1 a). $E[S] = E_N[E[S|N]]$

$$= E_N\left[E\left[\sum_{i=1}^{N} X_i \mid N\right]\right]$$

$$= E_N\left[\sum_{i=1}^{N} E[X_i]\right] = E_N[Nr]$$

$$= r E_N[N]$$

b). $E[S^2] = E_N[E[S^2|N]]$

$$= E_N\left[E\left[\sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j \mid N\right]\right]$$

$$= E_N\left[\sum_{i=1}^{N}\sum_{j=1}^{N} E[X_i X_j]\right]$$

$$= E_N\left[\sum_{i=1}^{N}\sum_{j=1}^{N} m\delta_{ij} + r^2(1-\delta_{ij})\right]$$

where $E[X_i X_j] = \begin{cases} E[X_i]E[X_j] & \text{if } i \neq j \\ E[X_i^2] & \text{if } i = j \end{cases}$

has been used

$$E[S^2] = E_N\left[mN + r^2N^2 - r^2N\right]$$

$$= mE_N[N] + r^2(E[N^2] - E[N])$$

2 Let $\underline{\underline{R}}^{-1} = \underline{\underline{A}}$

$$E[\underline{X}^T \underline{\underline{R}}^{-1} \underline{X}] = E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} X_i A_{ij} X_j^*\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} E[X_i A_{ij} X_j^*]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} E[X_i X_j^*]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} R_{ij}$$

since $R = R^T$, $R_{ij} = R_{ji}$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} A_{ij} R_{ji}\right)$$

Note that $\sum_{j=1}^{n} A_{ij} R_{ji} = 1$

to see this consider

$$AR = I$$

$$\Downarrow$$

$$\sum_{k=1}^{n} A_{ik} R_{kj} = \delta_{ij}$$

$$\Downarrow$$

$$\sum_{k=1}^{n} A_{ik} R_{ki} = 1$$

Hence $E[\underline{X}^T \underline{\underline{R}}^{-1} \underline{X}] = \sum_{i=1}^{n} 1 = n$

3 $\underline{Y} = \underline{\underline{U}}\,\underline{X}$

$$\underline{\underline{R}}_Y = E[\underline{Y}\,\underline{Y}^T] = E[\underline{\underline{U}}\,\underline{X}\,\underline{X}^T \underline{\underline{U}}^T] = \underline{\underline{U}}\,\underline{\underline{R}}_X \underline{\underline{U}}^T$$

Let $\underline{\underline{R}}_X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\underline{\underline{R}}_Y = \underline{\underline{U}}\,\underline{\underline{U}}^T = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$

Next consider similarity tfm. of R_Y $\quad \underline{\underline{S}}_Y^{-1} \underline{\underline{R}}_Y \underline{\underline{S}}_Y = \underline{\underline{\Lambda}}$

where $\underline{\underline{S}}_Y$ is the matrix whose columns are eigenvectors. We can rewrite similarity tfm. of $\underline{\underline{R}}_Y$ as

$$\underline{\underline{R}}_Y = \underline{\underline{U}}\,\underline{\underline{U}}^T = \underline{\underline{S}}_Y \Lambda \underline{\underline{S}}_Y^T$$
$$\Downarrow$$
$$\underline{\underline{U}} = \underline{\underline{S}}_Y \underline{\underline{\Lambda}}^{1/2}$$

$\underline{\underline{R}}_Y \underline{s}_i = \lambda_i \underline{s}_i \quad i=1,2 \qquad \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \underline{s} = \lambda \underline{s}$

1.5
.5

$$(1-\lambda)^2 - (0.5)^2 = \emptyset$$
$$1 - 2\lambda + \lambda^2 - 0.25 = \emptyset$$
$$0.75 - 2\lambda + \lambda^2 = \emptyset$$
$$(\lambda - 1.5)(\lambda - 0.5) = \emptyset$$

$$\lambda_1 = 1.5 \quad \lambda_2 = 0.5$$
$$-0.5 s_{11} + 0.5 s_{12} = \emptyset \Rightarrow s_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$
$$0.5 s_{21} + 0.5 s_{22} = \emptyset \Rightarrow s_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$U = \underline{\underline{S}}_Y \underline{\underline{\Lambda}}^{1/2} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} (1.5)^{1/2} & 0 \\ 0 & (0.5)^{1/2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

