STATICS

Assist. Prof. Dr. Cenk Üstündağ

4 Force System Resultants

Chapter Objectives

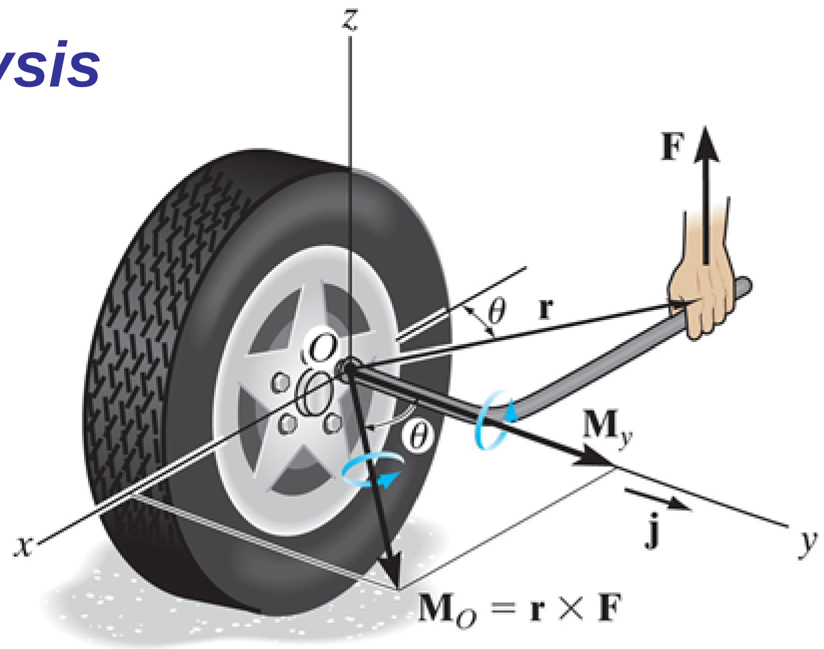
- Method for finding the moment of a force about a specified axis.
- Define the moment of a couple.
- Determine the resultants of non-concurrent force systems
- Reduce a simple distributed loading to a resultant force having a specified location

Chapter Outline

5. Moment of a Force about a Specified Axis
6. Moment of a Couple
7. Simplification of a Force and Couple System
8. Further Simplification of a Force and Couple System
9. Reduction of a Simple Distributed Loading

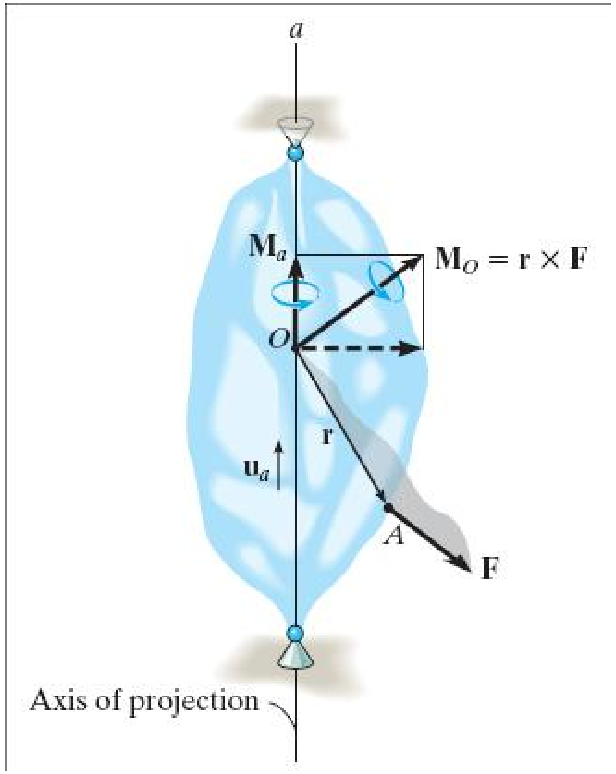
4.5 Moment of a Force about a Specified Axis

Scalar Analysis



In the figure above, the moment about the y-axis would be $M_y = F_z\ (d_x) = F\ (r \cos \theta)$. However, unless the force can easily be broken into components and the "d_x" found quickly, such calculations are not always trivial and vector analysis may be much easier (and less likely to produce errors).

4.5 Moment of a Force about a Specified Axis



Vector Analysis

Our goal is to find the moment of ***F*** (the tendency to rotate the body) about the a-axis.

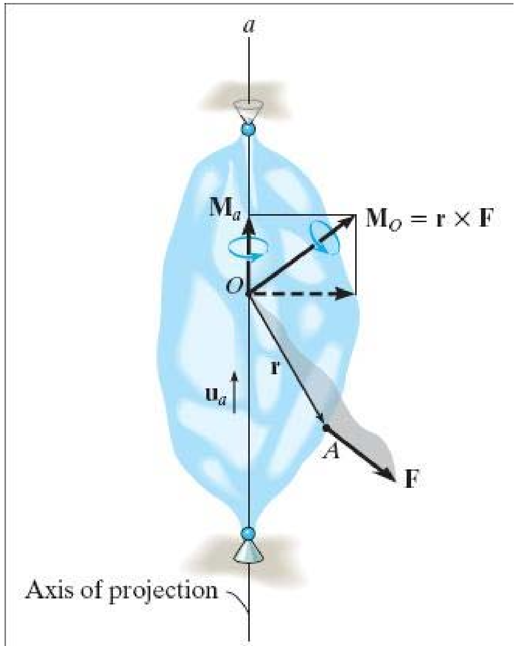
First compute the moment of ***F*** about any arbitrary point O that lies on the a'- a axis using the cross product.

$$\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$$

Now, find the component of $\boldsymbol{M_O}$ along the a-axis using the dot product.

$$M_{a'-a} = \boldsymbol{u_a} \cdot \boldsymbol{M_O}$$

4.5 Moment of a Force about a Specified Axis



Vector Analysis

M_a can also be obtained as

$$M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

The above equation is also called the triple scalar product.

In this equation,

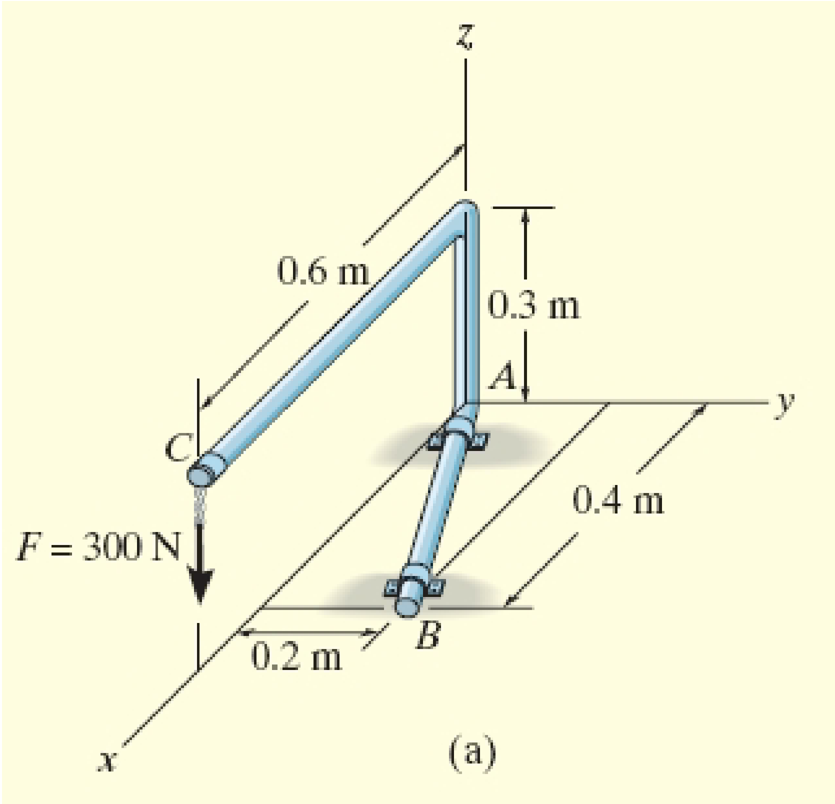
u_a represents the unit vector along the a-axis,

r is the position vector from any point on the a-axis to any point A on the line of action of the force, and

