant weight of a system of particles
- Consider system of n particles fixed within a region of space
- The weights of the particles can be replaced by a single (equivalent) resultant weight having defined point G of application



Center of Gravity and Center of Mass for a System of Particles

Center of Gravity

- Resultant weight = total weight of n particles

$$W_R = \sum W$$

- Sum of moments of weights of all the particles about x, y, z axes = moment of resultant weight about these axes
- Summing moments about the x axis,

$$\bar{x}W_R = \tilde{x}_1 W_1 + \tilde{x}_2 W_2 + \ldots + \tilde{x}_n W_n$$

- Summing moments about y axis,

$$\bar{y}W_R = \tilde{y}_1 W_1 + \tilde{y}_2 W_2 + \ldots + \tilde{y}_n W_n$$

Center of Gravity and Center of Mass for a System of Particles

Center of Gravity

- Although the weights do not produce a moment about z axis, by rotating the coordinate system 90° about x or y axis with the particles fixed in it and summing moments about the x axis,
  $\bar{z}W_R = \tilde{z}_1 W_1 + \tilde{z}_2 W_2 + \ldots + \tilde{z}_n W_n$
- Generally,

$$\bar{x} = \frac{\sum \tilde{x}m}{\sum m}; \bar{y} = \frac{\sum \tilde{y}m}{\sum m}, \bar{z} = \frac{\sum \tilde{z}m}{\sum m}$$

Center of Gravity and Center of Mass for a System of Particles

Center Mass

- Provided acceleration due to gravity g for every particle is constant, then W = mg

$$\bar{x} = \frac{\sum \tilde{x} m}{\sum m}; \bar{y} = \frac{\sum \tilde{y} m}{\sum m}, \bar{z} = \frac{\sum \tilde{z} m}{\sum m}$$

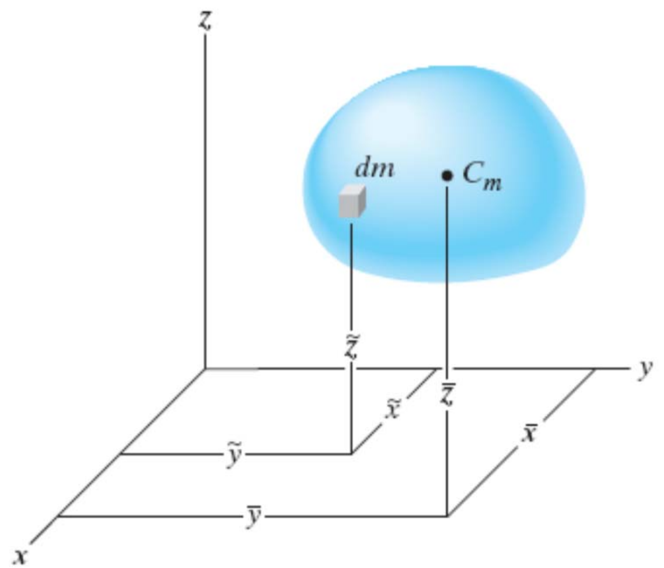
- By comparison, the location of the center of gravity coincides with that of center of mass
- Particles have weight only when under the influence of gravitational attraction, whereas center of mass is independent of gravity

Center of Gravity and Center of Mass for a System of Particles

Center Mass

- A rigid body is composed of an infinite number of particles
- Consider arbitrary particle having a weight of dW

$$\bar{x} = \frac{\int \tilde{x} dW}{\int dW}; \bar{y} = \frac{\int \tilde{y} dW}{\int dW}; \bar{z} = \frac{\int \tilde{z} dW}{\int dW}$$

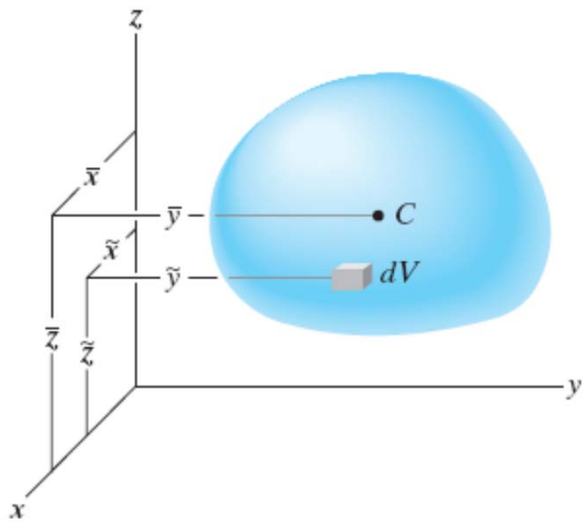


Center of Gravity and Center of Mass for a System of Particles

Centroid of a Volume

- Consider an object subdivided into volume elements dV, for location of the centroid,

$$\bar{x} = \frac{\int_V \tilde{x}\,dV}{\int_V dV};\ \bar{y} = \frac{\int_V \tilde{y}\,dV}{\int_V dV};\ \bar{z} = \frac{\int_V \tilde{z}\,dV}{\int_V dV}$$

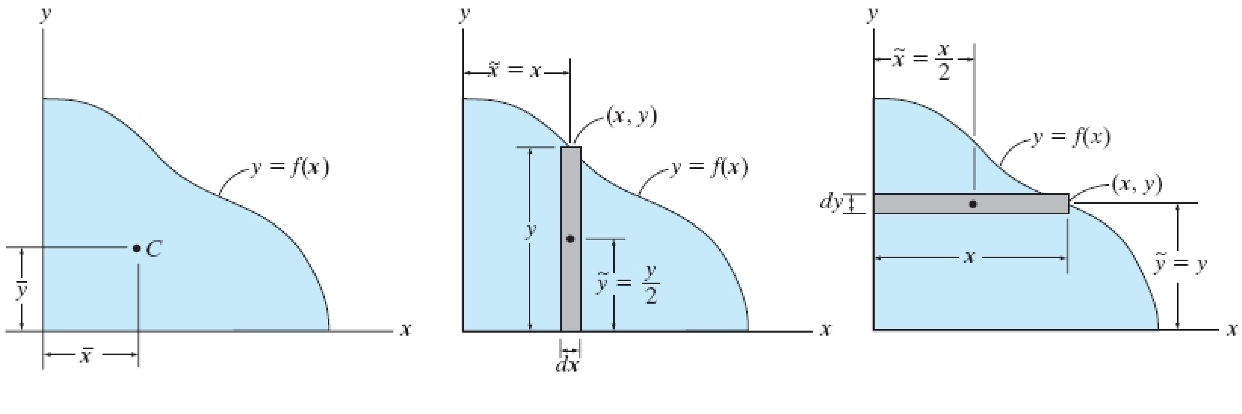


Center of Gravity and Center of Mass for a System of Particles

Centroid of an Area

- For centroid for surface area of an object, such as plate and shell, subdivide the area into differential elements dA

$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA};\ \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA};\ \bar{z} = \frac{\int_A \tilde{z}\, dA}{\int_A dA}$$



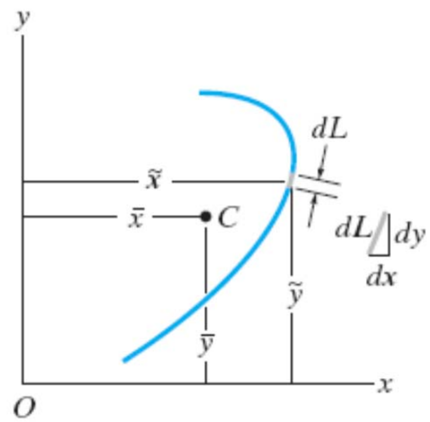
(a) (b) (c)

Center of Gravity and Center of Mass for a System of Particles

Centroid of a Line

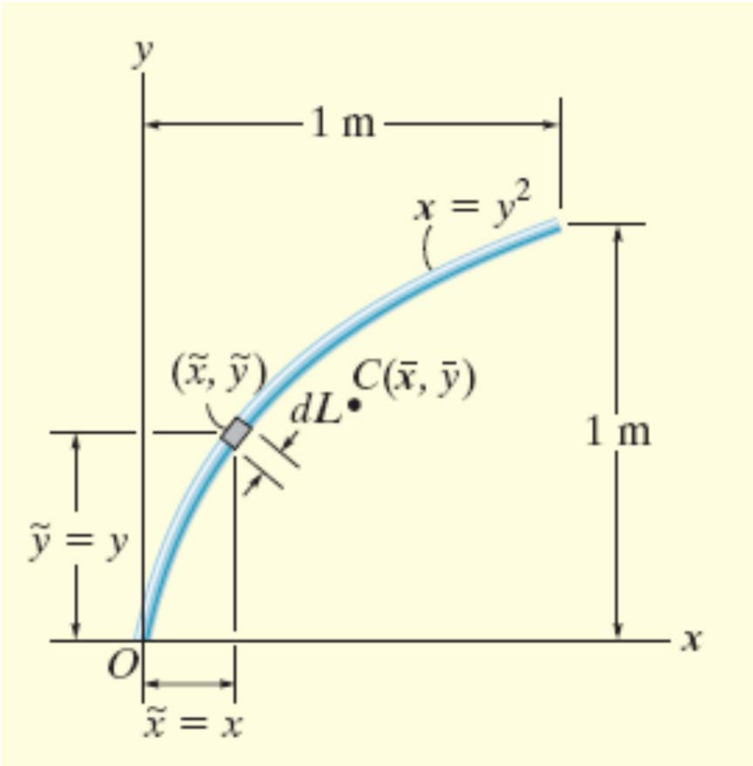
- If the geometry of the object takes the form of a line, the balance of moments of differential elements dL about each of the coordinate system yields

$$\bar{x} = \frac{\int_L \tilde{x}\,dL}{\int_L dL};\ \bar{y} = \frac{\int_L \tilde{y}\,dL}{\int_L dL};\ \bar{z} = \frac{\int_L \tilde{z}\,dL}{\int_L dL}$$



Example

Locate the centroid of the rod bent into the shape of a parabolic arc.



Example

Differential element

Located on the curve at the arbitrary point (x, y)

Area and Moment Arms

For differential length of the element dL

$$dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy$$

Since $x = y^2$ and then $dx/dy = 2y$

$$dL = \sqrt{(2y)^2 + 1}\,dy$$

The centroid is located at

$$\tilde{x} = x, \tilde{y} = y$$

Example

Integrations

$$\bar{x} = \frac{\int_L \tilde{x}\, dL}{\int_L dL} = \frac{\int_0^1 x\sqrt{4y^2+1}\,dy}{\int_0^1 \sqrt{4y^2+1}\,dy} = \frac{\int_0^1 y^2\sqrt{4y^2+1}\,dy}{\int_0^1 \sqrt{4y^2+1}\,dy}$$

$$= \frac{0.6063}{1.479} = 0.410m$$

$$\bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL} = \frac{\int_0^1 y\sqrt{4y^2+1}\,dy}{\int_0^1 \sqrt{4y^2+1}\,dy}$$

$$= \frac{0.8484}{1.479} = 0.574m$$

Composite Bodies

- Consists of a series of connected “simpler” shaped bodies, which may be rectangular, triangular or semicircular
- A body can be sectioned or divided into its composite parts
- Accounting for finite number of weights

$$\bar{x} = \frac{\sum \tilde{x} W}{\sum W} \quad \bar{y} = \frac{\sum \tilde{y} W}{\sum W} \quad \bar{z} = \frac{\sum \tilde{z} W}{\sum W}$$

Composite Bodies

Procedure for Analysis

Composite Parts

- Divide the body or object into a finite number of composite parts that have simpler shapes
- Treat the hole in composite as an additional composite part having negative weight or size

Moment Arms

- Establish the coordinate axes and determine the coordinates of the center of gravity or centroid of each part

Composite Bodies

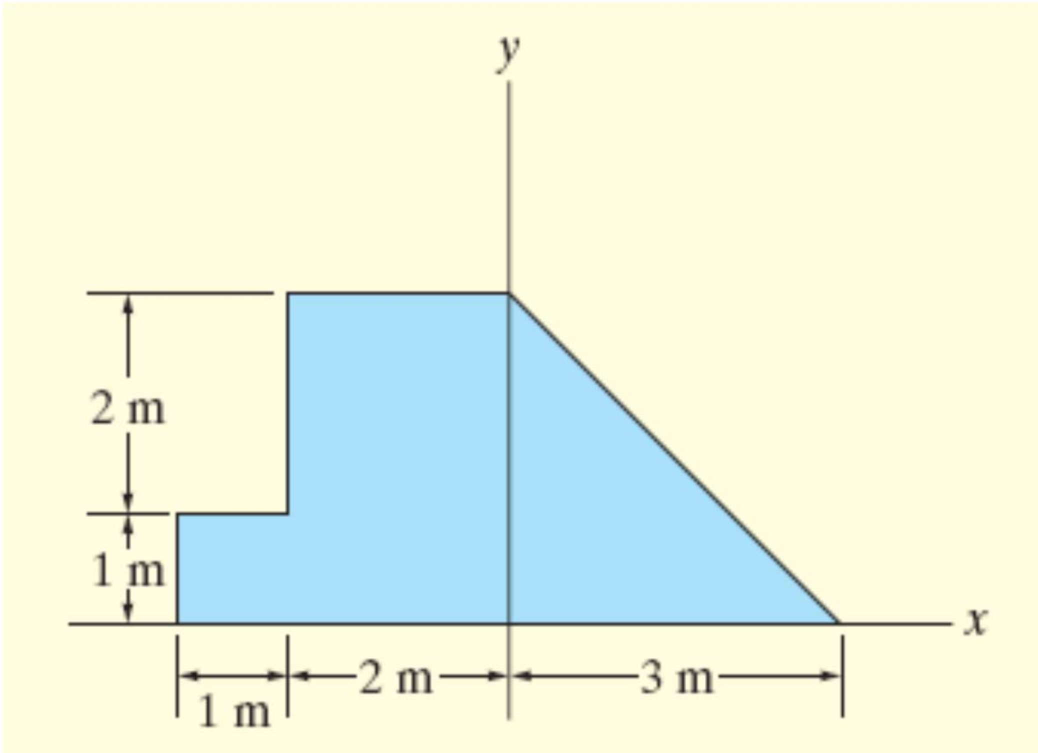
Procedure for Analysis

Summations

- Determine the coordinates of the center of gravity by applying the center of gravity equations
- If an object is symmetrical about an axis, the centroid of the objects lies on the axis

Example

Locate the centroid of the plate area.