if $\underline{\underline{R}}_X = \begin{bmatrix} 1 & 0.25 \\ 0.25 & 1 \end{bmatrix}$ then let's determine the transformation $\underline{\underline{V}}$ that will yield an identity correlation matrix at its output. Then $\underline{\underline{U}}\,\underline{\underline{V}}$ will be the desired tfm. matrix

$$\underline{Z} = \underline{\underline{V}}\,\underline{X} \Rightarrow \underline{\underline{I}} = \underline{\underline{R}}_Z = \underline{\underline{V}}\,\underline{\underline{R}}_X \underline{\underline{V}}^T$$

Let $\underline{\underline{S}}_X$ be the matrix whose columns are eigenvectors of $\underline{\underline{R}}_X$. Then $\underline{\underline{\Lambda}}_X^{-1/2} \underline{\underline{S}}_X^T \underline{\underline{R}}_X \underline{\underline{S}}_X \underline{\underline{\Lambda}}_X^{-1/2} = I \Rightarrow \underline{\underline{V}} = \underline{\underline{\Lambda}}_X^{-1/2} \underline{\underline{S}}_X^T$

$$\underline{\underline{R}}_x \underline{s}_i = \lambda_i \underline{s}_i \qquad i=1,2$$

$$\begin{bmatrix} 1 & 0.25 \\ 0.25 & 1 \end{bmatrix} \underline{s} = \lambda \underline{s}$$

$$(1-\lambda)^2 - (0.25)^2 = \emptyset$$

$$\lambda^2 - 2\lambda + 0.9375 = \emptyset$$

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$$\frac{2 \pm \sqrt{4 - 4(0.9375)}}{2} = \frac{2 \pm 0.5}{2}$$

$$\lambda_1 = 1.25$$

$$\lambda_2 = 0.75$$

$$\left.\begin{aligned} -0.25\, s_{11} + 0.25\, s_{12} &= \emptyset \\ 0.25\, s_{21} + 0.25\, s_{22} &= \emptyset \end{aligned}\right\} \qquad s_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad s_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\underline{\underline{V}} = \underline{\underline{\Lambda}}_x^{-1/2} \underline{\underline{S}}_x^T = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 \\ 0 & \frac{2}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{5}} & \frac{\sqrt{2}}{\sqrt{5}} \\ \frac{\sqrt{2}}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

$$\underline{\underline{U}}\,\underline{\underline{V}} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{5}} & \frac{\sqrt{2}}{\sqrt{5}} \\ \frac{\sqrt{2}}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{\sqrt{10}} + \frac{1}{\sqrt{6}} & \frac{\sqrt{6}}{\sqrt{10}} - \frac{1}{\sqrt{6}} \\ \frac{\sqrt{6}}{\sqrt{10}} - \frac{1}{\sqrt{6}} & \frac{\sqrt{6}}{\sqrt{10}} + \frac{1}{\sqrt{6}} \end{bmatrix}$$

5.9 From the Schwarz inequality

a) $E[(X_i-\mu_i)^2]\,E[(X_j-\mu_j)^2] \geq |E[(X_i-\mu_i)(X_j-\mu_j)]|^2$ $\quad 2\cdot 3 = \sigma_1^2\sigma_2^2 \ngeq |K_{12}|^2 = 16$

b) $\sigma_{33}^2 = E[X_3-\mu_3]^2 \geq 0 \neq -2$

c) $K_{12} = E[(X_1-\mu_1)(X_2-\mu_2)]$ must be real $\neq i+j$

d) $K_{23} = E[(X_2-\mu_2)(X_3-\mu_3)] = E[(X_3-\mu_3)(X_2-\mu_2)] = K_{32}$

But $K_{23}=3$ $\quad K_{32} = 12 \neq K_{23}$

(5.20) $\Phi(\underline{\Omega}) = e^{-\frac{1}{2}\underline{\Omega}^T \underline{K}\,\underline{\Omega}}$ $\qquad \underline{K} = \{K_{ij}\}_{4\times 4}$