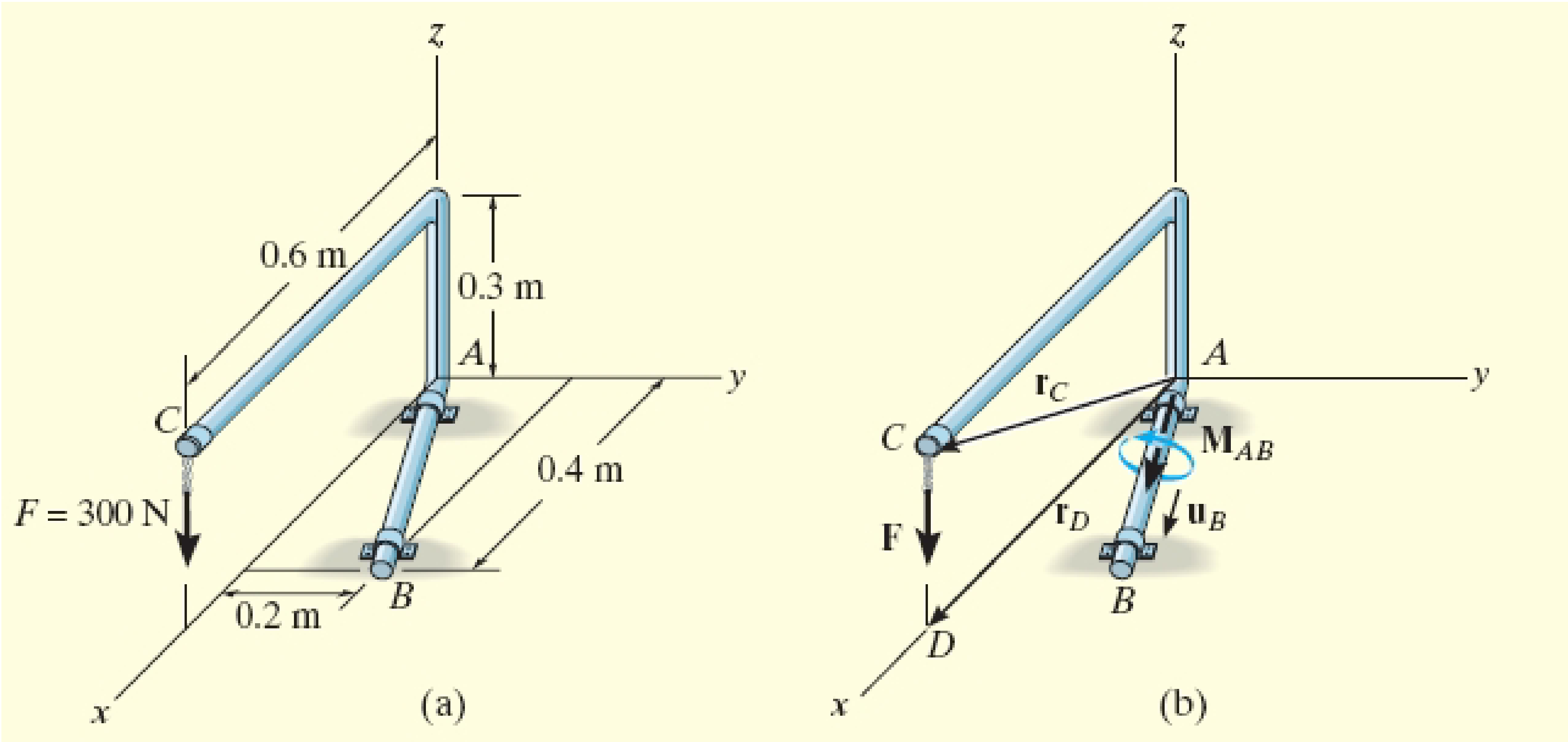
F is the force vector.

Example

Determine the moment produced by the force **F** which tends to rotate the rod about the *AB* axis.

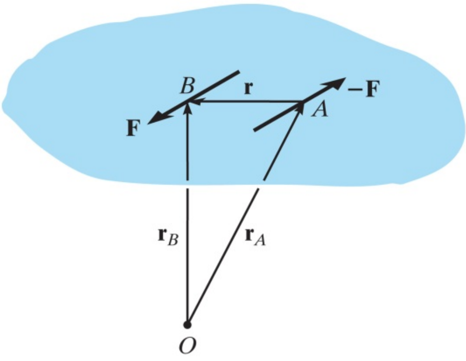


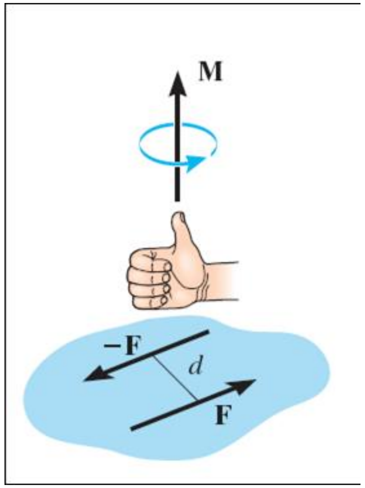
Solution



Solution on whiteboard

4.6 Moment of a Couple





A couple is defined as two parallel forces with the same magnitude but opposite in direction separated by a perpendicular distance "d."

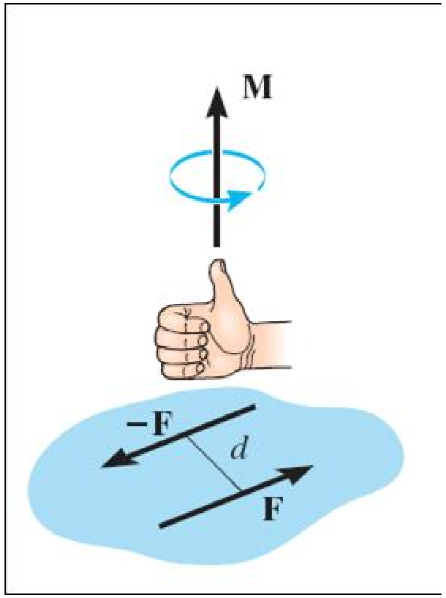
The moment of a couple is defined as

M_O = Fd (using a scalar analysis) or as

$\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$ (using a vector analysis).

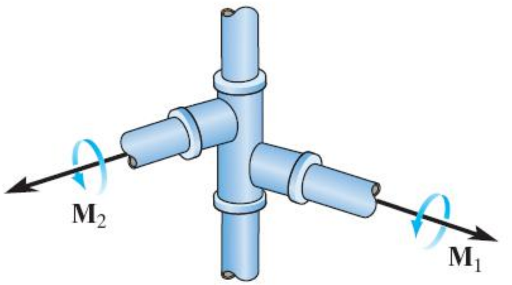
Here ***r*** is any position vector from the line of action of ***F*** to the line of action of ***F***.

4.6 Moment of a Couple

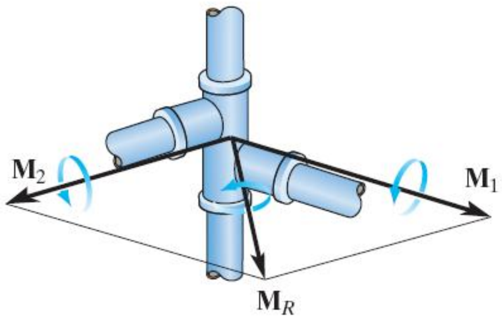


The net external effect of a couple is that the net force equals zero and the magnitude of the net moment equals F·d.

4.6 Moment of a Couple



Since the moment of a couple depends only on the distance between the forces, the moment of a couple is a free vector. It can be moved anywhere on the body and have the same external effect on the body.

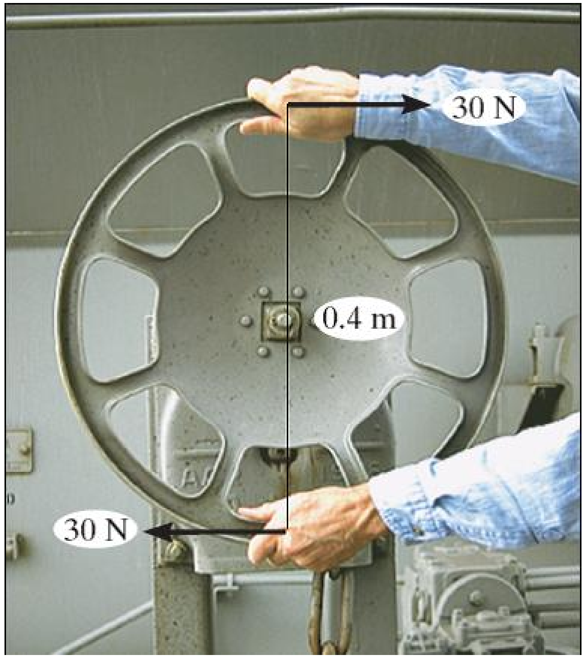


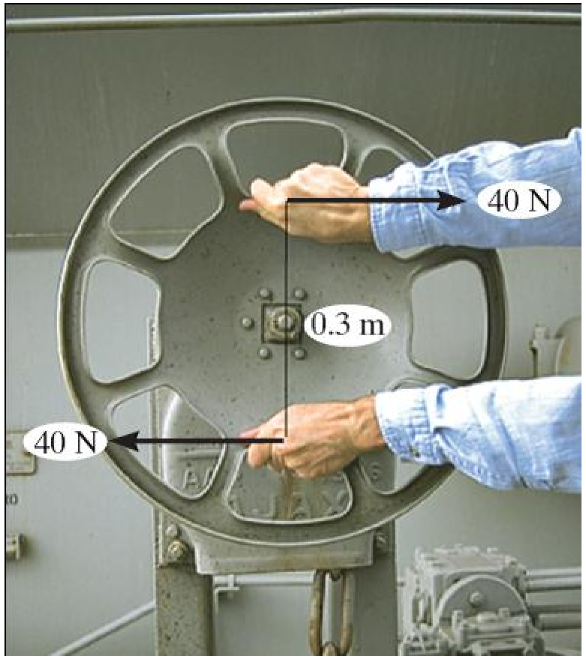
Moments due to couples can be added together using the same rules as adding any vectors.