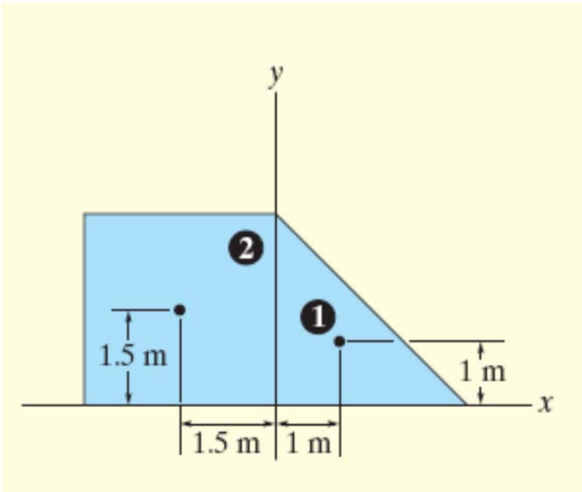


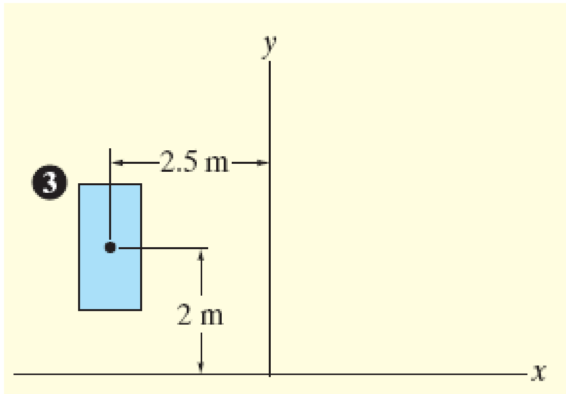
Solution

<u>Composite Parts</u>

Plate divided into 3 segments.

Area of small rectangle considered “negative”.





Solution

Moment Arm

Location of the centroid for each piece is determined and indicated in the diagram.

Segment	A (m^2)	$\tilde{x}$ (m)	$\tilde{y}$ (m)	$\tilde{x}A$ (m^3)	$\tilde{y}A$ (m^3)
1	$\frac{1}{2}(3)(3) = 4.5$	1	1	4.5	4.5
2	$(3)(3) = 9$	−1.5	1.5	−13.5	13.5
3	$-(2)(1) = -2$	−2.5	2	5	−4
	$\Sigma A = 11.5$			$\Sigma\tilde{x}A = -4$	$\Sigma\tilde{y}A = 14$

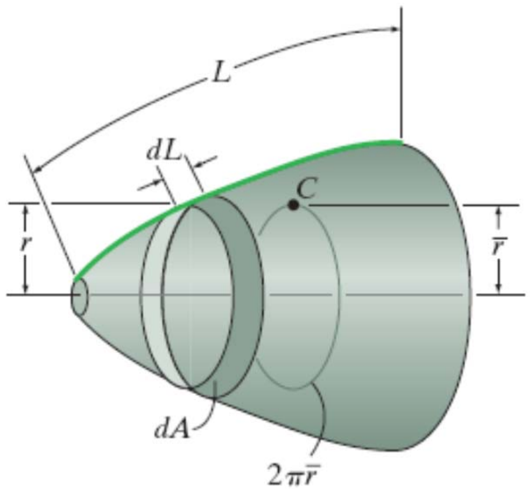
Summations

$$\bar{x} = \frac{\sum \tilde{x}A}{\sum A} = \frac{-4}{11.5} = -0.348mm$$

$$\bar{y} = \frac{\sum \tilde{y}A}{\sum A} = \frac{14}{11.5} = 1.22mm$$

Theorems of Pappus and Guldinus

- A surface area of revolution is generated by revolving a plane curve about a non-intersecting fixed axis in the plane of the curve
- A volume of revolution is generated by revolving a plane area about a nonintersecting fixed axis in the plane of area



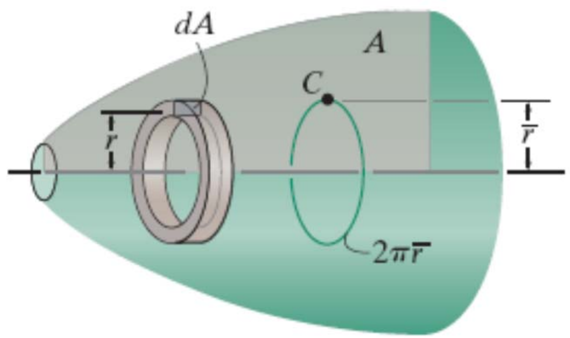
Theorems of Pappus and Guldinus

- The theorems of Pappus and Guldinus are used to find the surfaces area and volume of any object of revolution provided the generating curves and areas do not cross the axis they are rotated

Surface Area

- Area of a surface of revolution = product of length of the curve and distance traveled by the centroid in generating the surface area

$$A = \theta \bar{r} L$$

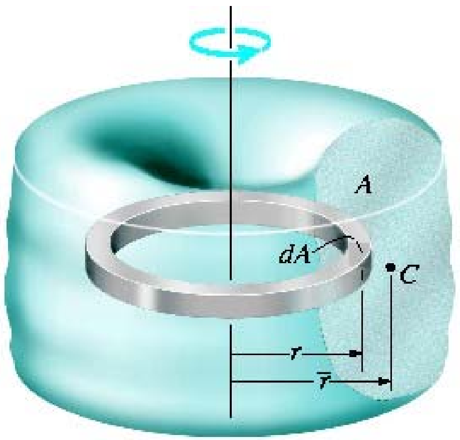


Theorems of Pappus and Guldinus

Volume

- Volume of a body of revolution = product of generating area and distance traveled by the centroid in generating the volume

$$V = \theta \bar{r} A$$



Example

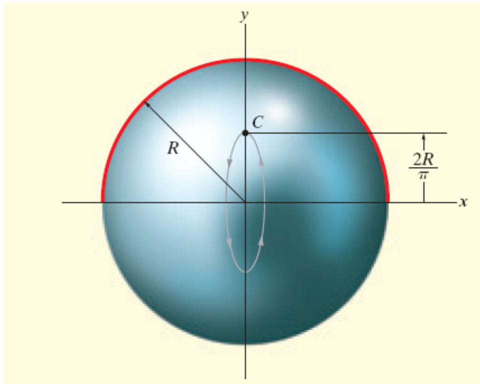
Show that the surface area of a sphere is A = 4πR^2 and its volume V = 4/3 πR^3.

Solution

Surface Area

Generated by rotating semi-arc about the x axis

For centroid, $\bar{r} = 2R/\pi$



For surface area, $A = \theta \tilde{r} L;$

$$A = 2\pi\left(\frac{2R}{\pi}\right)\pi R = 4\pi R^2$$

Solution

Volume

Generated by rotating semicircular area about the x axis

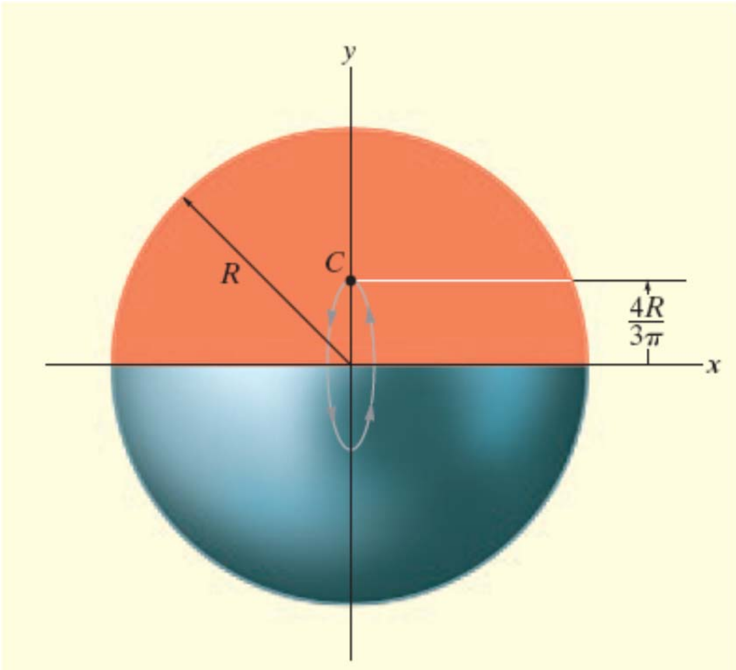
For centroid,

$$\bar{r} = 4R/3\pi$$

For volume,

$$V = \theta \tilde{r} A;$$

$$V = 2\pi\left(\frac{4R}{3\pi}\right)\left(\frac{1}{2}\pi R^2\right) = \frac{4}{3}\pi R^3$$



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8 Friction

Chapter Objectives

- Introduce the concept of dry friction
- To present specific applications of frictional force analysis on wedges, screws, belts, and bearings
- To investigate the concept of rolling resistance

Chapter Outline

1. Characteristics of Dry Friction
2. Problems Involving Dry Friction
3. Wedges
4. Frictional Forces on Screws
5. Frictional Forces on Flat Belts

Characteristics of Dry Friction

Friction

- Force that resists the movement of two contacting surfaces that slide relative to one another
- Acts tangent to the surfaces at points of contact with other body
- Opposing possible or existing motion of the body relative to points of contact
- Two types of friction – Fluid and Coulomb Friction

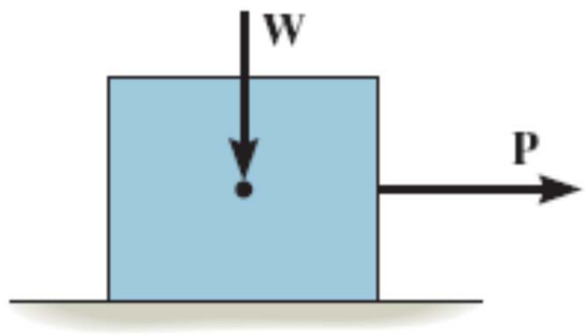
Characteristics of Dry Friction

- **Fluid friction** exist when the contacting surface are separated by a film of fluid (gas or liquid)
- Depends on velocity of the fluid and its ability to resist shear force
- **Coulomb friction** occurs between contacting surfaces of bodies in the absence of a lubricating fluid

Characteristics of Dry Friction

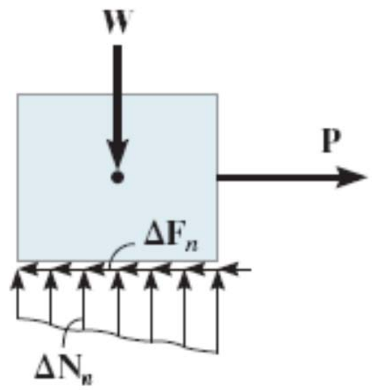
Theory of Dry Friction

- Consider the effects caused by pulling horizontally on a block of uniform weight **W** which is resting on a rough horizontal surface
- Consider the surfaces of contact to be nonrigid or deformable and other parts of the block to be rigid



Characteristics of Dry Friction

Theory of Dry Friction



- Normal force $\Delta\mathbf{N}_n$ and frictional force $\Delta\mathbf{F}_n$ act along the contact surface

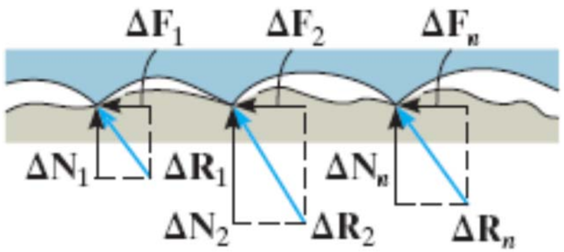
- For equilibrium, normal forces act upward to balance the block's weight **W**, frictional forces act to the left to prevent force **P** from moving the block to the right

Characteristics of Dry Friction

Theory of Dry Friction

- Many microscopic irregularities exist between the two surfaces of floor and the block

- Reactive forces $\Delta \mathbf{R}_n$ developed at each of the protuberances
- Each reactive force consist of both a frictional component $\Delta \mathbf{F}_n$ and normal component $\Delta \mathbf{N}_n$

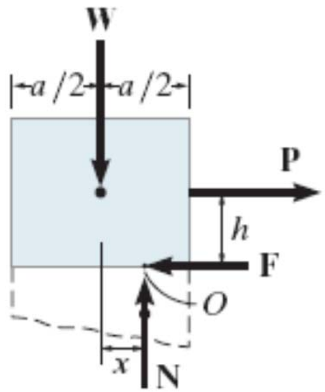


Characteristics of Dry Friction

Theory of Dry Friction

Equilibrium

- Effect of normal and frictional loadings are indicated by their resultant **N** and **F**
- Distribution of $\Delta \mathbf{F}_n$ indicates that **F** is tangent to the contacting surface, opposite to the direction of **P**
- Normal force **N** is determined from the distribution of $\Delta \mathbf{N}_n$



Resultant Normal and Frictional Forces

Characteristics of Dry Friction

Theory of Dry Friction

Equilibrium

- **N** is directed upward to balance **W**
- **N** acts a distance x to the right of the line of action of **W**
- This location coincides with the centroid or the geometric center of the loading diagram in order to balance the “tipping effect” caused by **P**