$\underline{K}$ is covariance matrix; $\underline{\Omega} = (\omega_1, \omega_2, \omega_3, \omega_4)^T$

$$\Phi(\underline{\Omega}) = \iiiint_{-\infty\,-\infty\,-\infty\,-\infty}^{\infty\,\infty\,\infty\,\infty} e^{j(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4)} f_{\underline{X}}(x_1, x_2, x_3, x_4)$$

$$\cdot\, dx_1\, dx_2\, dx_3\, dx_4$$

$$E[X_1 X_2 X_3 X_4] = \frac{\partial^4 \Phi}{\partial\omega_1 \partial\omega_2 \partial\omega_3 \partial\omega_4} \cdot \frac{1}{j^4}\Bigg|_{\underline{\Omega}=\underline{0}} \qquad \Phi \triangleq \Phi(\underline{\Omega})$$

The problem simplifies if we recognize that

$$\underline{\Omega}^T \underline{K}\,\underline{\Omega} = \sum_{i=1}^{4}\sum_{l=1}^{4} K_{il}\,\omega_l\,\omega_i$$

Then $\dfrac{\partial\Phi}{\partial\omega_1} = -\Phi \displaystyle\sum_{l=1}^{4} K_{1l}\,\omega_l \qquad$ at $\omega_1 = 0$

$$\frac{\partial^2\Phi}{\partial\omega_1\partial\omega_2} = -\Phi\left[K_{12} - \sum_{l=1}^{4}\sum_{k=1}^{4} K_{1l}K_{2k}\,\omega_l\,\omega_k\right]$$

at $\omega_1 = \omega_2 = 0$

$$\frac{\partial^3\Phi}{\partial\omega_1\partial\omega_2\partial\omega_3} = +\Phi\left[2K_{13}K_{23} + K_{23}\sum_{l=1}^{4} K_{1l}\,\omega_l + K_{13}\sum_{l=1}^{4} K_{2l}\,\omega_l\right] + \Phi\left[\left(K_{12} - \sum_{l=1}^{4}\sum_{k=1}^{4} K_{1l}K_{2k}\,\omega_l\,\omega_k\right)\cdot\sum_{l=1}^{4} K_{3l}\,\omega_l\right]$$

at $\omega_1 = \omega_2 = \omega_3 = 0$

Finally

$$\frac{\partial^4\Phi}{\partial\omega_1\partial\omega_2\partial\omega_3\partial\omega_4}\Bigg|_{\underline{\Omega}=0} = K_{12}K_{34} + K_{13}K_{24} + K_{23}K_{14}$$

6. Note that

$$X[n-1] = \sum_{m=1}^{n-1} \alpha^{n-1-m} W[m]$$

$$\alpha X[n-1] = \sum_{m=1}^{n-1} \alpha^{n-m} W[m]$$

$$\alpha X[n-1] + W[n] = \sum_{m=1}^{n} \alpha^{n-m} W[m] = X[n]$$

First we determine the mean function

$$\mu_X[n] = E[X[n]] = \sum_{m=1}^{n} \alpha^{n-m} E[W[m]]$$

$$= \sum_{m=1}^{n} \alpha^{n-m} p$$

$$= p\alpha^n \frac{1-(\alpha^{-1})^{n+1}}{1-\alpha^{-1}} = p\,\frac{\alpha^{n+1}-1}{\alpha-1}$$

Next, autocorrelation recursively

$$R_{XX}[n, n-1] = E[(\alpha X[n-1] + W[n])\, X[n-1]]$$

$$= \alpha E[X^2[n-1]] + \underbrace{E[W[n]]}_{\emptyset} E[X[n-1]]$$

$$R_{XX}[n, n-2] = E[(\alpha X[n-1] + W[n])\, X[n-2]]$$

$$= \alpha E[X[n-1]X[n-2]] + E[W[n]]E[X[n-2]]$$

$$= \alpha^2 E[X^2[n-2]]$$

Generalizing $R_{XX}[n, n-k] = \alpha^k E[X^2[n-k]]$

$$= \alpha^k(\sigma_X^2[n-k] + E^2[X[n-k]])$$

$$\sigma_X^2[n-k] = \sum_{m=1}^{n-k} \operatorname{var}\{\alpha^{n-k-m} W[m]\}$$ since $W[m]$ are i.i.d