4.6 Moment of a Couple

Equivalent Couples

- 2 couples are equivalent if they produce the same moment
- Forces of equal couples lie on the same plane or plane parallel to one another





4.6 Moment of a Couple

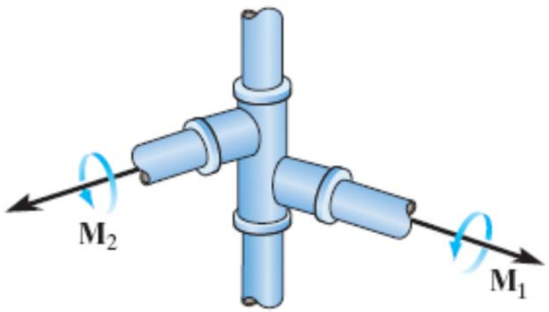
Resultant Couple Moment

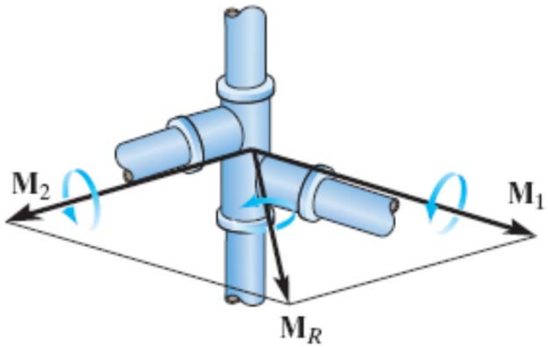
- Couple moments are free vectors and may be applied to any point P and added vectorially
- For resultant moment of two couples at point P,

$$\mathbf{M}_R = \mathbf{M}_1 + \mathbf{M}_2$$

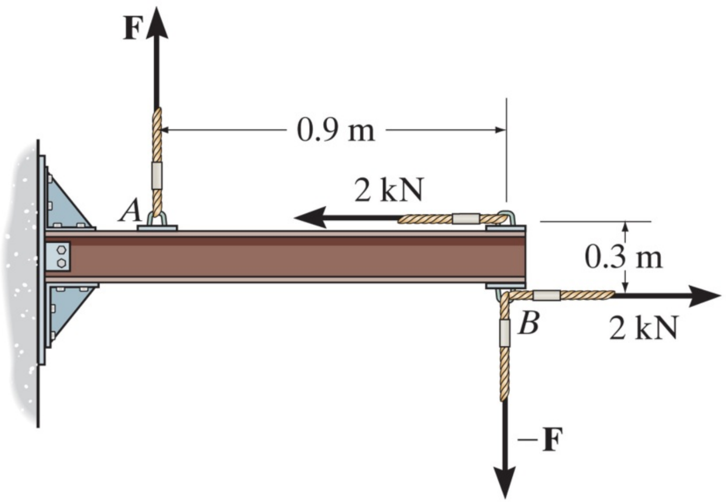
- For more than 2 moments,

$$\mathbf{M}_R = \sum(\mathbf{r} \times \mathbf{F})$$





Example



Given: Two couples act on the beam with the geometry shown.

Find: The magnitude of F so that the resultant couple moment is 1.5 kN·m clockwise.

Plan:

1) Add the two couples to find the resultant couple.
2) Equate the net moment to 1.5 kN·m clockwise to find F.

Solution

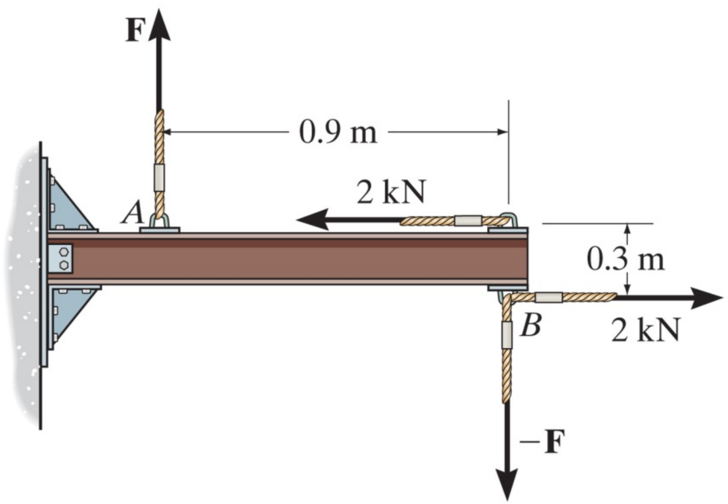
Solution:

The net moment is equal to:

$$+\Sigma M = -F\,(0.9) + (2)\,(0.3)$$

$$= -0.9\,F + 0.6$$

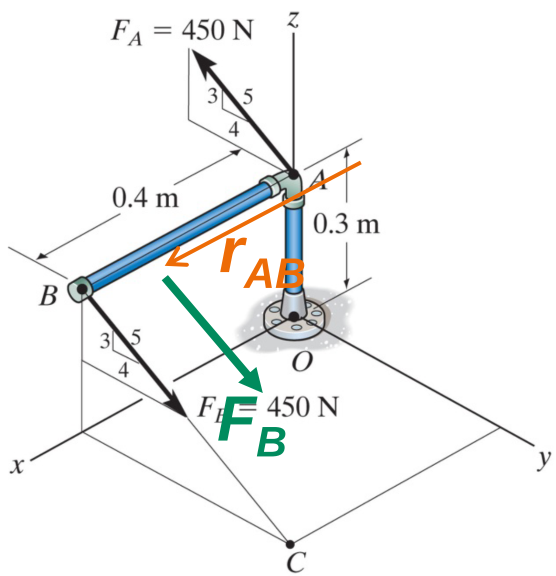
$$-1.5\ \text{kN}\cdot\text{m} \quad = -0.9\,F + 0.6$$



Solving for the unknown force F, we get

F = 2.33 kN

Example



Given: A 450 N force couple acting on the pipe assembly.

Find: The couple moment in Cartesian vector notation.

Plan:

1) Use $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$ to find the couple moment.

2) Set $\boldsymbol{r} = \boldsymbol{r}_{AB}$ and $\boldsymbol{F} = \boldsymbol{F}_B$.

3) Calculate the cross product to find $\boldsymbol{M}$.

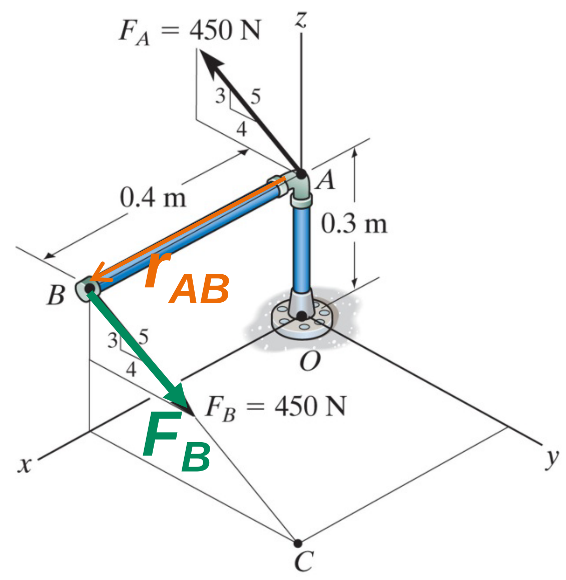
Solution

Solution:

$\boldsymbol{r}_{AB}$ = { 0.4 $\boldsymbol{i}$ } m

$\boldsymbol{F}_{B}$ = {0 $\boldsymbol{i}$ + 450(4/5) $\boldsymbol{j}$ – 450(3/5) $\boldsymbol{k}$} N

= {0 $\boldsymbol{i}$ + 360 $\boldsymbol{j}$ – 270 $\boldsymbol{k}$} N



$$\boldsymbol{M} = \boldsymbol{r}_{AB} \times \boldsymbol{F}_{B}$$

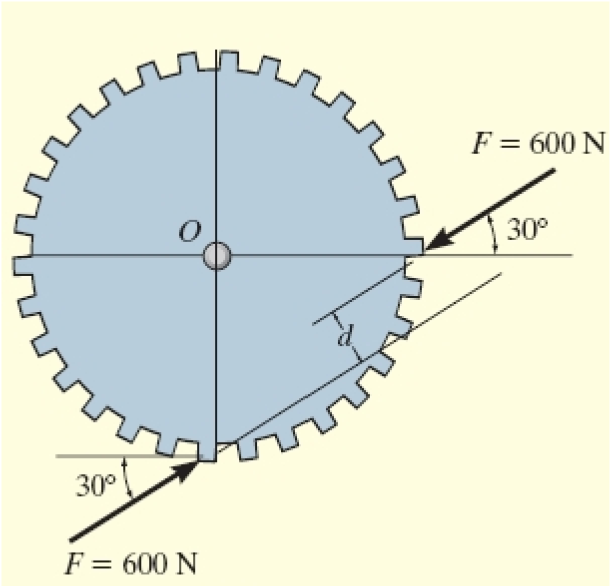
$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0.4 & 0 & 0 \\ 0 & 360 & -270 \end{vmatrix} \text{ N·m}$$

$$= [\{0(-270) - 0(360)\}\,\boldsymbol{i} - \{4(-270) - 0(0)\}\,\boldsymbol{j} + \{0.4(360) - 0(0)\}\,\boldsymbol{k}] \text{ N·m}$$

$$= \{0\,\boldsymbol{i} + 108\,\boldsymbol{j} + 144\,\boldsymbol{k}\} \text{ N·m}$$

