BBL501E - Probability & Stochastic Processes
Fall 2009 - Final Exam Soln.

1. i) $$\mu_X(t) = E\{X(t)\} = E_W\left[E[X(t)\mid W=w]\right]$$

$$= \int_{-\infty}^{\infty} \underbrace{E[X(t)\mid W=w]}_{\emptyset \text{ by work in class}} f_W(w)\,dw$$

$$= \emptyset$$

ii) $$R_{XX}(t, t+\tau) = E\left[\underbrace{E[X(t)X(t+\tau)\mid W=w]}_{\frac{\cos w\tau}{2} \text{ by work done in class}}\right]$$

$$= \int_{-\infty}^{\infty} \frac{\cos w\tau}{2} f_W(w)\,dw$$

$$= \frac{1}{2\pi}\int_{5\pi}^{7\pi} \frac{\cos w\tau}{2}\,dw$$

$$= \begin{cases} \frac{1}{2} & \tau = \emptyset \\ \frac{\sin 5\pi\tau - \sin 7\pi\tau}{4\pi} & \tau \neq \emptyset \end{cases}$$

iii). Require $R_{XX}(t, t+\tau) = \emptyset$ for $\tau \neq \emptyset$

since $R_{XX}(t, t+\tau) = \emptyset \Rightarrow C_{XX}(t, t+\tau) = \emptyset$ due to zero mean and $C_{XX}(t, t+\tau) = \emptyset \Rightarrow$ stat. ind. of $X(t)$ and $X(t+\tau)$ due to $X(t)$ being Gaussian.

$$\frac{\sin 5\pi\tau - \sin 7\pi\tau}{4\pi} = \emptyset$$ is satisfied for $\tau = \frac{1}{12}$

since $\sin\frac{5\pi}{12} = \sin\Big(\underbrace{\pi - \frac{5\pi}{12}}_{\frac{7\pi}{12}}\Big)$

2. i) $R_{xx}(t_1,t_2) = \alpha \min(t_1,t_2)$

$100 = \alpha \min(10,20) \Rightarrow \alpha = 10$

$$P_r\{X(10) > 20\} = \int_{20}^{\infty} \frac{1}{\sqrt{2\pi(10)(10)}} \exp\left(-\frac{1}{2}\frac{x^2}{(10)(10)}\right) dx$$

$u = \frac{x}{10} \quad du = \frac{dx}{10}$

$$P_r\{X(10) > 20\} = \int_{2}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right) du$$

$$= 1 - F(2) = 0.0227$$

ii) $P_r\{X(20) > 20 \mid X(10) = 10\}$

$$= P_r\{X(20) - X(10) > 10 \mid X(10) = 10\}$$

$$= P_r\{X(20) - X(10) > 10\}$$

$$= \int_{10}^{\infty} \frac{1}{\sqrt{2\pi(10)(10)}} \exp\left(-\frac{1}{2}\frac{x^2}{(10)(10)}\right) dx$$

$u = \frac{x}{10} \quad du = \frac{dx}{10}$

$$= \int_{1}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right) du$$

$$= 1 - F(1) = 0.1587$$

iii) Since $R_{xx}(t, t+\tau) = \alpha \min(t, t+\tau)$ depends on t.

not W.S.S. $\Rightarrow$ not S.S.S $\Rightarrow$ neither.

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i) White noise has $S_{xx}(\omega) = \sigma_w^2 = 1$

$$R_{xx}(\tau) = \sigma_x^2 \delta(\tau) = \delta(\tau)$$

ii) $R_{yy}(\tau) = e^{-2|\tau|}$

$$S_{yy}(\omega) = \mathcal{F}\{e^{-2|\tau|}\} = \frac{4}{4+\omega^2}$$

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

$$\frac{4}{4+\omega^2} = |H(\omega)|^2 \; 1$$

$$|H(j\omega)| = \frac{2}{\sqrt{4+\omega^2}}$$

$$H(j\omega) = 2\frac{1}{2+j\omega} \longleftrightarrow h(t) = 2e^{-2t}u(\tau)$$

iii) $R_{xy}(t_1,t_2) = R_{xy}(\tau) = h^*(-\tau) * R_{xx}(\tau)$

$$= 2e^{-2(-\tau)}u(-\tau) * \underbrace{R_{xx}(\tau)}_{\delta(\tau)}$$

$$= 2e^{2\tau}u(-\tau)$$

iv) $R_{yy}(\tau) = e^{-2|\tau|}$

$$P_y = R_{yy}(0) = 1$$

v) $$\int_{-2000\pi}^{2000\pi} \frac{4}{4+\omega^2}\, d\omega = 4\tan^{-1}\frac{x}{2}\Big|_0^{2000\pi}$$

$$= 4\tan^{-1}(1000\pi)$$

4. $F_Y(y) = P_r\{Y \leq y\} = P_r\{F_X(X) \leq y\}$

$$= \begin{cases} 0 & y < 0 \\ P_r\{X \leq F_X^{-1}(y)\} = y \\ 1 & y \geq 1 \end{cases}$$

Note that $F_X(.)$ is not necessarily invertible for all X but in this case it is since it is monotonically increasing.

$F_Y(y)$ is the distribution function of standard uniform r.v.