$$= \sum_{m=1}^{n-k} \alpha^{2(n-k-m)} \sigma_W^2[m] = \frac{1-\alpha^{2(n-k)}}{1-\alpha^2}\, p(1-p)$$

$$R_{XX}[n, n-k] = \alpha^k\left(\frac{1-\alpha^{2(n-k)}}{1-\alpha^2}\,\frac{1}{4} + \frac{1}{4}\left(\frac{\alpha^{n-k+1}-1}{\alpha-1}\right)^2\right)$$

$$C_{XX}[n, n-k] = \alpha^k \left(\frac{1}{4} \frac{1-\alpha^{2(n-k)}}{1-\alpha^2} + \frac{1}{4} \left(\frac{\alpha^{n-k+1} - 1}{\alpha - 1} \right)^2 \right)$$

$$- \frac{1}{4} \frac{(\alpha^{n+1} - 1)(\alpha^{n-k+1} - 1)}{(\alpha - 1)^2}$$

$$= \frac{1}{4} \left[\frac{\alpha^k - \alpha^{2n+k}}{1-\alpha^2} + \frac{\alpha^{2n-k+2} - 2\alpha^{n+1} + \alpha^k}{(1-\alpha)^2} \right.$$

$$\left. - \frac{\alpha^{2n-k+2} - \alpha^{n+1} - \alpha^{n-k+1} + 1}{(\alpha - 1)^2} \right]$$

$$= \frac{1}{4} \left[\frac{(\alpha^k - \alpha^{2n-k})}{1-\alpha^2} + \frac{\alpha^{n-k+1} + \alpha^k - \alpha^{n+1} - 1}{(\alpha - 1)^2} \right]$$

$$= \frac{1}{4} \left[\frac{\alpha^k - \alpha^{2n-k}}{1-\alpha^2} + \frac{(\alpha^{n-k+1} - 1)(1 - \alpha^k)}{(\alpha - 1)^2} \right]$$

Note that this result is consistent with $\sigma_X^2[n] = C_{XX}[n, n]$

b). In this case $E[X[n]] = \emptyset$ since $E[W[n]] = \emptyset$

Here $C_{XX}[n, n-k] = R_{XX}[n, n-k] = \alpha^k \sigma_X^2[n-k]$

$$= \frac{1-\alpha^{2(n-k)}}{1-\alpha^2} \sigma_W^2$$

where σ_W^2 is the variance of each random variable.

6.12

(a) $\mu_X[n] \triangleq E[X[n]] = \sum_{\xi_i} P[\{\xi_i\}]\, X[n, \xi_i]$ ← outcome ξ_i

$$= \frac{1}{3}\left(3\delta[n] + u[n-1] + \cos\frac{\pi n}{2}\right).$$

(b) $R_X[m,n] \triangleq E[X[m]X^*[n]]$

$$= \sum_{\xi_i} P[\{\xi_i\}]\, X[m, \xi_i]\, X^*[n, \xi_i]$$

$$= \frac{1}{3}\left(9\delta[m]\delta[n] + u[m-1]u[n-1] + \cos\frac{\pi m}{2} \cdot \cos\frac{\pi n}{2}\right)$$

(c) $X[0] = \begin{cases} 3 & \frac{1}{3} & \text{"a"} \\ 1 & \frac{1}{3} & \text{"b"} \\ 0 & \frac{1}{3} & \text{"c"} \end{cases}$

value prob outcome

$X[1] = \begin{cases} 0 & \frac{1}{3} & \text{"a"} \\ 1 & \frac{1}{3} & \text{"b"} \\ 0 & \frac{1}{3} & \text{"c"} \end{cases}$

so $P[X[0]=3, X[1]=0] = P[\{a\}] = \frac{1}{3}$

$P[X[0]=3] = \frac{1}{3}$ and $P[X[1]=0] = \frac{2}{3}$ (a or b)

so joint pmf does not factor, i.e. $X[0]$ & $X[1]$ are not independent.