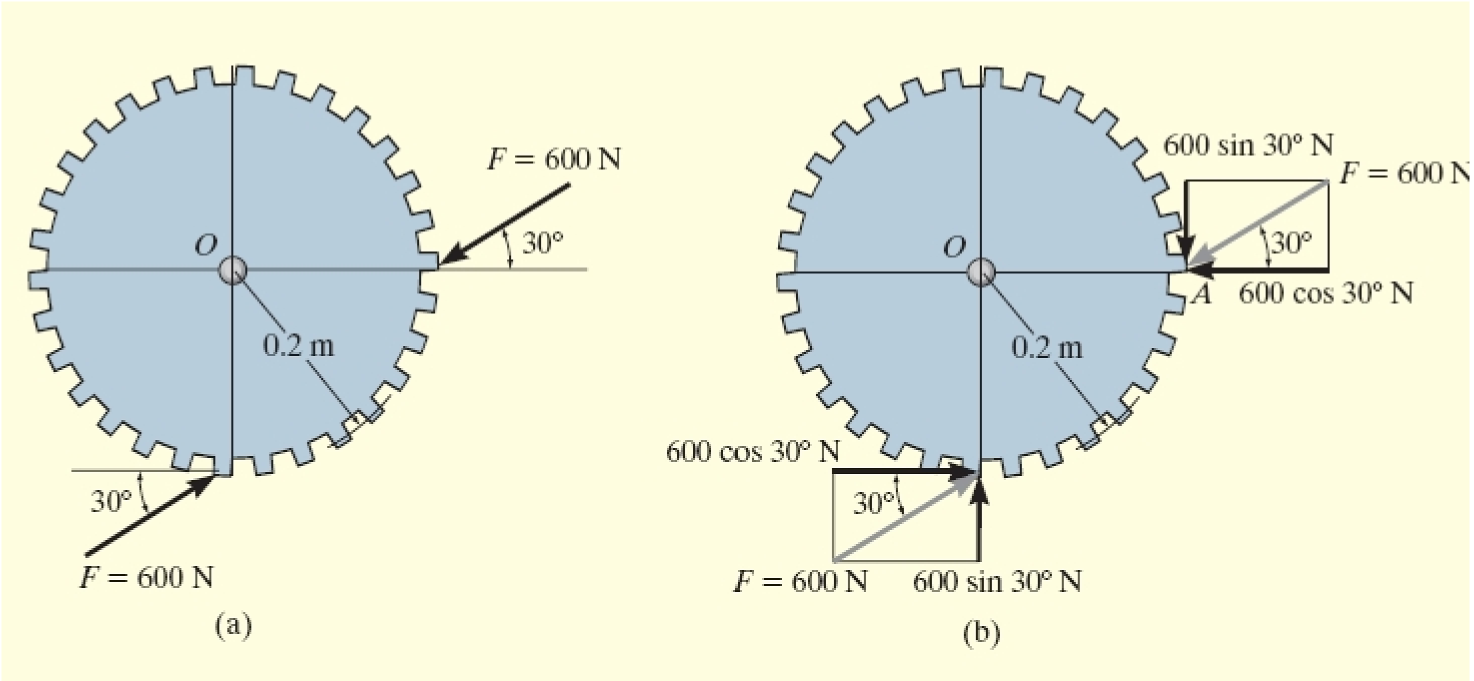
Example

Determine the magnitude and direction of the couple moment acting on the gear.



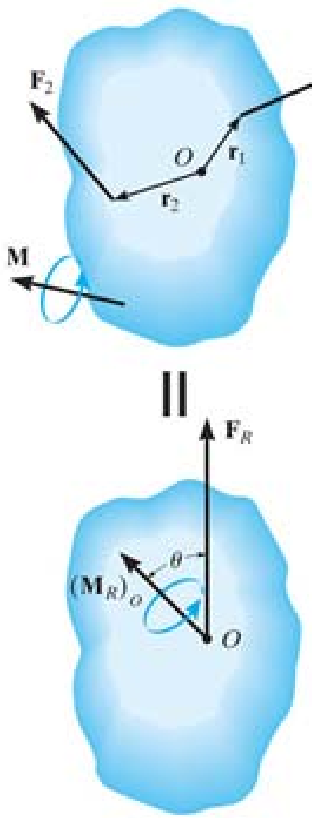
(c)

Solution



Solution on whiteboard

4.7 Simplification of a Force and Couple System

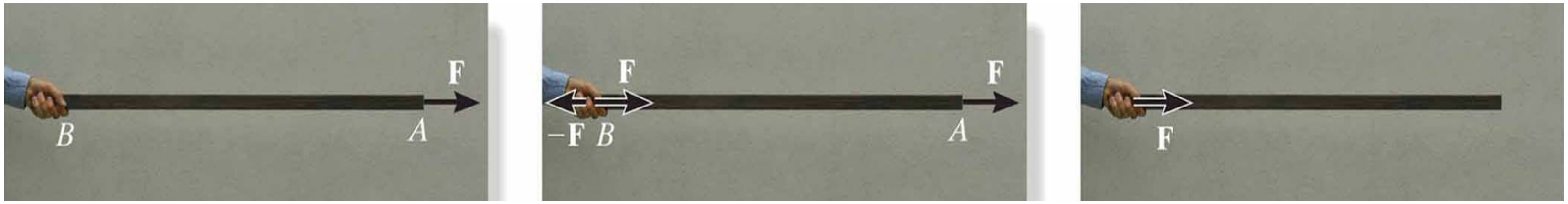


When a number of forces and couple moments are acting on a body, it is easier to understand their overall effect on the body if they are combined into a single force and couple moment having the same external effect.

The two force and couple systems are called equivalent systems since they have the same external effect on the body.

4.7 Simplification of a Force and Couple System

MOVING A FORCE ON ITS LINE OF ACTION

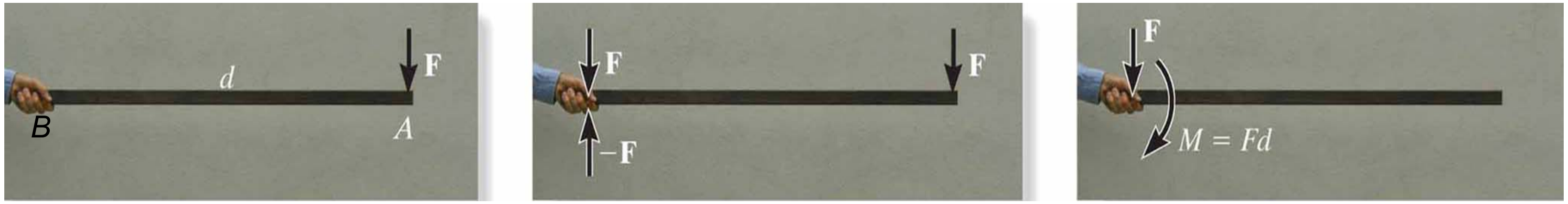


Moving a force from A to B, when both points are on the vector's line of action, does not change the external effect.

Hence, a force vector is called a sliding vector. (But the internal effect of the force on the body does depend on where the force is applied).

4.7 Simplification of a Force and Couple System

MOVING A FORCE OFF OF ITS LINE OF ACTION

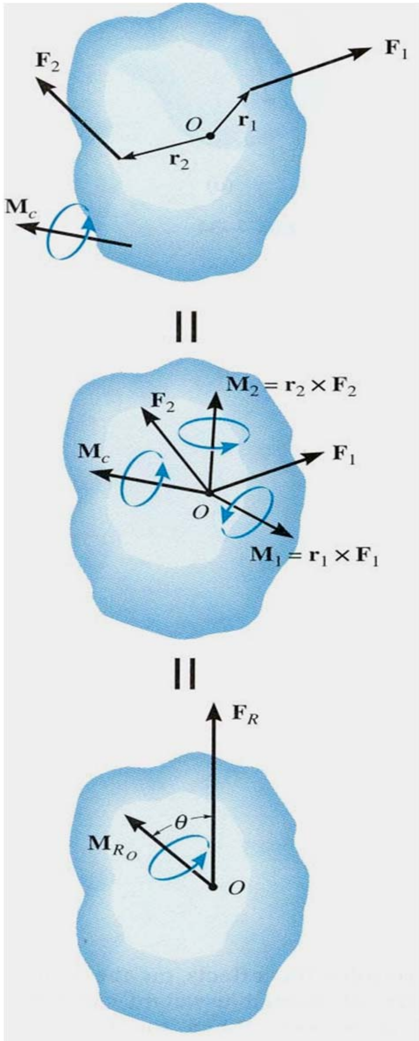


When a force is moved, but not along its line of action, there is a change in its external effect!

Essentially, moving a force from point A to B (as shown above) requires creating an additional couple moment. So moving a force means you have to “add” a new couple.

Since this new couple moment is a “free” vector, it can be applied at any point on the body.

4.7 Simplification of a Force and Couple System



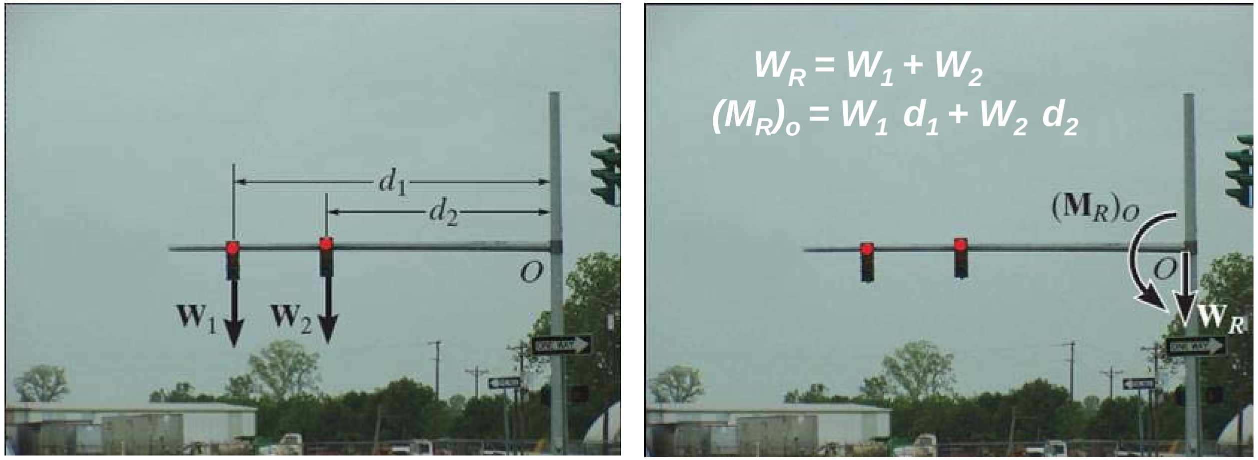
When several forces and couple moments act on a body, you can move each force and its associated couple moment to a common point O.