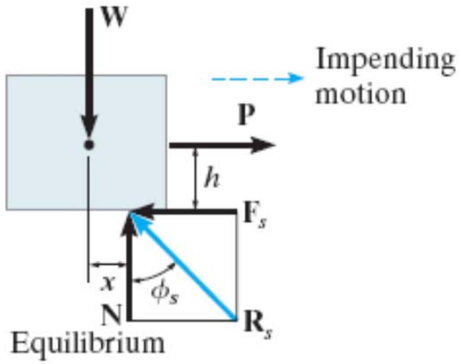
Characteristics of Dry Friction

Theory of Dry Friction

Impending Motion

- As P is slowly increased, F correspondingly increase until it attains a certain maximum value F, called the limiting static frictional force
- Limiting static frictional force F_s is directly proportional to the resultant normal force N

$$F_s = \mu_s N$$



Characteristics of Dry Friction

Theory of Dry Friction

Impending Motion

- Constant of proportionality μ_s is known as the coefficient of static friction
- Angle Φ_s that $\mathbf{R}_s$ makes with $\mathbf{N}$ is called the angle of static friction

$$\phi_s = \tan^{-1}\left(\frac{F_s}{N}\right) = \tan^{-1}\left(\frac{\mu_s N}{N}\right) = \tan^{-1}\mu_s$$

Characteristics of Dry Friction

Theory of Dry Friction

Typical Values of μ_s

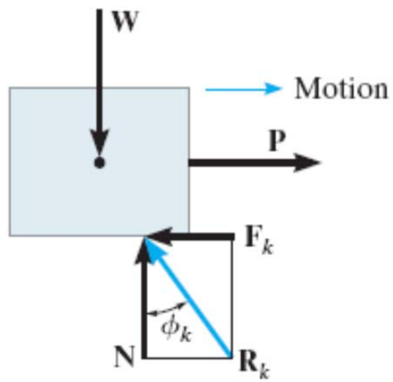
Contact Materials	Coefficient of Static Friction μ_s
Metal on ice	0.03 – 0.05
Wood on wood	0.30 – 0.70
Leather on wood	0.20 – 0.50
Leather on metal	0.30 – 0.60
Aluminum on aluminum	1.10 – 1.70

Characteristics of Dry Friction

Theory of Dry Friction

<u>Motion</u>

- When **P** is greater than F_s, the frictional force is slightly smaller value than F_s, called kinetic frictional force
- The block will not be held in equilibrium ($P > F_s$) but slide with increasing speed

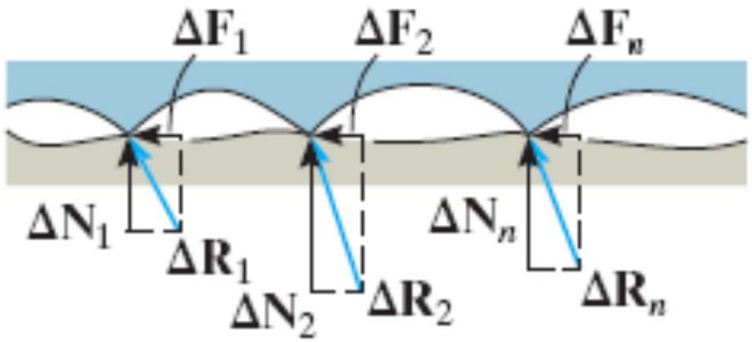


Characteristics of Dry Friction

Theory of Dry Friction

Motion

- The drop from F_s (static) to F_k (kinetic) can by explained by examining the contacting surfaces
- When P > F_s, P has the capacity to shear off the peaks at the contact surfaces



Characteristics of Dry Friction

Theory of Dry Friction

- Resultant frictional force $\mathbf{F}_k$ is directly proportional to the magnitude of the resultant normal force **N**

  $F_k = \mu_k N$

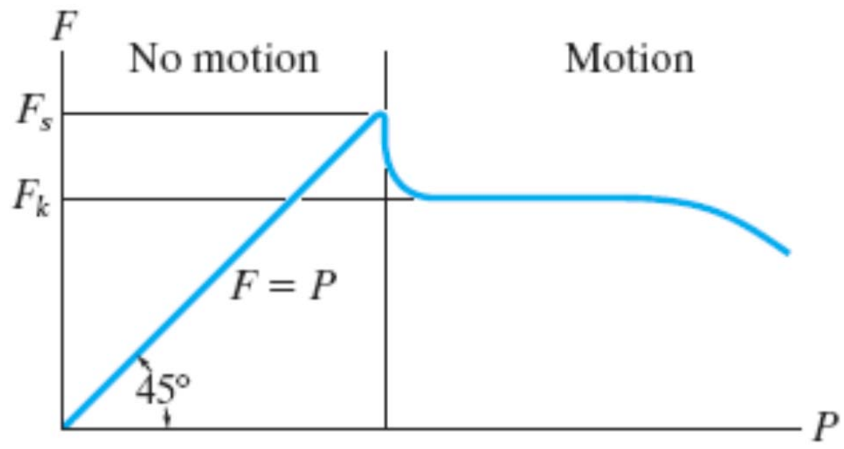
- Constant of proportionality μ_k is coefficient of kinetic friction
- μ_k are typically 25% smaller than μ_s
- Resultant $\mathbf{R}_k$ has a line of action defined by Φ_k, angle of kinetic friction

$$\phi_k = \tan^{-1}\left(\frac{F_k}{N}\right) = \tan^{-1}\left(\frac{\mu_k N}{N}\right) = \tan^{-1}\mu_k$$

Characteristics of Dry Friction

Theory of Dry Friction

- *F* is a *static frictional force* if equilibrium is maintained
- *F* is a *limiting static frictional force* when it reaches a maximum value needed to maintain equilibrium
- *F* is termed a *kinetic frictional force* when sliding occurs at the contacting surface



Characteristics of Dry Friction

Characteristics of Dry Friction

- The frictional force acts tangent to the contacting surfaces
- The max static frictional force F_s is independent of the area of contact
- The max static frictional force is greater than kinetic frictional force
- When slipping, the max static frictional force is proportional to the normal force and kinetic frictional force is proportional to the normal force

Problems Involving Dry Friction

Types of Friction Problems

- In all cases, geometry and dimensions are assumed to be known
- 3 types of mechanics problem involving dry friction

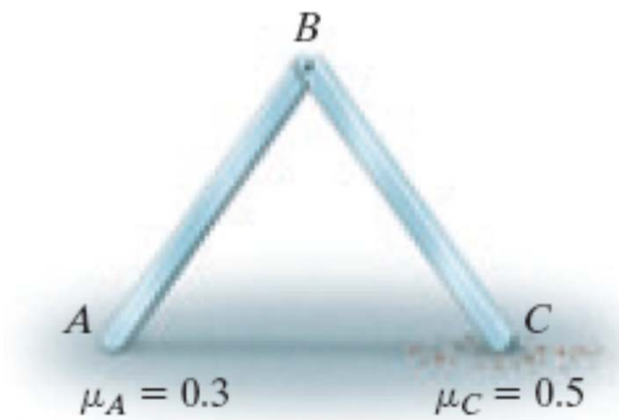
  - Equilibrium
  - Impending motion at all points
  - Impending motion at some points

Problems Involving Dry Friction

Types of Friction Problems

<u>Equilibrium</u>

- Total number of unknowns = Total number of available equilibrium equations
- Frictional forces must satisfy $F \leq \mu_s N$; otherwise, slipping will occur and the body will not remain in equilibrium
- We must determine the frictional forces at A and C to check for equilibrium



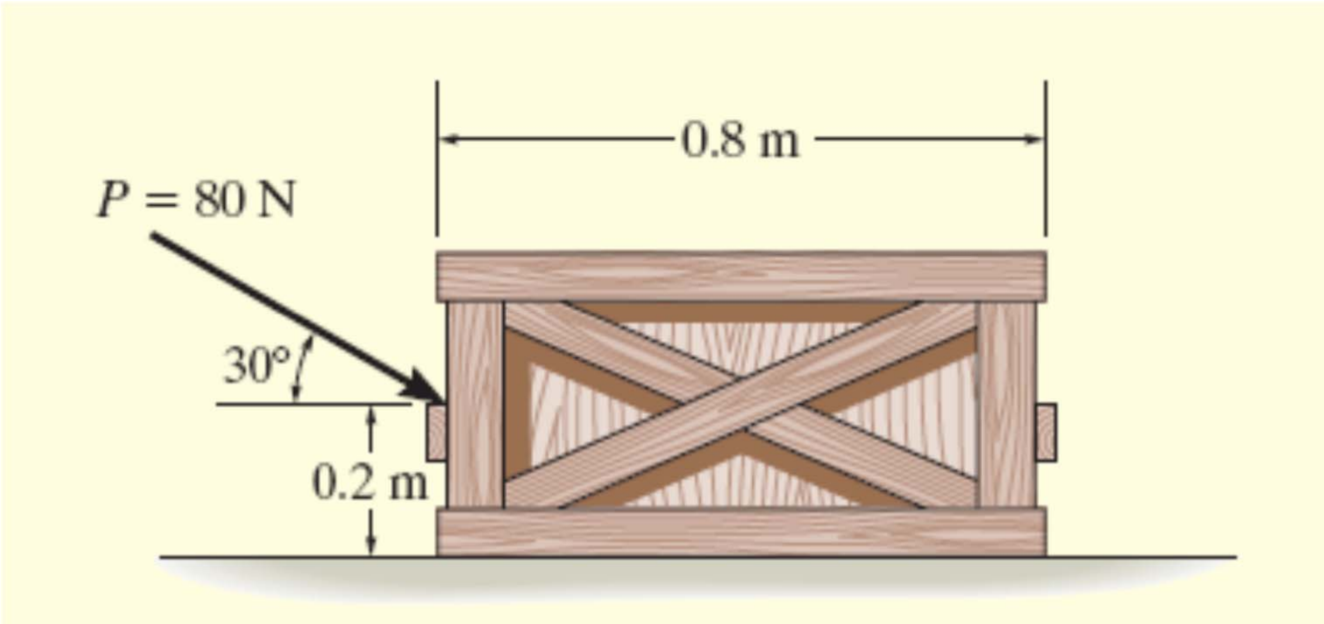
Problems Involving Dry Friction

Equilibrium Versus Frictional Equations

- Frictional force always acts so as to oppose the relative motion or impede the motion of the body over its contacting surface
- Assume the sense of the frictional force that require F to be an “equilibrium” force
- Correct sense is made after solving the equilibrium equations
- If F is a negative scalar, the sense of **F** is the reverse of that assumed

Example

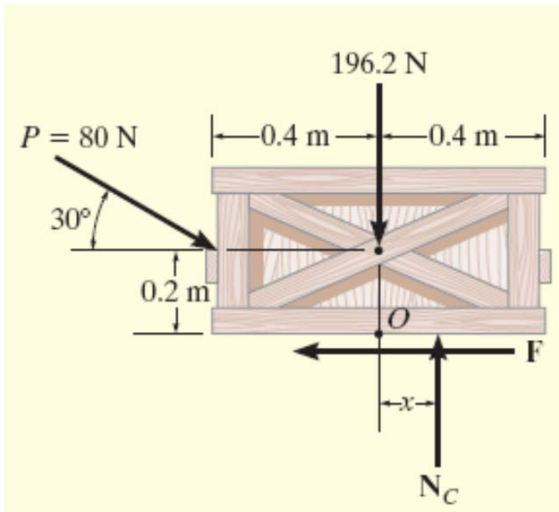
The uniform crate has a mass of 20kg. If a force P = 80N is applied on to the crate, determine if it remains in equilibrium. The coefficient of static friction is μ = 0.3.



Solution

Resultant normal force **N**C act a distance x from the crate's center line in order to counteract the tipping effect caused by **P.**

3 unknowns to be determined by 3 equations of equilibrium.



Solution

$$+\rightarrow \sum F_x = 0;$$

$$80\cos 30^\circ N - F = 0$$

$$+\uparrow \sum F_y = 0;$$

$$-80\sin 30^\circ N + N_C - 196.2N = 0$$

$$\sum M_O = 0;$$

$$80\sin 30^\circ N(0.4m) - 80\cos 30^\circ N(0.2m) + N_C(x) = 0$$

Solving

$$F = 69.3N, N_{C=236N} \quad x = -0.00908m = -9.08mm$$

Solution

Since x is negative, the resultant force acts (slightly) to the left of the crate's center line.

No tipping will occur since $x \le 0.4m$

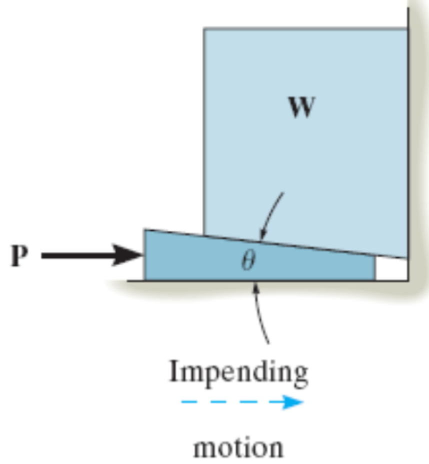
Max frictional force which can be developed at the surface of contact

$$F_{max} = \mu_s N_C = 0.3(236N) = 70.8N$$

Since $F = 69.3N < 70.8N$, the crate will not slip thou it is close to doing so.