6.15

$$X[n,\zeta] = \sum_{i=-\infty}^{\infty} A(\zeta_i)\, p[n-i]$$

where $p[n] = \begin{cases} \frac{1}{4} & n=-1 \\ \frac{1}{2} & n=0 \\ \frac{1}{4} & n=+1 \\ 0 & \text{elsewhere} \end{cases}$

a) $$E[X[n]] = \sum_{i=-1}^{+1} E[A_i]\, p[n-i]$$

$$= \lambda\left[\frac{1}{4}+\frac{1}{2}+\frac{1}{4}\right] = \lambda$$

b) $$E[(X-\lambda)^2] = E[X^2] - \lambda^2$$

$$E[X^2] = \sum_{i=n-1}^{n+1}\sum_{j=n-1}^{n+1} p[n-i]\,p[n+j]\,E[A_iA_j]$$

$$= \sum_{\substack{i=n-1\\ j=i}}^{n+1} p^2[n-i]\, E\{A_i^2\} + \sum_{\substack{i,j\\ i\neq j}} p[n-i]\,p[n-j]$$ $E[A_i]E[A_j]$

$$= \left[\frac{1}{16}+\frac{1}{4}+\frac{1}{16}\right](\lambda^2+\lambda) + \left[\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{4}\right]\lambda^2$$

so $\sigma^2 = 3\lambda/8$

and $X[n] \sim N(\lambda, 3\lambda/8)$.

c) Since the X's will be correlated, we need to specify the correlation coefficient 'ρ' to

Complete our expression for the joint pdf.

$$E[X_n X_{n+1}] = E\left[\sum_{i=n-1}^{n+1} A_i\, p[n-i] \sum_{j=n}^{n+2} A_j\, p[n+1-j]\right]$$

$$= \sum_{i=n-1}^{n+1} \sum_{j=n}^{n+2} E\{A_i A_j] \, p[n-i]\, p[n+1-j]$$

$$= (\lambda+\lambda^2)\left(\frac{1}{8}+\frac{1}{8}\right) + \lambda^2\left[\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4}\right]$$

$$= \frac{\lambda}{4} + \frac{\lambda^2}{4} + \frac{3\lambda^2}{4} = \frac{\lambda}{4} + \lambda^2$$

$$\rho = \frac{\mathrm{Cov}[X_n, X_{n+1}]}{\sigma_{X[n]}\,\sigma_{X[n+1]}} = \frac{E[X_n X_{n+1}] - \lambda^2}{\sigma^2} = \frac{2}{3}.$$

So the joint pdf becomes

$$f_{X[n],X[n+1]}(x_1,x_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu)^2}{\sigma^2} - 2\rho\frac{(x_1-\mu)(x_2-\mu)}{\sigma^2} + \frac{(x_2-\mu)^2}{\sigma^2}\right]\right\}$$

where $\mu = \lambda$,

$\sigma^2 = 3/8\,\lambda$,

and $\rho = 2/3$.

Alternate simple solution to 6.15:

6.15 a)&b) Solution using Characteristic Functions (CFs).

$$\Phi_X(\omega) \triangleq E\left[e^{+j\omega X}\right]$$
$$= E\left[e^{+j\omega\left(\frac{1}{4}A_{-1} + \frac{1}{2}A_0 + \frac{1}{4}A_1\right)}\right]$$
$$= \Phi_A^2\left(\frac{\omega}{4}\right)\Phi_A\left(\frac{\omega}{2}\right).$$

Now $\Phi_A(\omega) = E\left[e^{+j\omega A}\right]$ with $A \sim N(\lambda, \lambda)$,

so

$$= \exp\left(j\omega\lambda - \frac{1}{2}\omega^2\lambda\right)$$

$$\Rightarrow \Phi_A\left(\frac{\omega}{4}\right) = \exp\left(j\omega\frac{\lambda}{4} - \frac{1}{2}\omega^2\frac{\lambda}{16}\right)$$
$$\& \quad \Phi_A\left(\frac{\omega}{2}\right) = \exp\left(j\omega\frac{\lambda}{2} - \frac{1}{2}\omega^2\lambda/4\right)$$

Thus $\Phi_X(\omega) = \exp\left(j\omega\lambda - \frac{1}{2}\omega^2\frac{3}{8}\lambda\right)$

and $X[n]$ is $N\left(\lambda, \frac{3}{8}\lambda\right)$.

$$X[n] = \rho X[n-1] + W[n], \quad -\infty < n < +\infty$$

6.19 (a) from the equation above, we can see that, $X[n-1]$ is a linear combination of $W[n-i]$s, $i > 0$. However, since $W[n]$ are independent, therefore $X[n-1]$ and $W[n]$ are independent.