Now you can add all the forces and couple moments together and find one resultant force-couple moment pair.

$$\mathbf{F}_R = \Sigma \mathbf{F}$$

$$\mathbf{M}_{R_O} = \Sigma \mathbf{M}_c + \Sigma \mathbf{M}_O$$

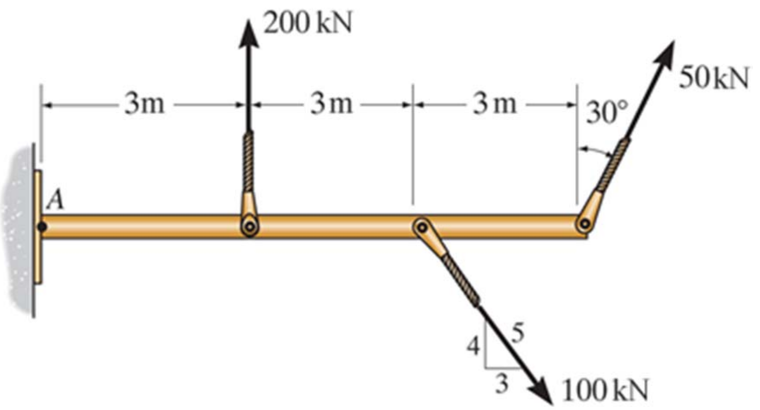
4.7 Simplification of a Force and Couple System



If the force system lies in the x-y plane (a 2-D case), then the reduced equivalent system can be obtained using the following three scalar equations.

$$F_{R_x} = \Sigma F_x$$
$$F_{R_y} = \Sigma F_y$$
$$M_{R_O} = \Sigma M_c + \Sigma M_O$$

Example



Given: A 2-D force system with geometry as shown.

Find: The equivalent resultant force and couple moment acting at A and then the equivalent single force location measured from A.

Plan:

1) Sum all the x and y components of the forces to find F_{RA}.
2) Find and sum all the moments resulting from moving each force component to A.
3) Shift F_{RA} to a distance d such that $d = M_{RA}/F_{Ry}$

Solution

$+\rightarrow \Sigma F_{Rx} = 50(\sin 30) + 100(3/5)$

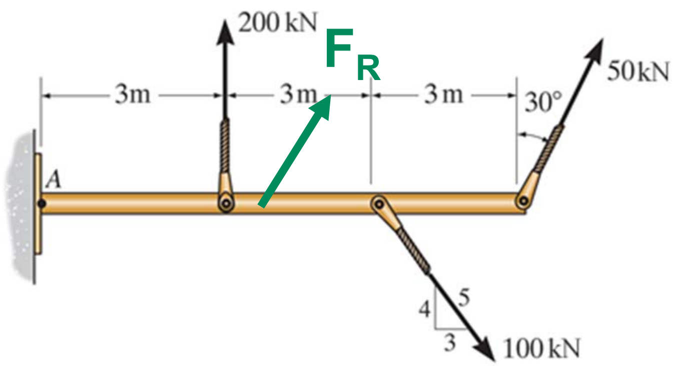
$= 85 \text{ kN}$

$+\uparrow \Sigma F_{Ry} = 200 + 50(\cos 30) - 100(4/5)$

$= 163.3 \text{ kN}$

$+ \circlearrowleft M_{RA} = 200\ (3) + 50\ (\cos 30)\ (9)$

$- 100\ (4/5)\ 6 = 509.7 \text{ kN.m} \circlearrowleft$



$$F_R = (85^2 + 163.3^2)^{1/2} = 184 \text{ kN}$$

$$\theta = \tan^{-1}(163.3/85) = 62.5°$$

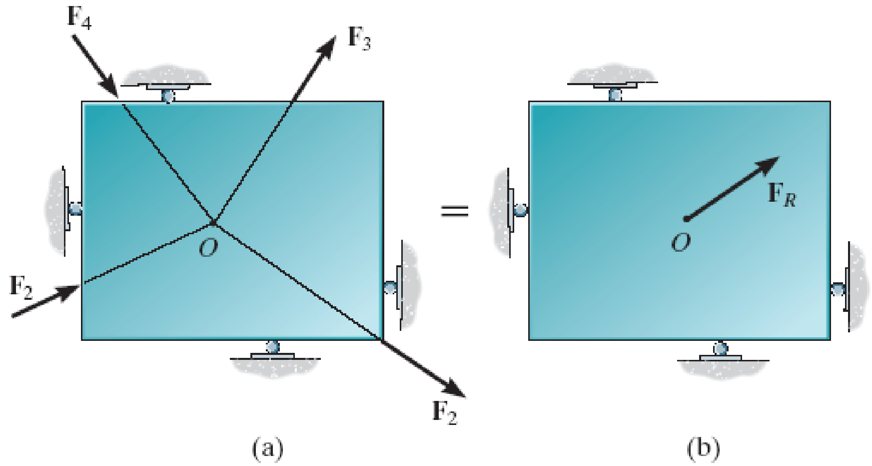
The equivalent single force F_R can be located at a distance d measured from A.

$$d = M_{RA}/F_{Ry} = 509.7 / 163.3 = 3.12 \text{ m}$$

4.8 Further Simplification of a Force and Couple System

Concurrent Force System

- A *concurrent force system* is where lines of action of all the forces intersect at a common point *O*

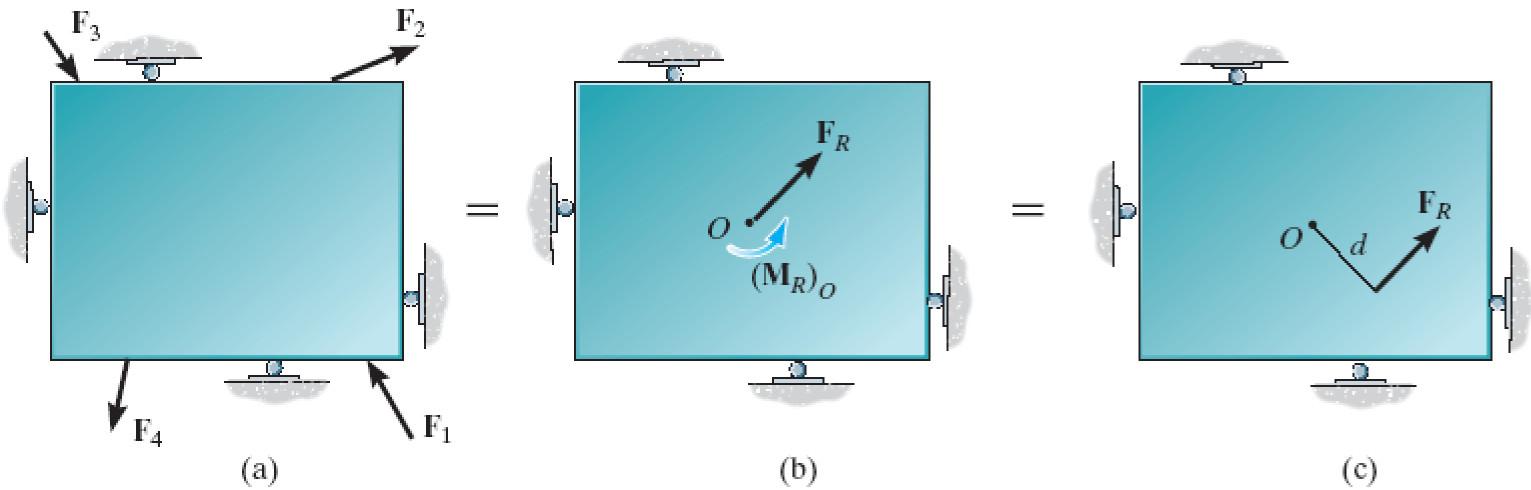


$$F_R = \sum F$$

4.8 Further Simplification of a Force and Couple System

Coplanar Force System

- Lines of action of all the forces lie in the same plane
- Resultant force of this system also lies in this plane

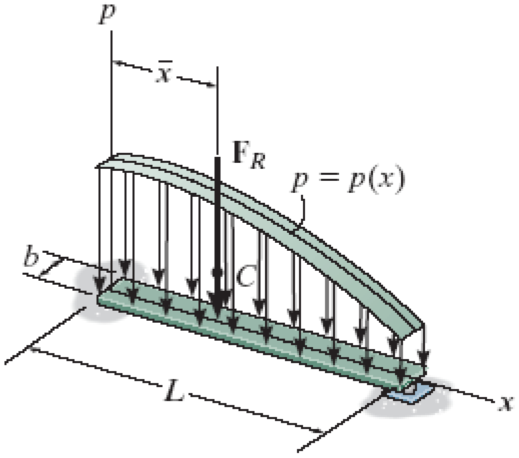


4.9 Reduction of a Simple Distributed Loading

- Large surface area of a body may be subjected to distributed loadings
- Loadings on the surface is defined as pressure
- Pressure is measured in Pascal (Pa): 1 Pa = 1N/m^2

Uniform Loading Along a Single Axis

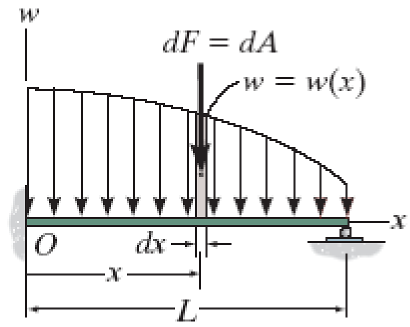
- Most common type of distributed loading is uniform along a single axis



4.9 Reduction of a Simple Distributed Loading

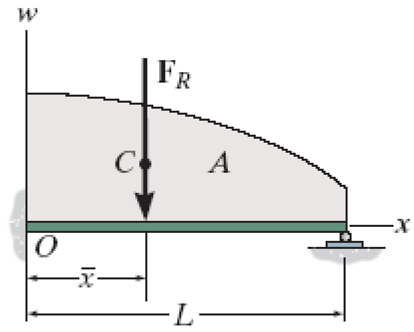
Magnitude of Resultant Force

- Magnitude of *d***F** is determined from differential *area dA* under the loading curve.
- For length *L*,



$$F_R = \int_L w(x)dx = \int_A dA = A$$

- *Magnitude of the resultant force is equal to the total area A under the loading diagram*.