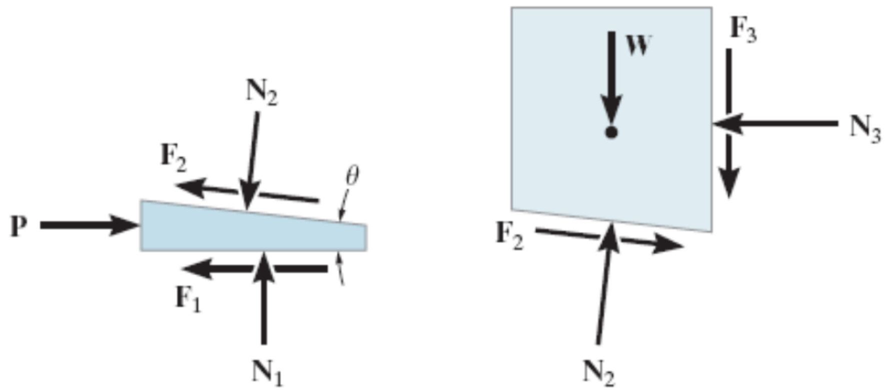
Wedges

- A simple machine used to transform an applied force into much larger forces, directed at approximately right angles to the applied force
- Used to give small displacements or adjustments to heavy load
- Consider the wedge used to lift a block of weight **W** by applying a force **P** to the wedge



Wedges

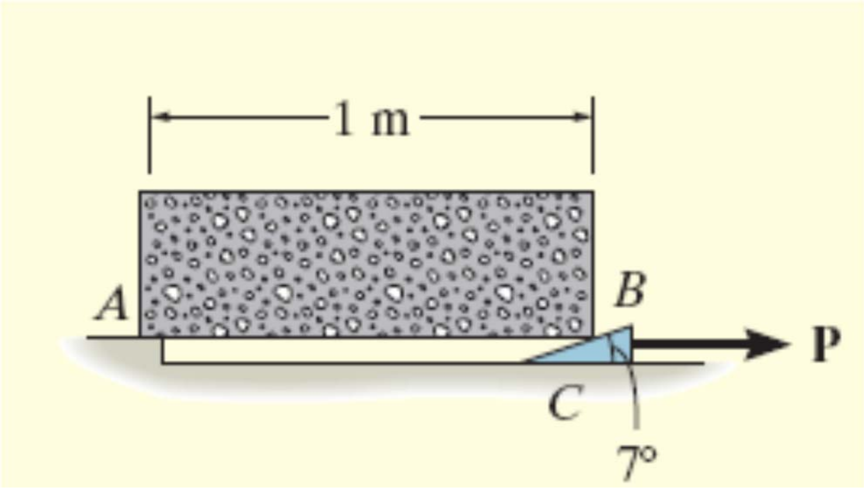
- FBD of the block and the wedge



- Exclude the weight of the wedge since it is small compared to weight of the block

Example

The uniform stone has a mass of 500kg and is held in place in the horizontal position using a wedge at B. if the coefficient of static friction μ_s = 0.3, at the surfaces of contact, determine the minimum force **P** needed to remove the wedge. Is the wedge self-locking? Assume that the stone does not slip at A.

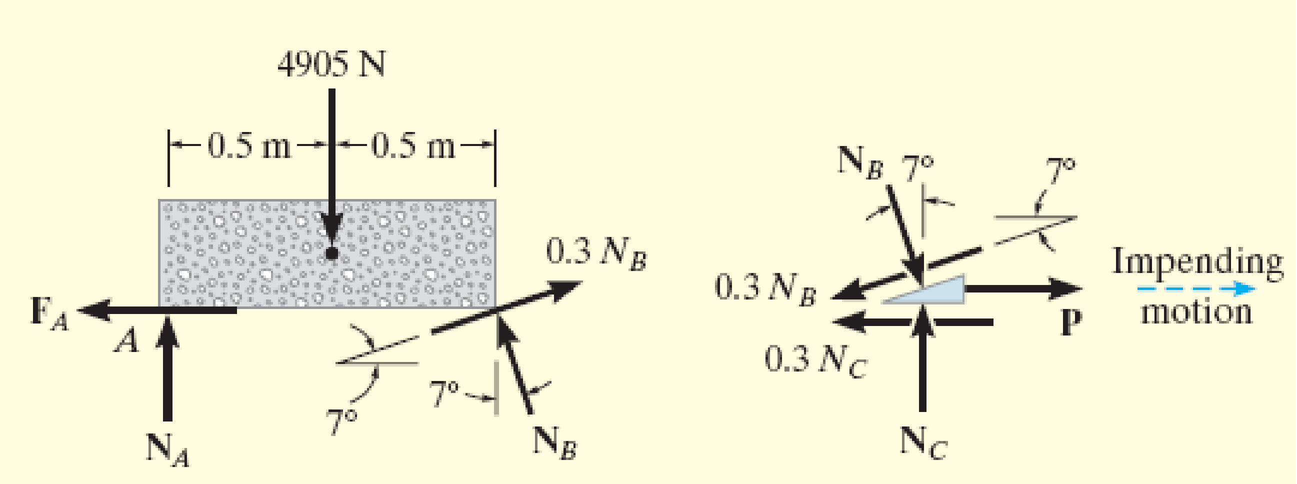


Solution

Minimum force P requires $F = \mu_s N_A$ at the surfaces of contact with the wedge.

FBD of the stone and the wedge as below.

On the wedge, friction force opposes the motion and on the stone at A, $F_A \leq \mu_s N_A$, slipping does not occur.



Solution

5 unknowns, 3 equilibrium equations for the stone and 2 for the wedge.

$$\sum M_A = 0;$$

$$-4905N(0.5m) + (N_B \cos 7^\circ N)(1m) + (0.3N_B \sin 7^\circ N)(1m) = 0$$

$$N_B = 2383.1N$$

$$+\rightarrow \sum F_x = 0;$$

$$2383.1\sin 7^\circ - 0.3(2383.1\cos 7^\circ) + P - 0.3N_C = 0$$

$$+\uparrow \sum F_y = 0;$$

$$N_C - 2383.1\cos 7^\circ N - 0.3(2383.1\sin 7^\circ) = 0$$

$$N_C = 2452.5N$$

$$P = 1154.9N = 1.15kN$$

Solution

Since P is positive, the wedge must be pulled out.

If P is zero, the wedge would remain in place (self-locking) and the frictional forces developed at B and C would satisfy

$$F_B < \mu_s N_B$$

$$F_C < \mu_s N_C$$

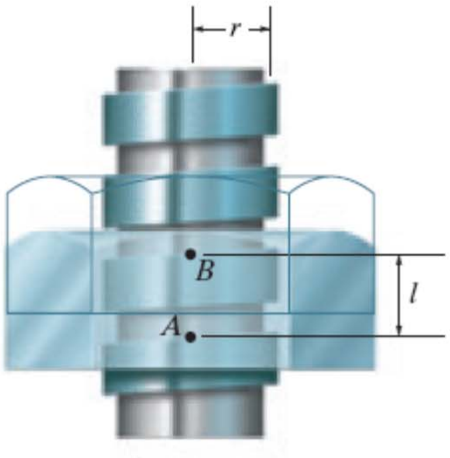
Frictional Forces on Screws

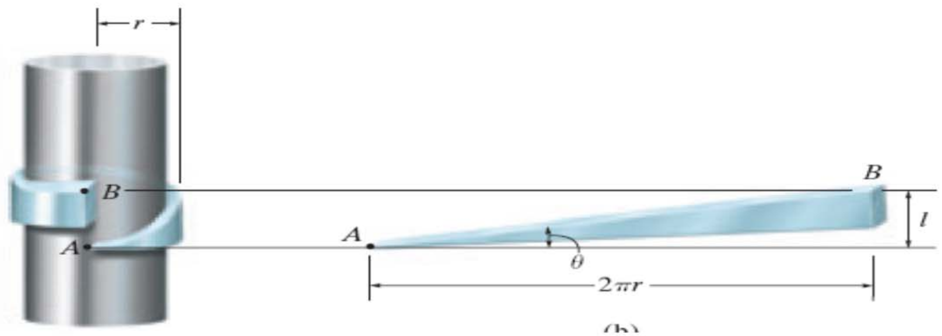
- Screws used as fasteners
- Sometimes used to transmit power or motion from one part of the machine to another
- A square-ended screw is commonly used for the latter purpose, especially when large forces are applied along its axis
- A screw is thought as an inclined plane or wedge wrapped around a cylinder

Frictional Forces on Screws

- A nut initially at A on the screw will move up to B when rotated 360° around the screw
- This rotation is equivalent to translating the nut up an inclined plane of height l and length 2πr, where r is the mean radius of the head
- Applying the force equations of equilibrium, we have

$$M = rW\tan(\phi_s + \theta)$$



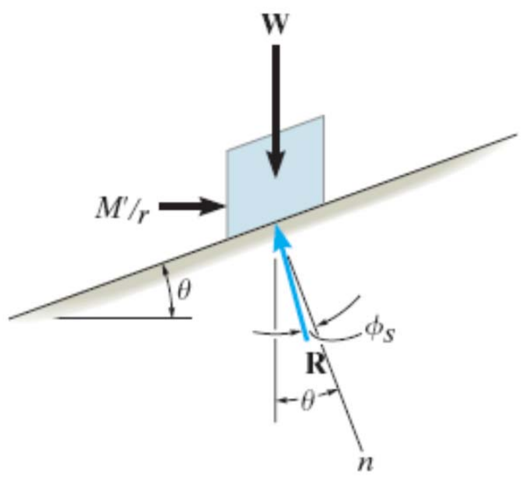


Frictional Forces on Screws

Downward Screw Motion

- If the surface of the screw is very slippery, the screw may rotate downward if the magnitude of the moment is reduced to say M' < M
- This causes the effect of **M**' to become **S**'

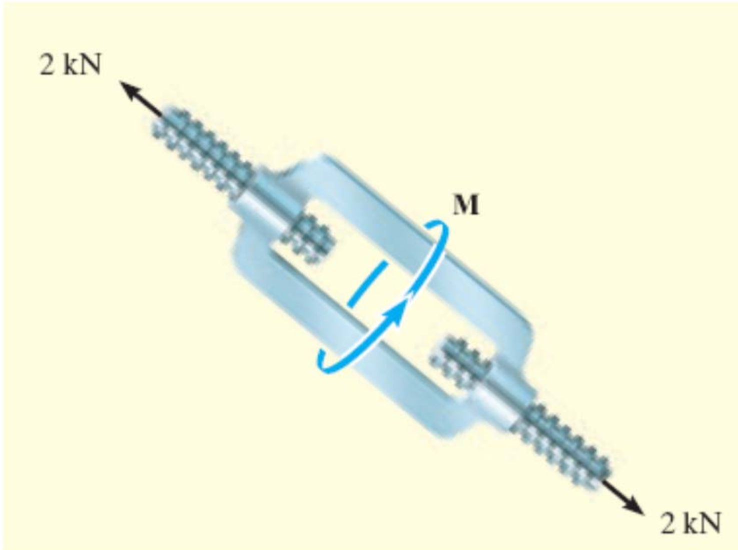
$$M' = Wr\tan(\theta - \Phi)$$



Downward screw motion ($\theta > \phi_s$)

Example

The turnbuckle has a square thread with a mean radius of 5mm and a lead of 2mm. If the coefficient of static friction between the screw and the turnbuckle is $\mu_s = 0.25$, determine the moment **M** that must be applied to draw the end screws closer together. Is the turnbuckle self-locking?



Solution

Since friction at two screws must be overcome, this requires

$$M = 2\left[Wr\tan(\theta + \phi)\right]$$

$$W = 2000N, r = 5mm, \phi_s = \tan^{-1}\mu_s = \tan^{-1}(0.25) = 14.04^\circ$$

$$\theta = \tan^{-1}(\ell / 2\pi r) = \tan^{-1}(2mm / 2\pi(5mm)) = 3.64^\circ$$

Solving

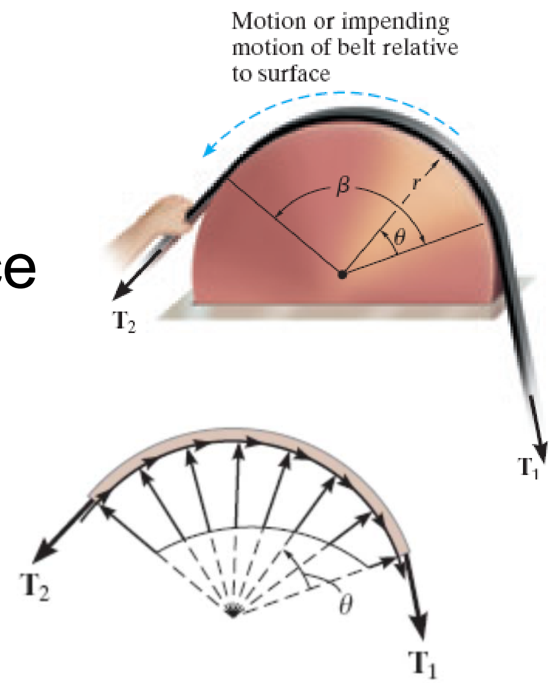
$$M = 2\left[(2000N)(5mm)\tan(14.04^\circ + 3.64^\circ)\right]$$

$$= 6374.7N.mm = 6.37N.m$$

When the moment is removed, the turnbuckle will be self-locking

Frictional Forces on Flat Belts

- It is necessary to determine the frictional forces developed between the contacting surfaces
- Consider the flat belt which passes over a fixed curved surface
- Obviously $T_2 > T_1$
- Consider FBD of the belt segment in contact with the surface
- **N** and **F** vary both in magnitude and direction

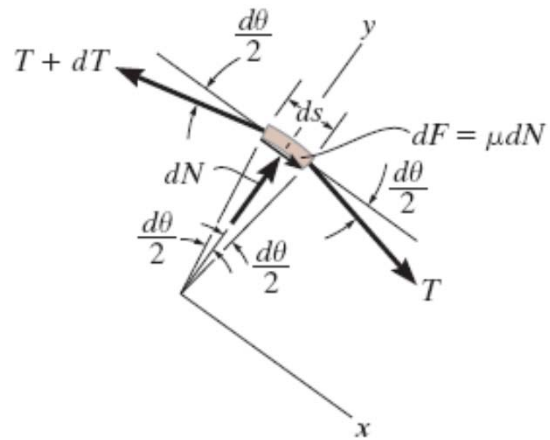


Frictional Forces on Flat Belts

- Consider FBD of an element having a length ds
- Assuming either impending motion or motion of the belt, the magnitude of the frictional force

  dF = μ dN

- Applying equilibrium equations



$$\sum F_x = 0;$$

$$T\cos\left(\frac{d\theta}{2}\right) + \mu dN - (T + dT)\cos\left(\frac{d\theta}{2}\right) = 0$$

$$\sum F_y = 0;$$

$$dN - (T + dT)\sin\left(\frac{d\theta}{2}\right) - T\sin\left(\frac{d\theta}{2}\right) = 0$$

Frictional Forces on Flat Belts

- We have

$$\mu dN = dT$$

$$dN = Td\theta$$

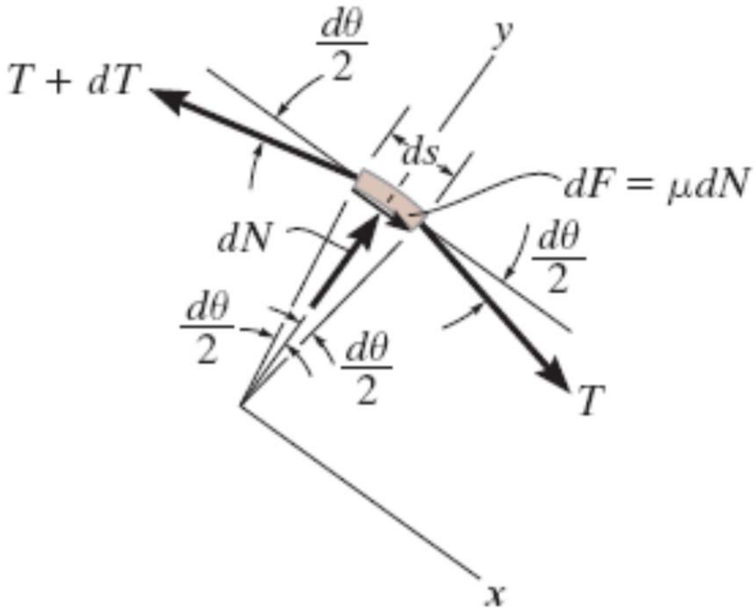
$$\frac{dT}{T} = \mu d\theta$$

$$T = T_1, \theta = 0, T = T_2, \theta = \beta$$

$$\int_{T_1}^{T_2} \frac{dT}{T} = \mu \int_0^\beta d\theta$$

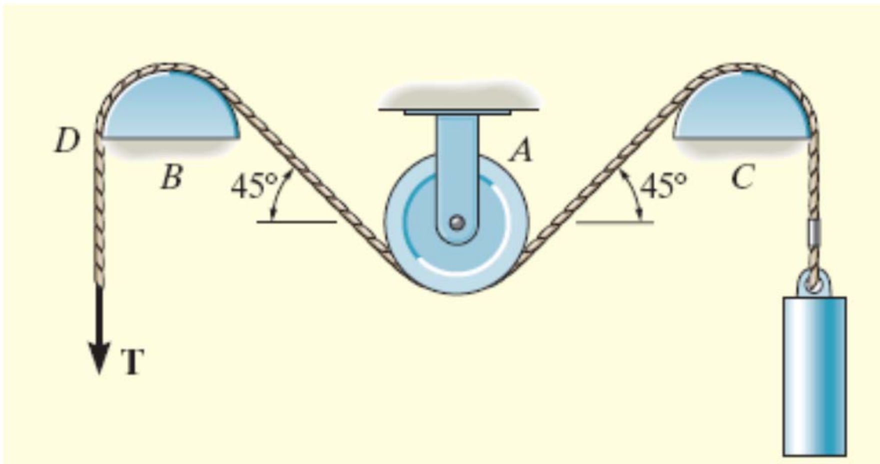
$$In \frac{T_2}{T_1} = \mu\beta$$

$$T_2 = T_1 e^{\mu\beta}$$



Example

The maximum tension that can be developed In the cord is 500N. If the pulley at A is free to rotate and the coefficient of static friction at fixed drums B and C is μ_s = 0.25, determine the largest mass of cylinder that can be lifted by the cord. Assume that the force **F** applied at the end of the cord is directed vertically downward.



Example

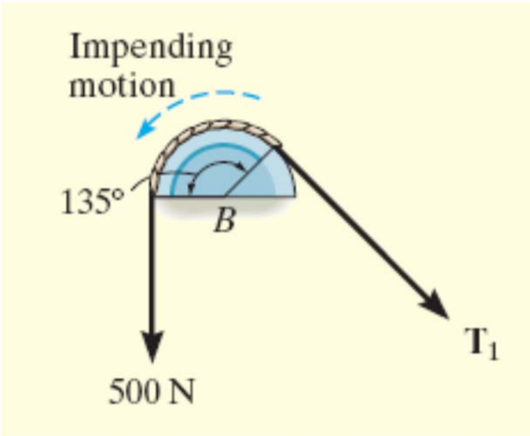
Weight of W = mg causes the cord to move CCW over the drums at B and C.

Max tension T_2 in the cord occur at D where T_2 = 500N

For section of the cord passing over the drum at B

180° = π rad, angle of contact between drum and cord

β = (135° /180°)π = 3/4π rad



$$T_2 = T_1 e^{\mu_s \beta};$$

$$500N = T_1 e^{0.25[(3/4)\pi]}$$

$$T_1 = \frac{500N}{e^{0.25[(3/4)\pi]}} = \frac{500N}{1.80} = 277.4N$$

Example

For section of the cord passing over the drum at C

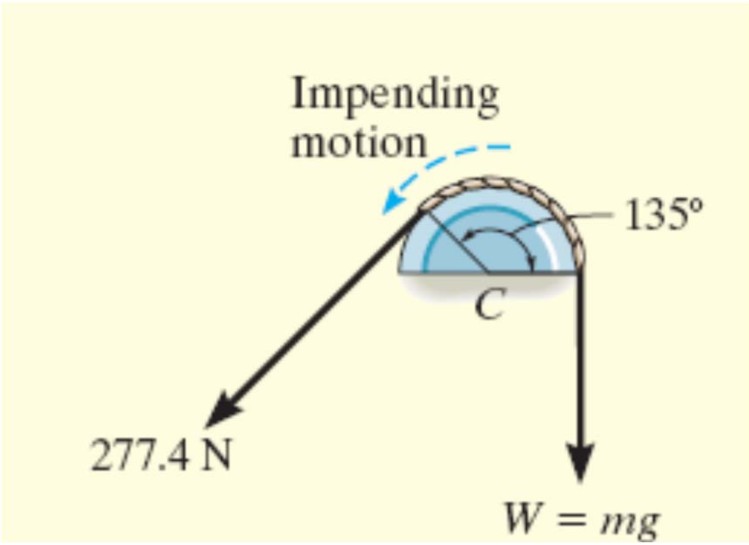
W < 277.4N

$$T_2 = T_1 e^{\mu_s \beta};$$

$$277.4 = We^{0.25[(3/4)\pi]}$$

$$W = 153.9N$$

$$m = \frac{W}{g} = \frac{153.9N}{9.81m/s^2} = 15.7kg$$



STATICS

Assist. Prof. Dr. Cenk Üstündağ

9 Cables

Cables

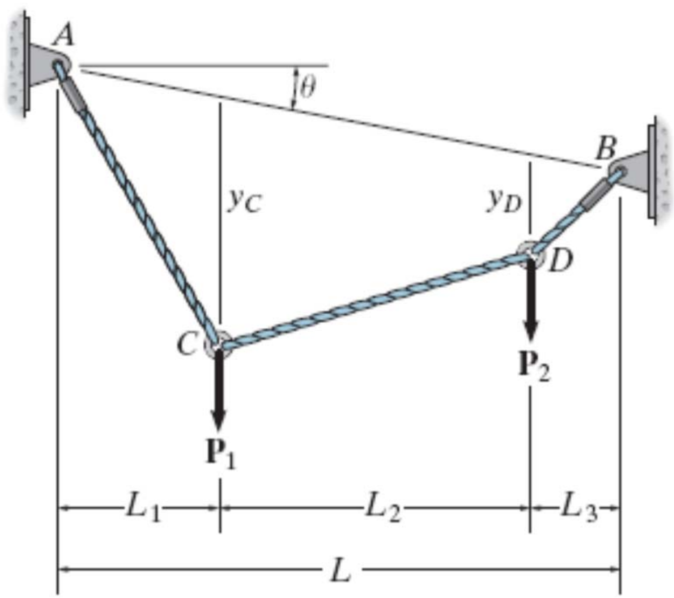
- Cables are often used in engineering structures for support and to transmit loads from one member to another. When used to support suspension roofs and bridges, cables form the main load-carrying element in the structure.
- In the force analysis of such systems, the weight of the cable itself may be neglected; however, when cables are used as guys for radio antennas, electrical transmission lines and derricks, the cable weight may become important and must be included in the structural analysis.

Cables

- Assumptions when deriving the relations between force in cable & its slope
  - Cable is perfectly flexible & inextensible
- Due to its flexibility, cable offers *no resistance to shear or bending*
- The force acting the cable is always tangent to the cable at points along its length

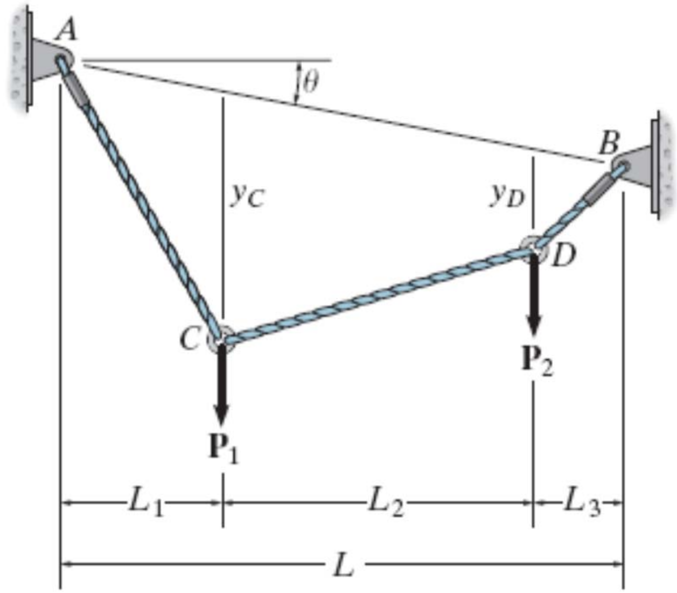
Cable subjected to concentrated loads

- When a cable of negligible weight supports several concentrated loads, the cable takes the form of several straight line segments
- Each of the segment is subjected to a constant tensile force
- θ specifies the angle of the chord AB
- L = cable length



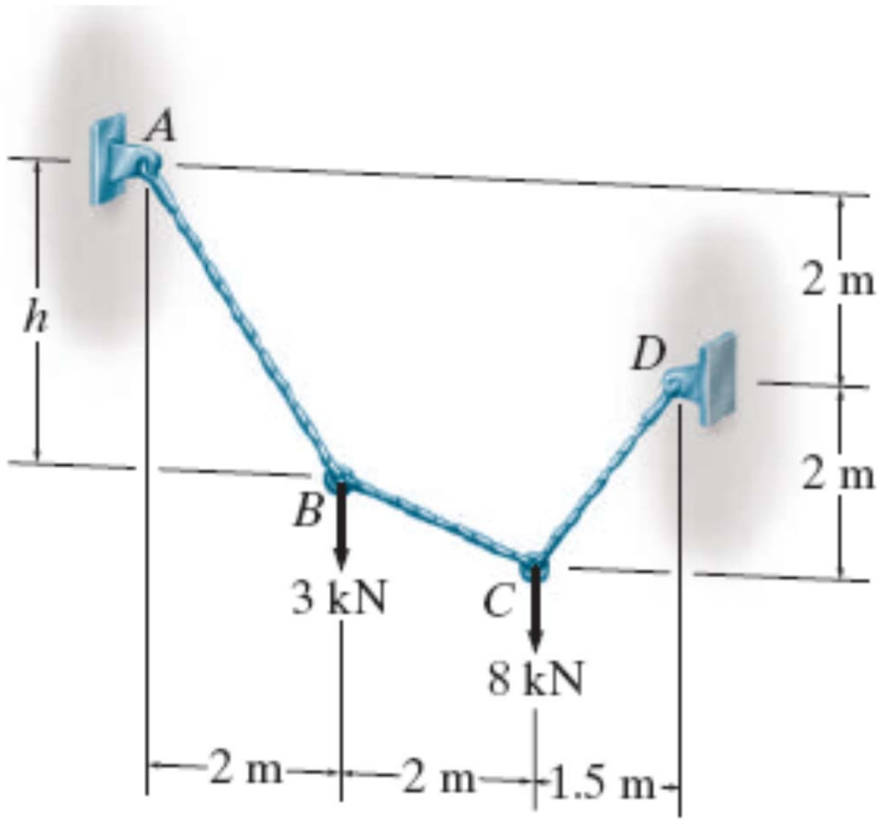
Cable subjected to concentrated loads

- If L_1, L_2 & L_3 and loads P_1 & P_2 are known, determine the 9 unknowns consisting of the tension of in each of the 3 segments, the 4 components of reactions at A & B and the sags y_C & y_D
- For solutions, we write 2 equations of equilibrium at each of 4 points A, B, C & D
- Total 8 equations
- The last equation comes from the geometry of the cable



Example

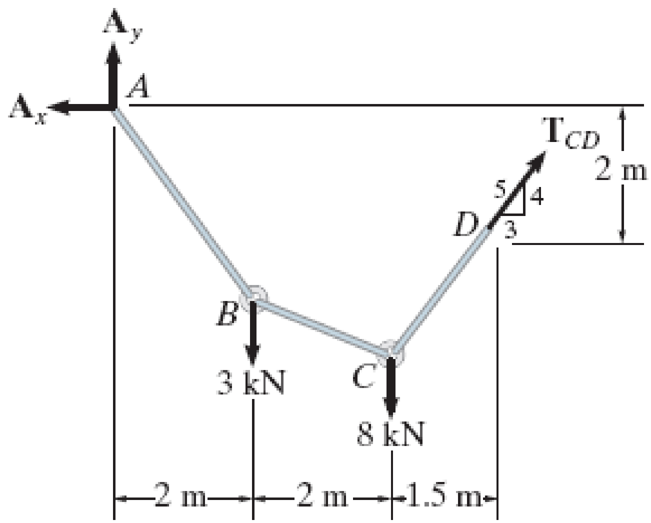
Determine the tension in each segment of the cable. Also, what is the dimension *h*?