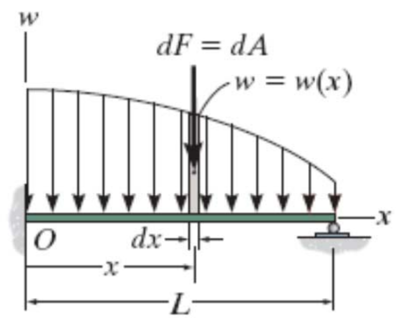


4.9 Reduction of a Simple Distributed Loading

Location of Resultant Force

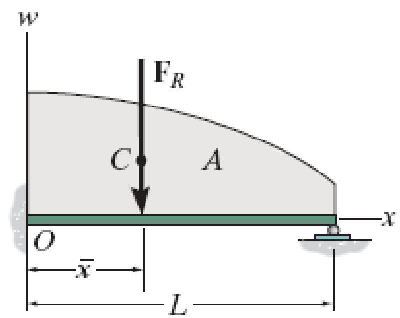
- $M_R = \sum M_O$
- $d\mathbf{F}$ produces a moment of $xdF = x\, w(x)\, dx$ about O
- For the entire plate,

$$M_{Ro} = \Sigma M_O \qquad \bar{x}F_R = \int_L x w(x) dx$$

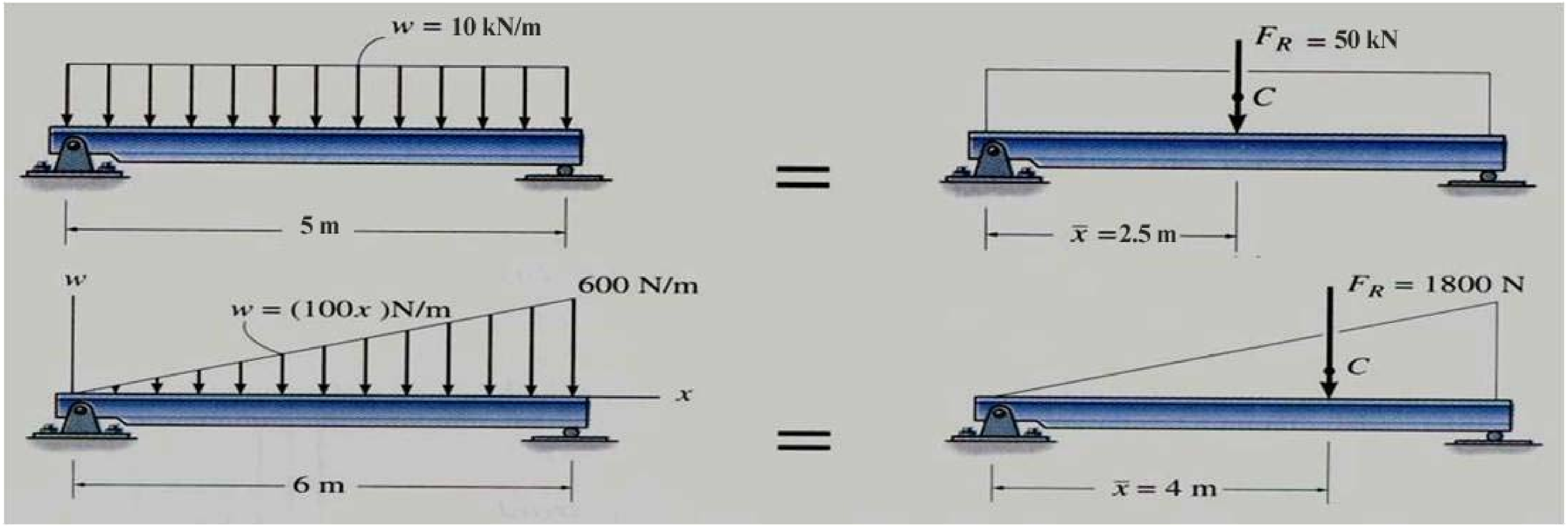


- Solving for $\bar{x}$

$$\bar{x} = \frac{\int_L x w(x) dx}{\int_L w(x) dx} = \frac{\int_A x dA}{\int_A dA}$$

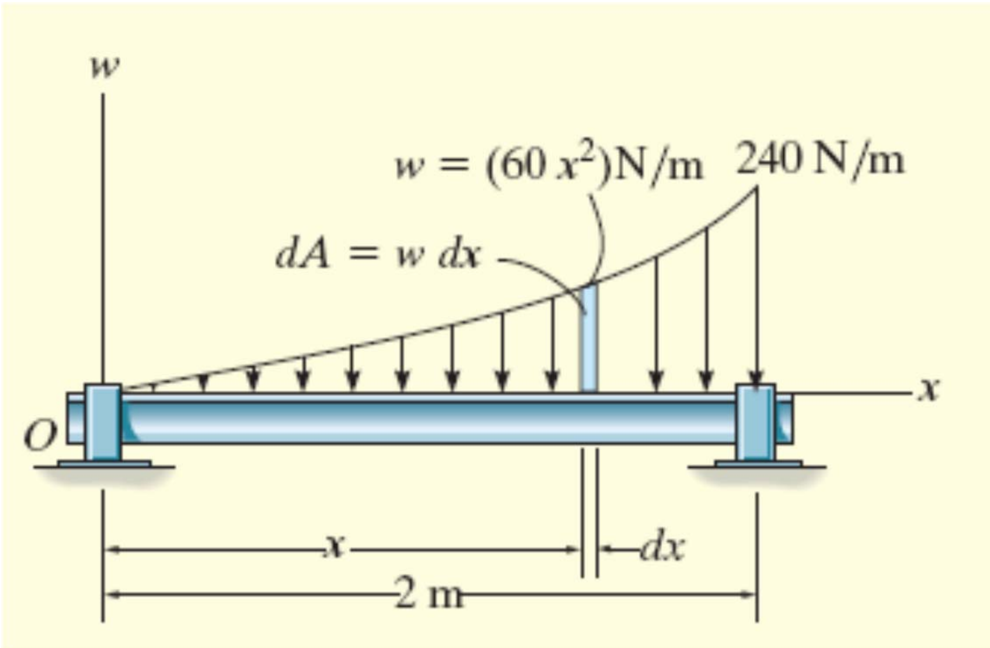


4.9 Reduction of a Simple Distributed Loading



Example

Determine the magnitude and location of the equivalent resultant force acting on the shaft.



Solution

For the colored differential area element,

$$dA = w\,dx = 60x^2\,dx$$

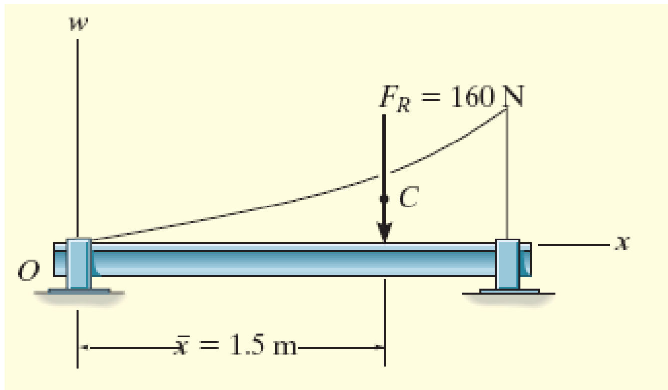
For resultant force

$$F_R = \Sigma F;$$

$$F_R = \int_A dA = \int_0^2 60x^2\,dx$$

$$= 60\left[\frac{x^3}{3}\right]_0^2 = 60\left[\frac{2^3}{3} - \frac{0^3}{3}\right]$$

$$= 160N$$



Solution

For location of line of action,

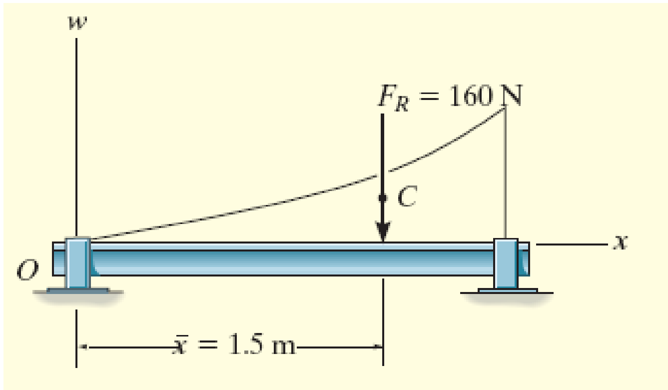
$$\bar{x} = \frac{\int_A x\,dA}{\int_A dA} = \frac{\int_0^2 x(60x^2)\,dx}{160} = \frac{60\left[\dfrac{x^4}{4}\right]_0^2}{160} = \frac{60\left[\dfrac{2^4}{4} - \dfrac{0^4}{4}\right]}{160}$$

$$= 1.5m$$

Checking,

$$A = \frac{ab}{3} = \frac{2m(240N/m)}{3} = 160$$

$$\bar{x} = \frac{3}{4}a = \frac{3}{4}(2m) = 1.5m$$



STATICS

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5 Equilibrium of a Rigid Body