Solution

By inspection, there are

- ➔ 4 unknown external reactions (A_x, A_y, D_x and D_y)
- ➔ 3 unknown cable tensions



These unknowns along with the sag h can be determined from available equilibrium equations applied to points A through D.

A more direct approach to the solution is to recognize that the slope of cable CD is specified.

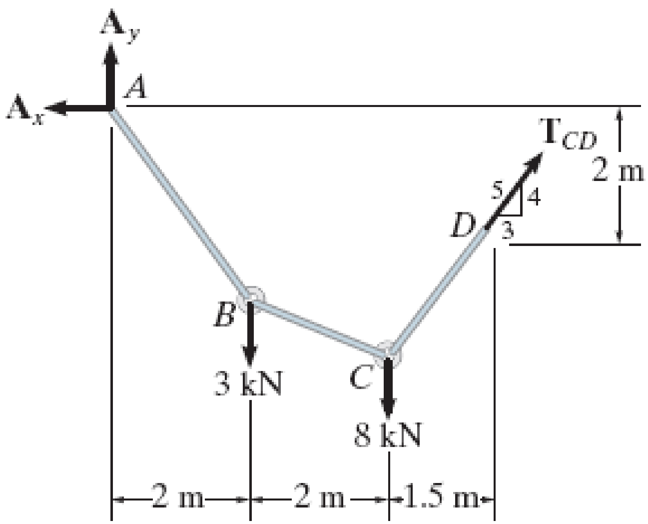
Solution

With anti-clockwise as positive

$$\Sigma M_A = 0$$

$$T_{CD}(3/5)(2m) + T_{CD}(4/5)(5.5m) - 3kN(2m) - 8kN(4m) = 0$$

$$T_{CD} = 6.79kN$$



Solution

Now we can analyze the equilibrium of points C and B in sequence.

Point C:

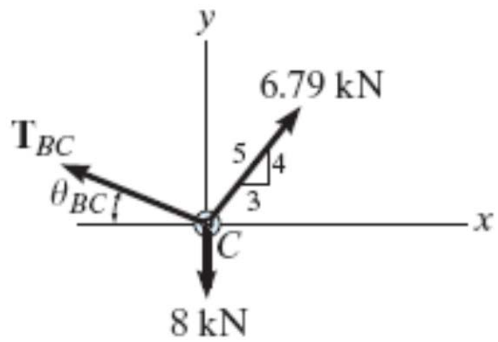
$\xrightarrow{+} \Sigma F_x = 0$

$6.79\text{kN}(3/5) - T_{BC}\cos\theta_{BC} = 0$

$+\uparrow \Sigma F_y = 0$

$6.79\text{kN}(4/5) - 8\text{kN} + T_{BC}\sin\theta_{BC} = 0$

$\theta_{BC} = 32.3^{\circ}$ and $T_{BC} = 4.82\text{kN}$



Solution

Point B:

$\underset{\rightarrow}{+}\Sigma F_x = 0$

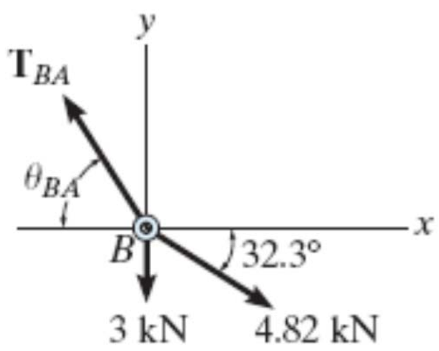
$-T_{BA}\cos\theta_{BA} + 4.82\text{kN}\cos 32.3^\circ = 0$

$+\uparrow \Sigma F_y = 0$

$T_{BA}\sin\theta_{BA} - 4.82\text{kN}\sin 32.3^\circ - 3\text{kN} = 0$

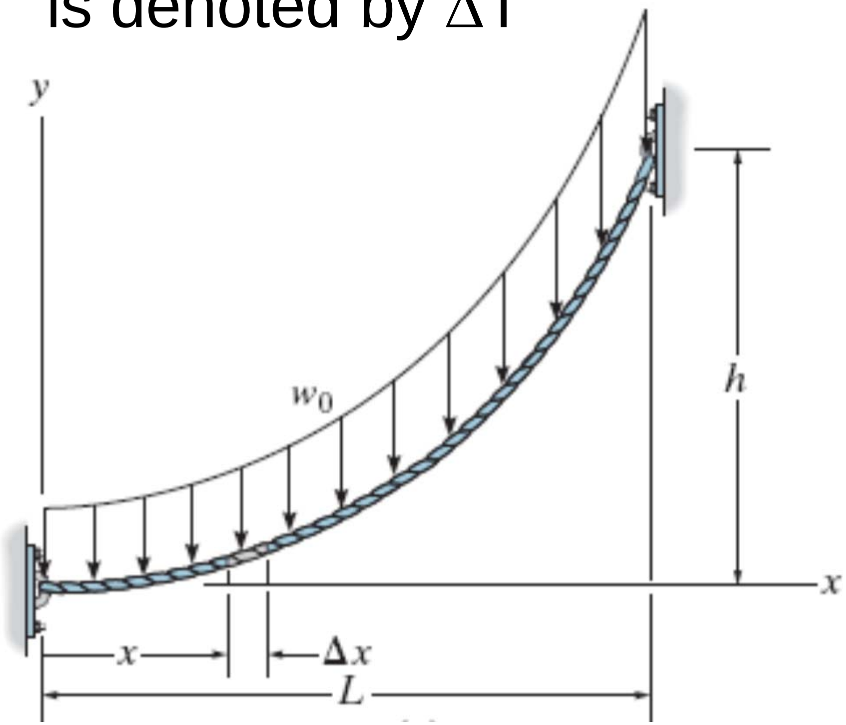
$\theta_{BA} = 53.8^\circ$ and $T_{BA} = 6.90\text{kN}$

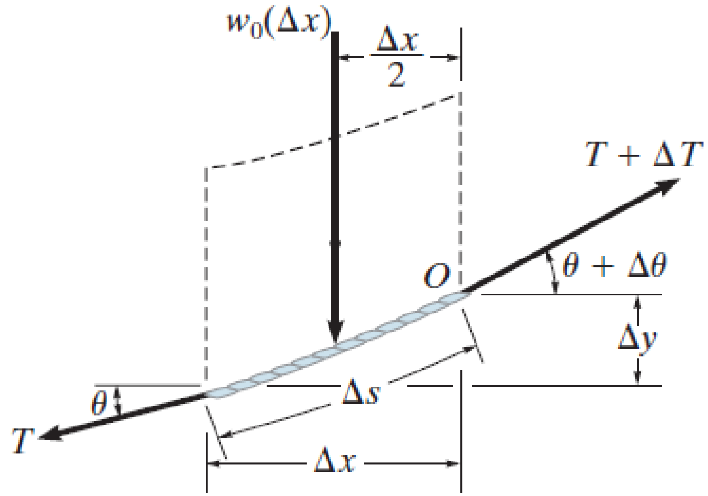
$h = (2\text{m})\tan 53.8^\circ = 2.74\text{m}$



Cable subjected to a uniform distributed load

- The x,y axes have their origin located at the lowest point on the cable such that the slope is zero at this point
- Since the tensile force in the cable changes continuously in both magnitude & direction along the cable's length, this load is denoted by ΔT





Cable subjected to a uniform distributed load

- The distributed load is represented by its resultant force $w_0\Delta x$ which acts at $\Delta x/2$ from point O
- Applying equations of equilibrium yields:

$$\xrightarrow{+} \Sigma F_x = 0$$

$$-T\cos\theta + (T+\Delta T)\cos(\theta+\Delta\theta) = 0$$

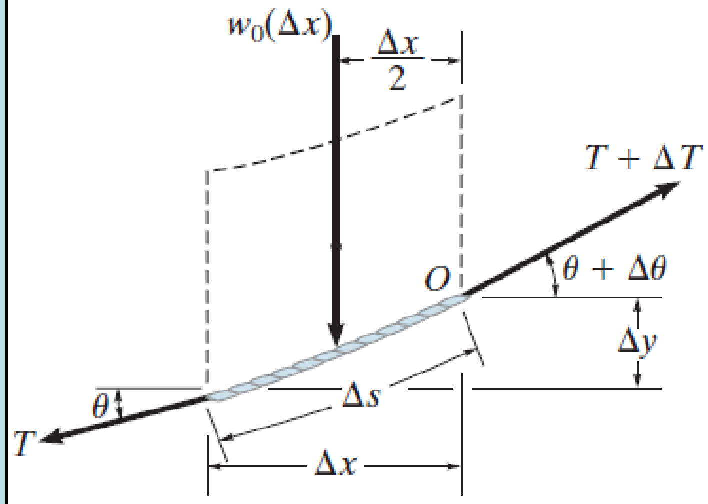
$$+\uparrow \Sigma F_y = 0$$

$$-T\sin\theta - w_0(\Delta x) + (T+\Delta T)\sin(\theta+\Delta\theta) = 0$$

With anti-clockwise as positive

$$\Sigma M_0 = 0$$

$$w_0(\Delta x)(\Delta x/2) - T\cos\theta\Delta y + T\sin\theta\Delta x = 0$$



Cable subjected to a uniform distributed load

- Dividing each of these equations by Δx and taking the limit as $\Delta x \rightarrow 0$, hence, $\Delta y \rightarrow 0$, $\Delta\theta \rightarrow 0$ and $\Delta T \rightarrow 0$, we obtain:

$$\frac{d(T\cos\theta)}{dx} = 0 \qquad \text{eqn 1}$$

$$\frac{d(T\sin\theta)}{dx} = w_0 \qquad \text{eqn 2}$$

$$\frac{dy}{dx} = \tan\theta \qquad \text{eqn 3}$$

Cable subjected to a uniform distributed load

- Integrating eqn 1 where $T = T_0$ at $x = 0$, we have:

$$T\cos\theta = T_0 \qquad \text{eqn 4}$$

which indicates the *horizontal component of force* at any point along the cable remains *constant*

- Integrating eqn 2, realizing that $T\sin\theta = 0$ at $x = 0$, we have:

$$T\sin\theta = w_0 x \qquad \text{eqn 5}$$

Cable subjected to a uniform distributed load

- Dividing eqn 5 by eqn 4 eliminates T
- Then using eqn 3, we can obtain the slope at any point

$$\tan\theta = \frac{dy}{dx} = \frac{w_0 x}{T_0} \qquad \text{eqn 6}$$

- Performing a second integration with y = 0 at x = 0 yields

$$y = \frac{w_0}{2T_0} x^2 \qquad \text{eqn 7}$$

Cable subjected to a uniform distributed load

- This is the equation of parabola
- The constant T_0 may be obtained by using the boundary condition y = h at x = L
- Thus

$$T_0 = \frac{w_0 L^2}{2h} \qquad \text{eqn 8}$$

- Substituting into eqn 7

$$y = \frac{h}{L^2} x^2 \qquad \text{eqn 9}$$

Cable subjected to a uniform distributed load

- From eqn 4, the maximum tension in the cable occurs when θ is maximum; i.e., at $x = L$.
- Hence from eqn 4 and 5

$$T_{max} = \sqrt{T_0^2 + (w_0 L)^2} \qquad \text{eqn 10}$$

- Using eqn 8 we can express T_{max} in terms of w_0

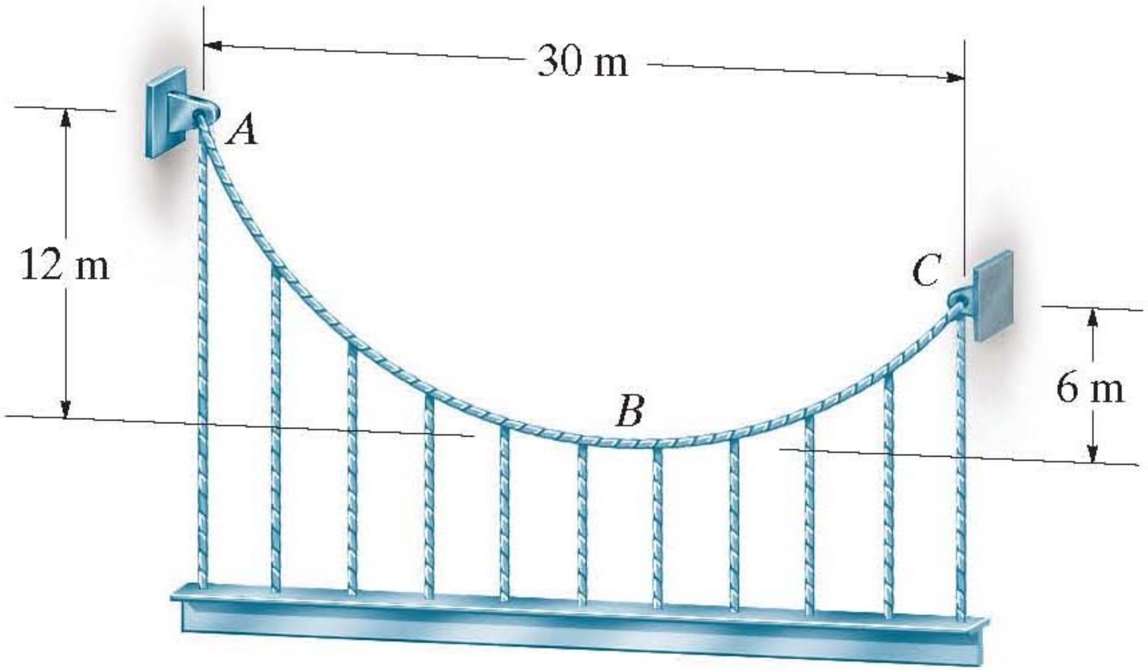
$$T_{max} = w_0 L \sqrt{1 + (L/2h)^2} \qquad \text{eqn 11}$$

Cable subjected to a uniform distributed load

- We have neglect the weight of the cable which is uniform along the length
- A cable subjected to its own weight will take the form of a catenary curve
- If the sag-to-span ratio is small, this curve closely approximates a parabolic shape

Example 5.2

The cable supports a girder which weighs 12kN/m. Determine the tension in the cable at points A, B & C.