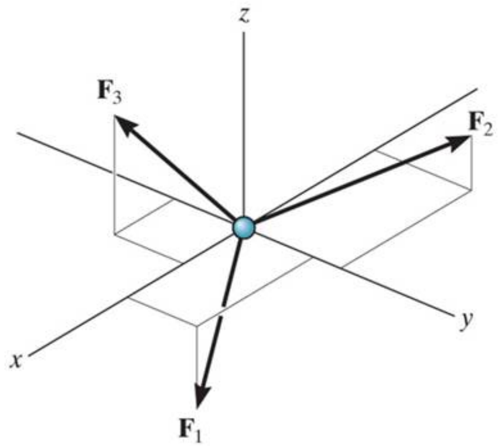
Chapter Objectives

- Develop the equations of equilibrium for a rigid body
- Concept of the free-body diagram for a rigid body
- Solve rigid-body equilibrium problems using the equations of equilibrium

Chapter Outline

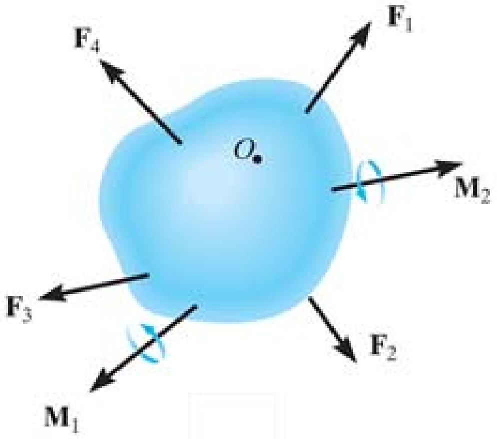
1. Conditions for Rigid Equilibrium
2. Free-Body Diagrams
3. Equations of Equilibrium

5.1 Conditions for Rigid-Body Equilibrium



Forces on a particle

In contrast to the forces on a particle, the forces on a rigid-body are not usually concurrent and may cause rotation of the body (due to the moments created by the forces).



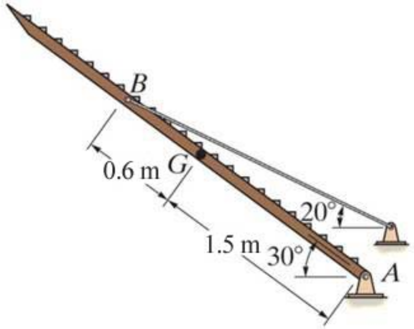
Forces on a rigid body

For a rigid body to be in equilibrium, the net force as well as the net moment about any arbitrary point O must be equal to zero.

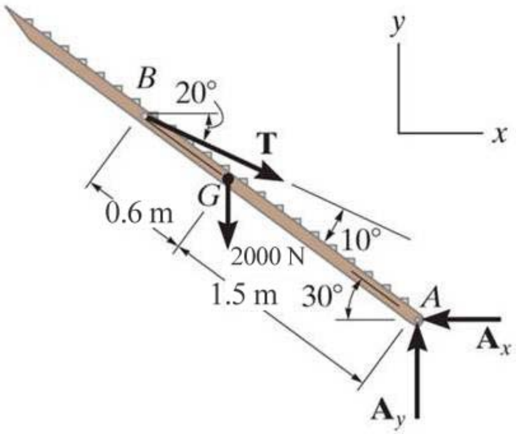
$\sum \boldsymbol{F} = 0$ (no translation)

and $\sum \boldsymbol{M_O} = 0$ (no rotation)

5.2 Free Body Diagrams



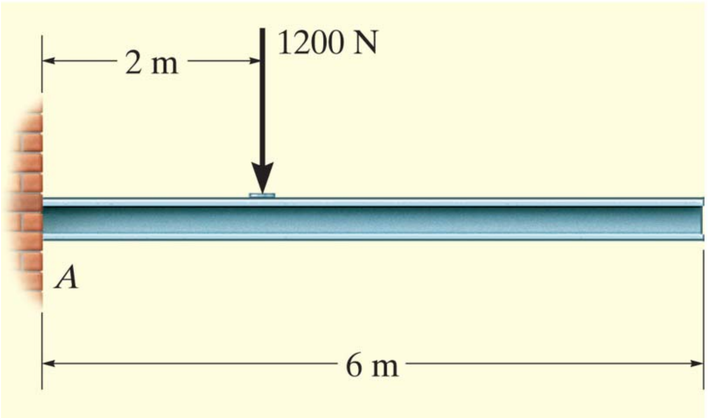
For analyzing an actual physical system, first we need to create an idealized model (above right).



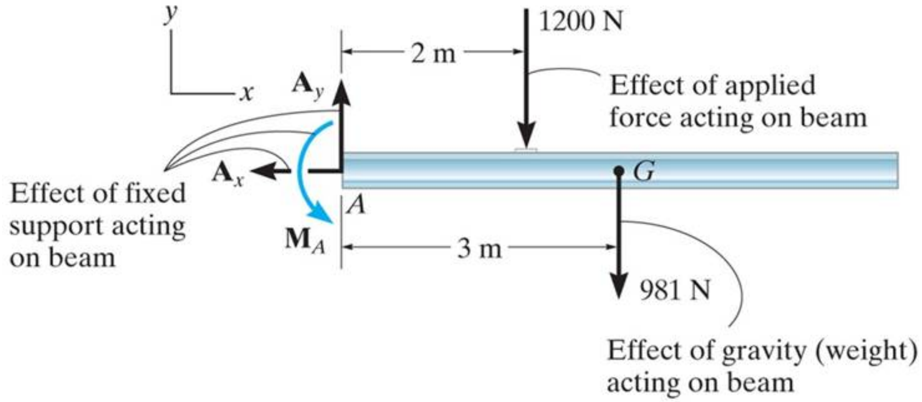
Then we need to draw a free-body diagram (FBD) showing all the external (active and reactive) forces.

Finally, we need to apply the equations of equilibrium to solve for any unknowns.

5.2 Free Body Diagrams



Idealized model

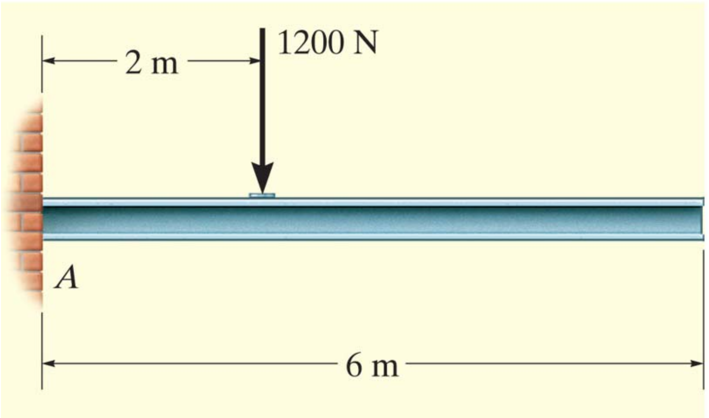


Free-body diagram (FBD)

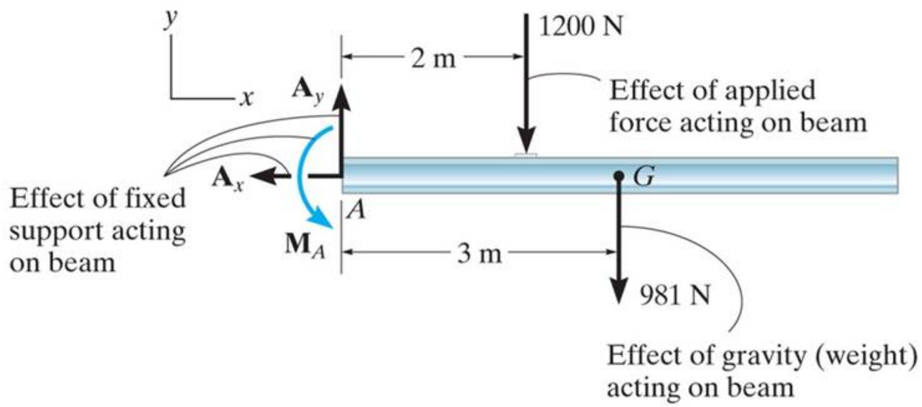
1. Draw an outlined shape. Imagine the body to be isolated or cut "free" from its constraints and draw its outlined shape.

2. Show all the external forces and couple moments. These typically include: a) applied loads, b) support reactions, and c) the weight of the body.