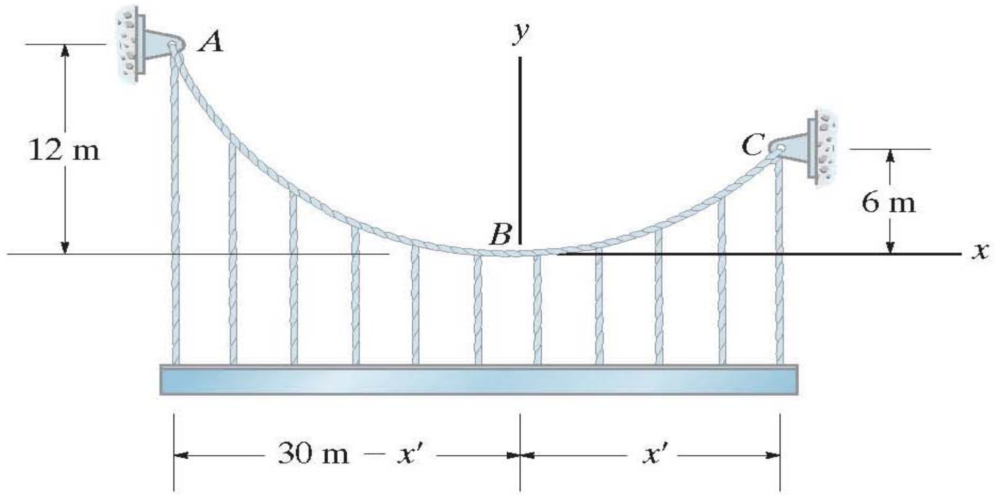
Solution

The origin of the coordinate axes is established at point B, the lowest point on the cable where slope is zero,

$$y = \frac{w_0}{2T_0}x^2 = \frac{12\text{kN/m}}{2T_0}x^2 = \frac{6}{T_0}x^2 \quad (1)$$

Assuming point C is located x’ from B we have:

$$6 = \frac{6}{T_0}x'^2 \Rightarrow T_0 = 1.0x'^2 \quad (2)$$



Solution

For point A,

$$12 = \frac{6}{T_0}[-(30 - x')]^2$$

$$12 = \frac{6}{1.0x'^2}[-(30 - x')]^2$$

$$x'^2 + 60x' - 900 = 0 \Rightarrow x' = 12.43m$$

Thus from equations 2 and 1, we have:

$$T_0 = 1.0(12.43)^2 = 154.4kN$$

$$\frac{dy}{dx} = \frac{12}{154.4}x = 0.7772x \quad (3)$$

Solution

At point A,

$$x = -(30 - 12.43) = -17.57\,\mathrm{m}$$

$$\tan\theta_A = \left.\frac{dy}{dx}\right|_{x=-17.57} = 0.7772(-17.57) = -1.366$$

$$\theta_A = -53.79^{o}$$

We have,

$$T_A = \frac{T_0}{\cos\theta_A} = \frac{154.4}{\cos(-53.79^{o})} = 261.4\,\mathrm{kN}$$

Solution

At point B, x = 0

$$\tan\theta_B = \left.\frac{dy}{dx}\right|_{x=0} = 0 \Rightarrow \theta_B = 0^o$$

$$T_B = \frac{T_0}{\cos\theta_B} = \frac{154.4}{\cos 0^o} = 154.4\text{kN}$$

At point C, x = 12.43m

$$\tan\theta_C = \left.\frac{dy}{dx}\right|_{x=12.43} = 0.7772(12.43) = 0.9657$$

$$\theta_C = 44.0^o$$

$$T_C = \frac{T_0}{\cos\theta_C} = \frac{154.4}{\cos 44.0^o} = 214.6\text{kN}$$

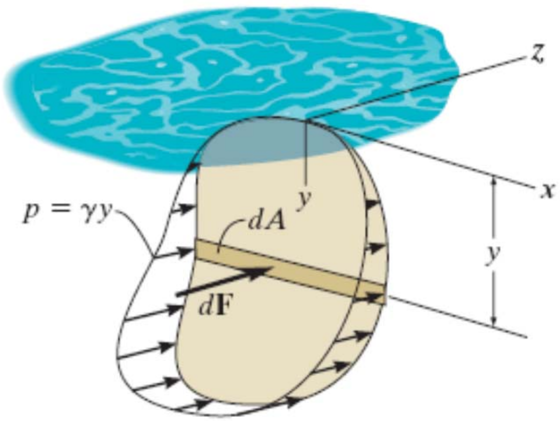
STATICS

Assist. Prof. Dr. Cenk Üstündağ

10 Moments of Inertia

Definition of Moments of Inertia for Areas

- Centroid for an area is determined by the first moment of an area about an axis
- Second moment of an area is referred as the moment of inertia
- Moment of inertia of an area originates whenever one relates the normal stress σ or force per unit area



Definition of Moments of Inertia for Areas

Moment of Inertia

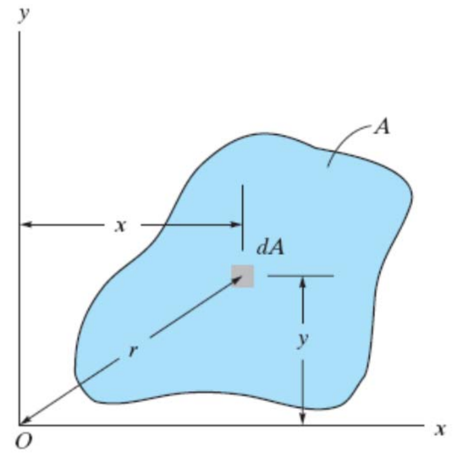
- Consider area A lying in the x-y plane
- Be definition, moments of inertia of the differential plane area dA about the x and y axes

$$dI_x = y^2 dA \quad dI_y = x^2 dA$$

- For entire area, moments of inertia are given by

$$I_x = \int_A y^2 dA$$

$$I_y = \int_A x^2 dA$$



Definition of Moments of Inertia for Areas

Moment of Inertia

- Formulate the second moment of dA about the pole O or z axis
- This is known as the polar axis

$$dJ_O = r^2 dA$$

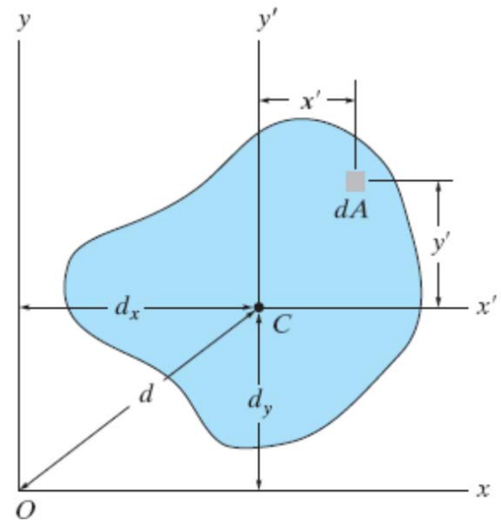
where r is perpendicular from the pole (z axis) to the element dA

- Polar moment of inertia for entire area,

$$J_O = \int_A r^2 dA = I_x + I_y$$

Parallel Axis Theorem for an Area

- For moment of inertia of an area known about an axis passing through its centroid, determine the moment of inertia of area about a corresponding parallel axis using the parallel axis theorem
- Consider moment of inertia of the shaded area
- A differential element dA is located at an arbitrary distance y' from the centroidal x' axis



Parallel Axis Theorem for an Area

- The fixed distance between the parallel x and x′ axes is defined as d_y
- For moment of inertia of dA about x axis

$$dI_x = \left(y' + d_y\right)^2 dA$$

- For entire area

$$I_x = \int_A \left(y' + d_y\right)^2 dA$$

$$= \int_A y'^2\, dA + 2d_y \int_A y'\, dA + d_y^2 \int_A dA$$

- First integral represent the moment of inertia of the area about the centroidal axis

Parallel Axis Theorem for an Area

- Second integral = 0 since x′ passes through the area′s centroid C $$\int y'\,dA = \bar{y}\int dA = 0; \quad \bar{y} = 0$$

- Third integral represents the total area A $$I_x = \bar{I}_x + Ad_y^2$$

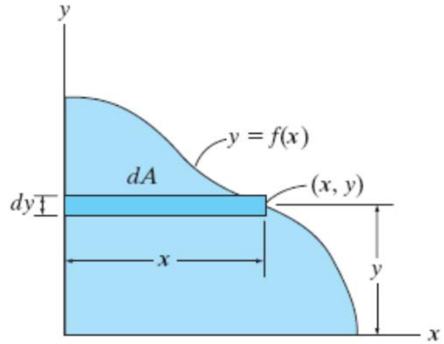
- Similarly $$I_y = \bar{I}_y + Ad_x^2$$

- For polar moment of inertia about an axis perpendicular to the x-y plane and passing through pole O (z axis) $$J_O = \bar{J}_C + Ad^2$$

Radius of Gyration of an Area

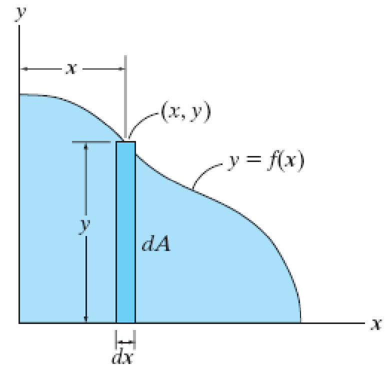
- Radius of gyration of a planar area has units of length and is a quantity used in the design of columns in structural mechanics
- For radii of gyration

$$k_x = \sqrt{\frac{I_x}{A}} \quad k_y = \sqrt{\frac{I_y}{A}} \quad k_z = \sqrt{\frac{J_O}{A}}$$



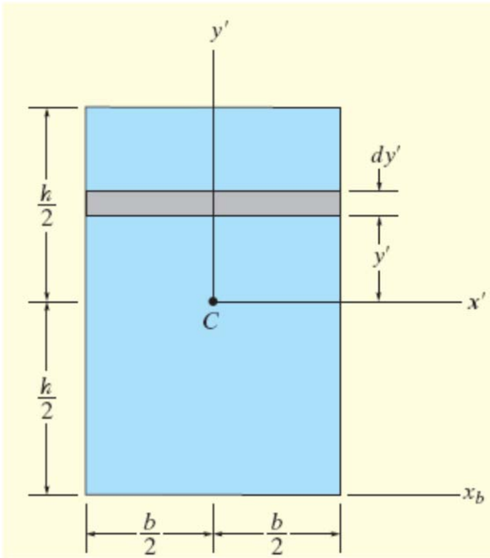
- Similar to finding moment of inertia of a differential area about an axis

$$I_x = k_x^2 A \quad dI_x = y^2 dA$$



Example

Determine the moment of inertia for the rectangular area with respect to (a) the centroidal x' axis, (b) the axis x_b passing through the base of the rectangular, and (c) the pole or z' axis perpendicular to the x'-y' plane and passing through the centroid C.



Solution

Part (a)

Differential element chosen, distance y' from x' axis.

Since $dA = b\,dy'$,

$$\bar{I}_x = \int_A y'^2\,dA = \int_{-h/2}^{h/2} y'^2\,(b\,dy') = \int_{-h/2}^{h/2} y'^2\,dy = \frac{1}{12}bh^3$$

Part (b)

By applying parallel axis theorem,

$$I_{x_b} = \bar{I}_x + Ad^2 = \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3$$

Solution

Part (c)

For polar moment of inertia about point C,

$$\bar{I}_{y'} = \frac{1}{12} hb^3$$

$$J_C = \bar{I}_x + \bar{I}_{y'} = \frac{1}{12} bh(h^2 + b^2)$$

Moments of Inertia for Composite Areas

- Composite area consist of a series of connected simpler parts or shapes
- Moment of inertia of the composite area = algebraic sum of the moments of inertia of all its parts

Procedure for Analysis

<u>Composite Parts</u>

- Divide area into its composite parts and indicate the centroid of each part to the reference axis

<u>Parallel Axis Theorem</u>

- Moment of inertia of each part is determined about its centroidal axis

Moments of Inertia for Composite Areas

Procedure for Analysis

Parallel Axis Theorem

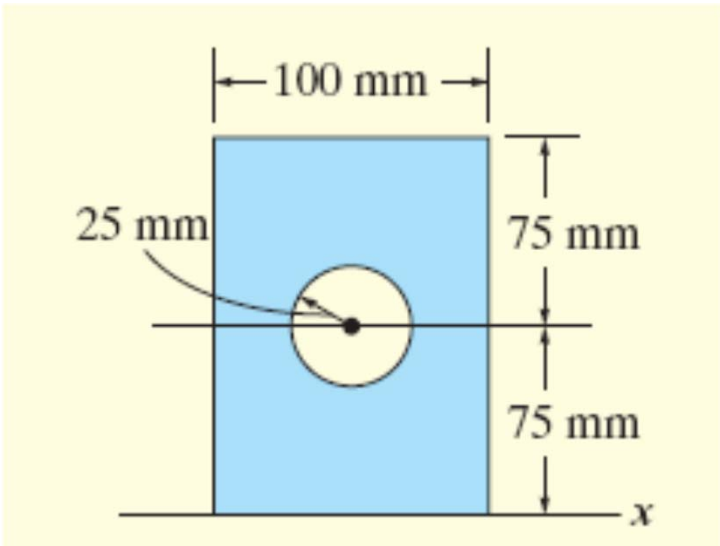
- When centroidal axis does not coincide with the reference axis, the parallel axis theorem is used

Summation

- Moment of inertia of the entire area about the reference axis is determined by summing the results of its composite parts

Example

Compute the moment of inertia of the composite area about the x axis.

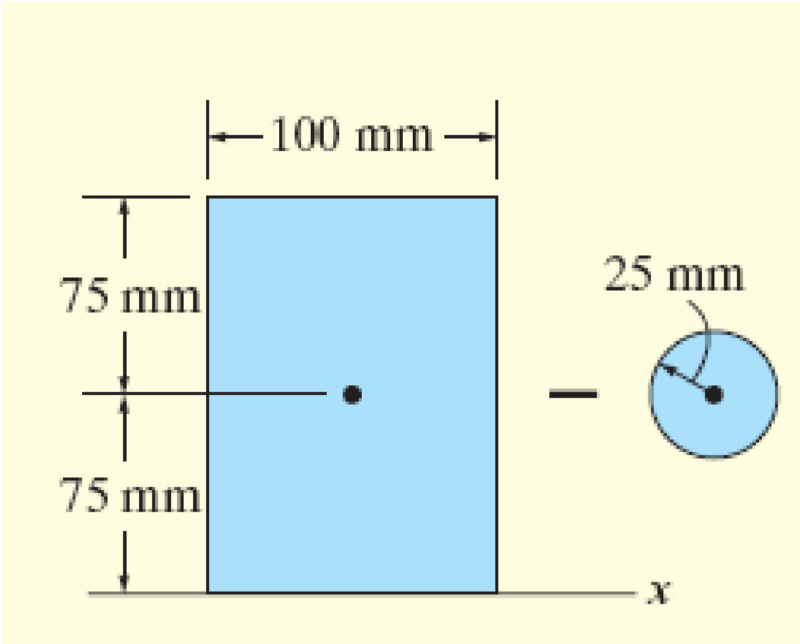


Solution

<u>Composite Parts</u>

Composite area obtained by subtracting the circle form the rectangle.

Centroid of each area is located in the figure below.



Solution

<u>Parallel Axis Theorem</u>

Circle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4\left(10^6\right)mm^4$$

Rectangle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5\left(10^6\right)mm^4$$

Solution

Summation

For moment of inertia for the composite area,

$$I_x = -11.4\left(10^6\right) + 112.5\left(10^6\right)$$

$$= 101\left(10^6\right) mm^4$$

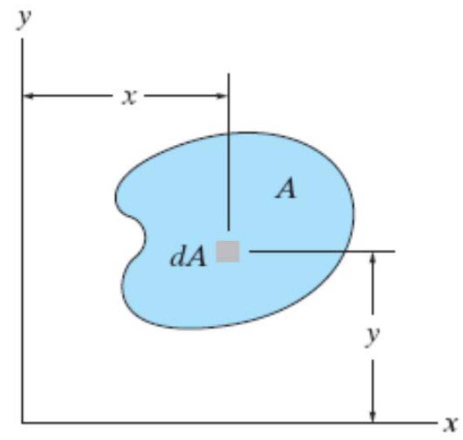
Product of Inertia for an Area

- Moment of inertia for an area is different for every axis about which it is computed
- First, compute the product of the inertia for the area as well as its moments of inertia for given x, y axes
- Product of inertia for an element of area dA located at a point (x, y) is defined as

$$dI_{xy} = xydA$$

- Thus for product of inertia,

$$I_{xy} = \int_A xydA$$

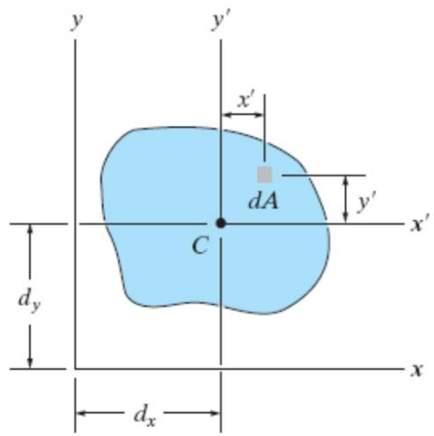


Product of Inertia for an Area

Parallel Axis Theorem

- For the product of inertia of dA with respect to the x and y axes

$$dI_{xy} = \int_A (x'+d_x)(y'+d_y)dA$$



- For the entire area,

$$dI_{xy} = \int_A (x'+d_x)(y'+d_y)dA$$

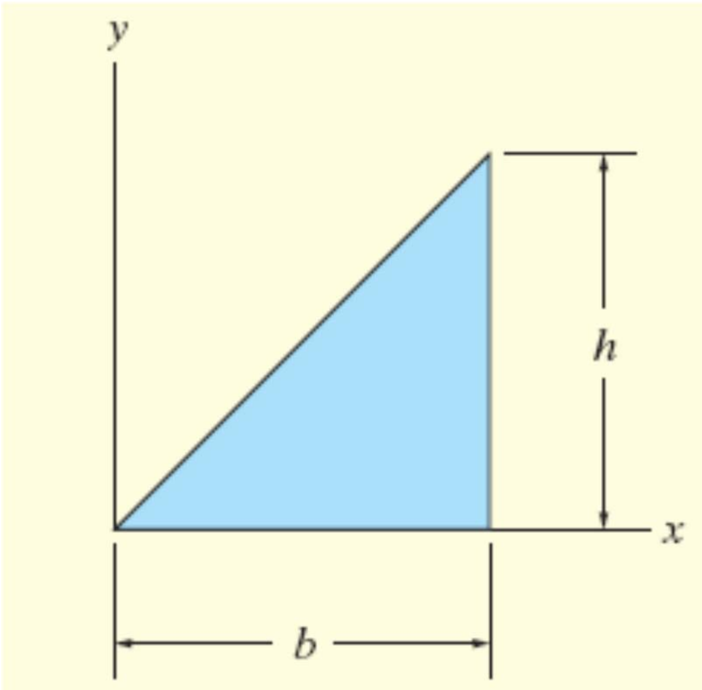
$$= \int_A x'\,y'\,dA + d_x \int_A y'dA + d_y \int_A x'dA + d_x d_y \int_A dA$$

- Forth integral represent the total area A,

$$I_{xy} = \bar{I}_{x'y'} + A d_x d_y$$

Example

Determine the product I_{xy} of the triangle.



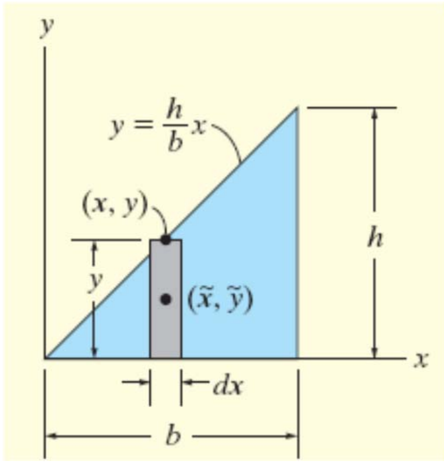
Solution

Differential element has thickness dx and area dA = y dx

Using parallel axis theorem,

$$dI_{xy} = d\bar{I}_{xy} + dA\tilde{x}\tilde{y}$$

$(\tilde{x}, \tilde{y})$ locates centroid of the element or origin of x′, y′ axes



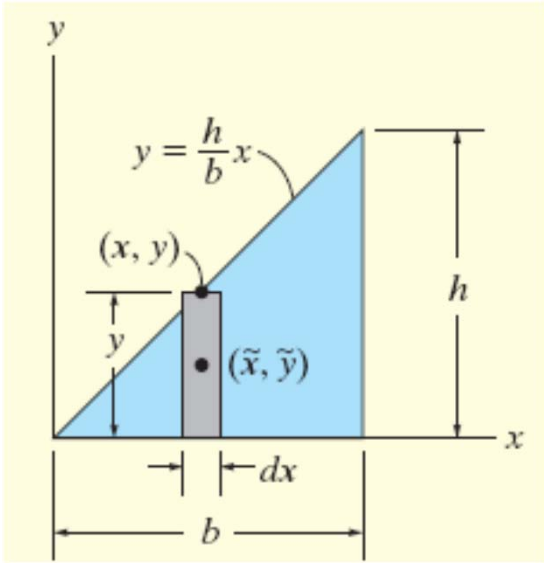
Solution

Due to symmetry, $d\bar{I}_{xy} = 0 \quad \tilde{x} = x, \tilde{y} = y/2$

$$dI_{xy} = 0 + (ydx)x\left(\frac{y}{2}\right) = \left(\frac{h}{b}xdx\right)x\left(\frac{h}{2b}x\right) = \frac{h^2}{2b^2}x^3dx$$

Integrating we have

$$I_{xy} = \frac{h^2}{2b^2}\int_0^b x^3dx = \frac{b^2h^2}{8}$$



Solution

Differential element has thickness dy and area dA = (b - x) dy.

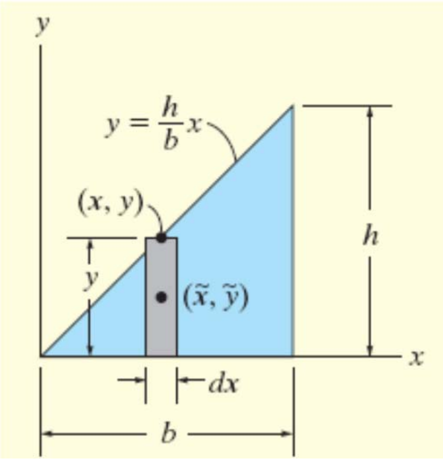
For centroid,

$$\tilde{x} = x + (b - x)/2 = (b + x)/2, \tilde{y} = y$$

For product of inertia of element

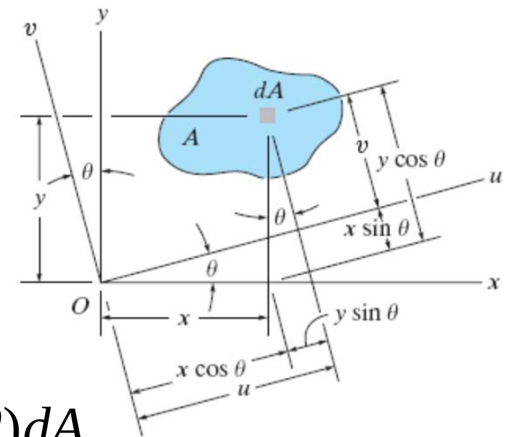
$$dI_{xy} = d\tilde{I}_{xy} + dA\tilde{x}\tilde{y} = 0 + (b - x)dy\left(\frac{b + x}{2}\right)y$$

$$= \left(b - \frac{b}{h}y\right)dy\left[\frac{b + (b/h)y}{2}\right]y = \frac{1}{2}y\left(b^2 - \frac{b^2}{h^2}y^2\right)dy$$



Moments of Inertia for an Area about Inclined Axes

- In structural and mechanical design, necessary to calculate I_u, I_v and I_{uv} for an area with respect to a set of inclined u and v axes when the values of θ, I_x, I_y and I_{xy} are known
- Use transformation equations which relate the x, y and u, v coordinates



$$u = x\cos\theta + y\sin\theta$$

$$v = y\cos\theta - x\sin\theta$$

$$dI_u = v^2 dA = (y\cos\theta - x\sin\theta)^2 dA$$

$$dI_v = u^2 dA = (x\cos\theta + y\sin\theta)^2 dA$$

$$dI_{uv} = uvdA = (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)dA$$

Moments of Inertia for an Area about Inclined Axes

- Integrating,

$$I_u = I_x \cos^2\theta + I_y \sin^2\theta - 2I_{xy}\sin\theta\cos\theta$$

$$I_v = I_x \sin^2\theta + I_y \cos^2\theta + 2I_{xy}\sin\theta\cos\theta$$

$$I_{uv} = I_x \sin\theta\cos\theta - I_y \sin\theta\cos\theta + 2I_{xy}(\cos^2\theta - \sin^2\theta)$$

- Simplifying using trigonometric identities,

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

Moments of Inertia for an Area about Inclined Axes

- We can simplify to

$$I_u = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_v = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{uv} = \frac{I_x - I_y}{2}\sin 2\theta + 2I_{xy}\cos 2\theta$$

- Polar moment of inertia about the *z* axis passing through point *O* is,

$$J_O = I_u + I_v = I_x + I_y$$

Moments of Inertia for an Area about Inclined Axes

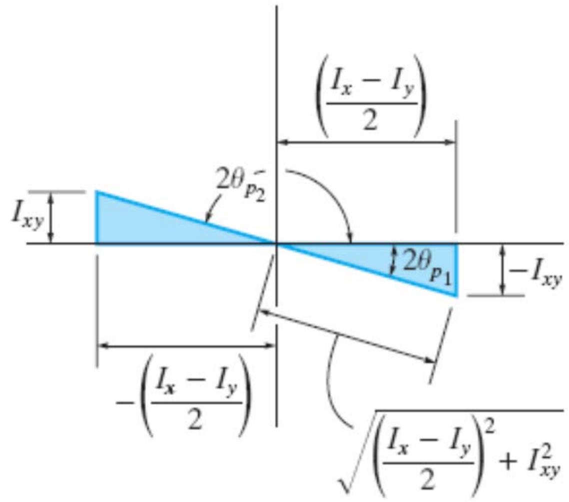
Principal Moments of Inertia

- I_u, I_v and I_{uv} depend on the angle of inclination θ of the u, v axes
- The angle $\theta = \theta_p$ defines the orientation of the principal axes for the area

$$\frac{dI_u}{d\theta} = -2\left(\frac{I_x - I_y}{2}\right)\sin 2\theta - 2I_{xy}\cos 2\theta = 0$$

$$\theta = \theta_p$$

$$\tan 2\theta_p = \frac{-I_{xy}}{\left(I_x - I_y\right)/2}$$



Moments of Inertia for an Area about Inclined Axes

Principal Moments of Inertia

- Substituting each of the sine and cosine ratios, we have

$$I_{\min}^{\max} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

- Result can gives the max or min moment of inertia for the area
- It can be shown that $I_{uv} = 0$, that is, the product of inertia with respect to the principal axes is zero
- Any symmetric axis represent a principal axis of inertia for the area