5.2 Free Body Diagrams



Idealized model

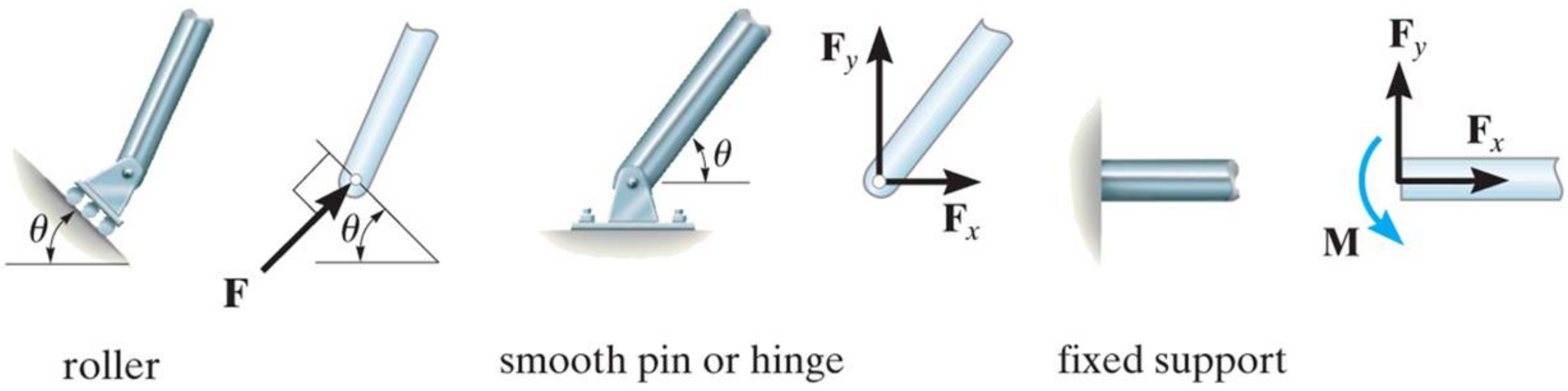


Free-body diagram (FBD)

3. Label loads and dimensions on the FBD: All known forces and couple moments should be labeled with their magnitudes and directions. For the unknown forces and couple moments, use letters like A_x, A_y, M_A. Indicate any necessary dimensions.

5.2 Free Body Diagrams

Support reactions in 2-D



Common types of supports

As a general rule, if a support prevents translation of a body in a given direction, then a force is developed on the body in the opposite direction.

Similarly, if rotation is prevented, a couple moment is exerted on the body in the opposite direction.

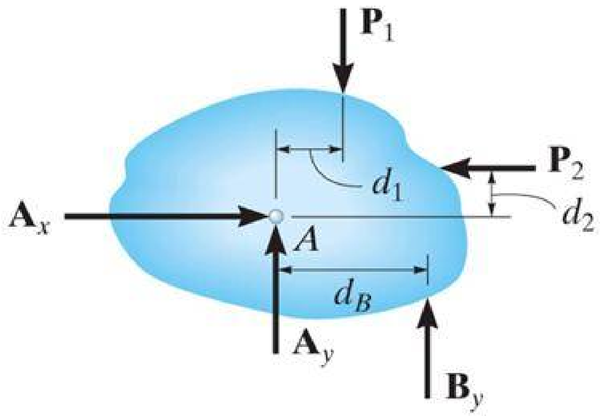
5.3 Equations of Equilibrium

- For equilibrium of a rigid body in 2D,

$$\sum F_x = 0; \sum F_y = 0; \sum M_O = 0$$

- $\sum F_x$ and $\sum F_y$ represent sums of x and y components of all the forces

- $\sum M_O$ represents the sum of the couple moments and moments of the force components



5.3 Equations of Equilibrium

Procedure for Analysis

Free-Body Diagram

- Force or couple moment having an unknown magnitude but known line of action can be assumed
- Indicate the dimensions of the body necessary for computing the moments of forces

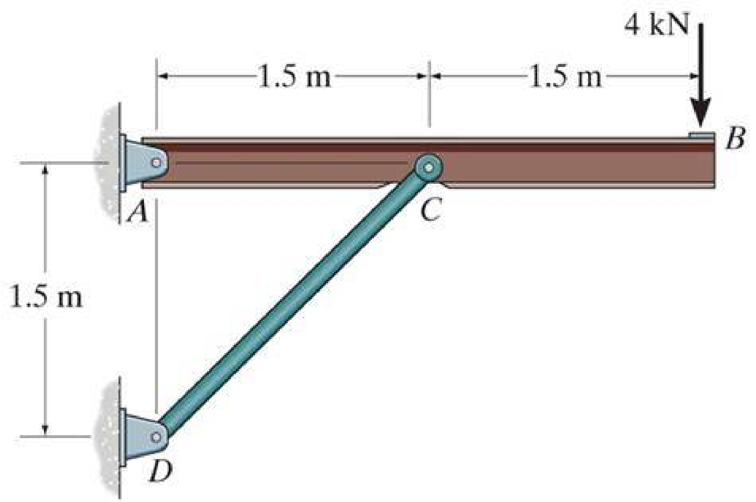
5.3 Equations of Equilibrium

Procedure for Analysis

Equations of Equilibrium

- Apply $\sum M_O = 0$ about a point O
- Unknowns moments of are zero about O and a direct solution the third unknown can be obtained
- Orient the x and y axes along the lines that will provide the simplest resolution of the forces into their x and y components
- Negative result scalar is opposite to that was assumed on the FBD

Example



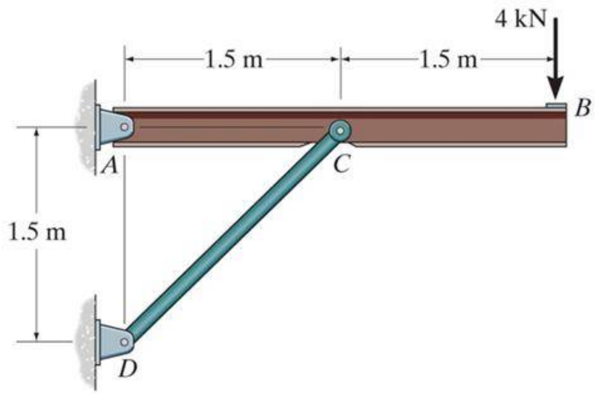
Given: The 4-kN load at B of the beam is supported by pins at A and C .

Find: The support reactions at A and C.

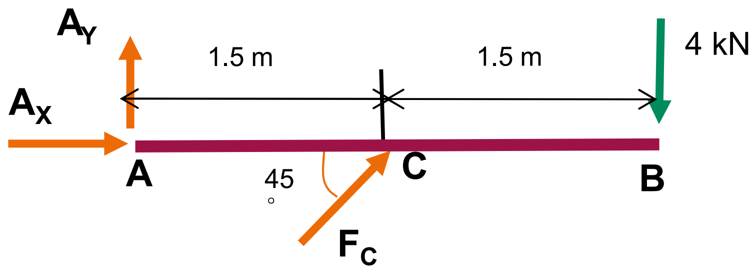
Plan:

1. Put the x and y axes in the horizontal and vertical directions, respectively.
2. Determine if there are any two-force members.
3. Draw a complete FBD of the boom.
4. Apply the E-of-E to solve for the unknowns.

Solution



FBD of the beam:



Note: Upon recognizing CD as a two-force member, the number of unknowns at C are reduced from two to one. Now, using E-o-f E, we get,

$$+\ \sum M_A = F_C \sin 45^\circ \times 1.5 - 4 \times 3 = 0$$

$$F_c = 11.31 \text{ kN or } 11.3 \text{ kN}$$

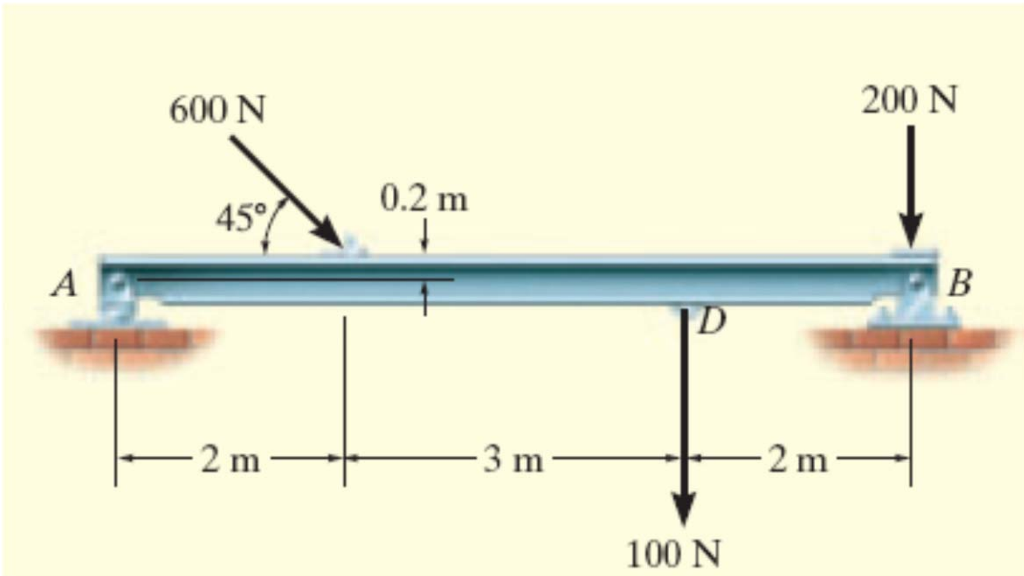
$$\rightarrow +\ \sum F_X = A_X + 11.31 \cos 45^\circ = 0; \quad A_X = -8.00 \text{ kN}$$

$$\uparrow +\ \sum F_Y = A_Y + 11.31 \sin 45^\circ - 4 = 0; \quad A_Y = -4.00 \text{ kN}$$

Note that the negative signs means that the reactions have the opposite directions to that shown on FBD.

Example

Determine the horizontal and vertical components of reaction for the beam loaded. Neglect the weight of the beam in the calculations.



Solution on whiteboard