(b) $\Phi_X(\omega) \triangleq E[e^{j\omega X}]$

$\because X[n] = \rho X[n-1] + W[n]$

$\therefore \Phi_{X[n]}(\omega) = \Phi_{X[n-1], W[n]}(\rho\omega, \omega)$

but $X[n-1]$ and $W[n]$ are independent, therefore:

$\Phi_X(\omega) = \Phi_X(\rho\omega)\, \Phi_W(\omega)$ since $X[n]$ is stationary.

(c) Since $W[n]$ is gaussian, with zero mean,

we have: $\Phi_W(\omega) = e^{-\frac{1}{2}\sigma_W^2 \omega^2}$

but, we know: $\Phi_X(\omega) = \Phi_X(\rho\omega)\, \Phi_W(\omega)$

Let $\omega \longrightarrow \rho\omega$; Then

$\Phi_X(\rho\omega) = \Phi_X(\rho^2\omega)\, \Phi_W(\rho\omega)$

and $\Phi_X(\rho^2\omega) = \Phi_X(\rho^3\omega)\, \Phi_W(\rho^2\omega)$

$\vdots$

$\Phi_X(\rho^{n-1}\omega) = \Phi_X(\rho^n\omega)\, \Phi_W(\rho^{n-1}\omega)$

multiplying all the terms on both sides:

$$\Phi_X(\omega) = \Phi_X(\rho^n \omega)\left[\Phi_W(\rho^{n-1}\omega)\,\Phi_W(\rho^{n-2}\omega)\cdots\Phi_W(\omega)\right]$$

$$= \Phi_X(\rho^n \omega)\left[e^{-\frac{1}{2}(\rho^{n-1}\omega)^2\sigma_w^2}\cdots e^{-\frac{1}{2}\omega^2\sigma_w^2}\right]$$

$$= \Phi_X(\rho^n \omega)\left[\exp\left(-\frac{\sigma_w^2\omega^2}{2}\right)\sum_{i=0}^{n-1}\rho^{2i}\right]$$

Now, let $n \to \infty$; then,

i) $\rho^n \to 0$, if $|\rho| < 1$, then

$$\Phi(\rho^n \omega) \longrightarrow \Phi(0) = 1$$

ii) $\sum_{i=1}^{n-1} \rho^{2i} = \frac{1}{1-\rho^2}$

Therefore:

$$\Phi_X(\omega) = e^{-\frac{\sigma_w^2\omega^2}{2(1-\rho^2)}}$$

(d) from (c), $\sigma_X^2 = \frac{\sigma_w^2}{1-\rho^2}$

6.22 $X[0]=0$, $X[n]=\rho X[n-1]+W[n]$ for $n\geq 1$ (1).

a) From (1), $X[n]=\sum_{i=1}^{n}\rho^{n-i}W[i]$, $n\geq 1$

so $E[X[n]] = E[\sum \rho^{n-i}W[i]] = \sum_{i=1}^{n}\rho^{n-i}E[W[i]]$

$= \sum \rho^{n-i}\cdot 0 = 0.$

b) $K_X[m,n] \triangleq E[(X[m]-\mu)(X[n]-\mu)]$

$= E[X[m]X[n]]$ since mean is zero.

$= E\left[\left(\sum_{i=1}^{m}\rho^{m-i}W[i]\right)\left(\sum_{j=1}^{n}\rho^{n-j}W[j]\right)\right]$

Let $m\leq n$; Also we know $E[W[i]W[j]]=0$ for $i\neq j$ since the $W[i]$ are independent & have mean 0.

Thus $K_X[m,n] = \sum_{i=1}^{m}\rho^{m-i}\rho^{n-i}E[W^2[i]]$ for $m\leq n$

Now $E[W^2[i]]=\sigma_W^2$ and

$$\sum_{i=1}^{m}\rho^{m+n-2i} = \rho^{n-m}\sum_{i=1}^{m}(\rho^2)^{m-i} = \rho^{n-m}\left(\frac{1-\rho^{2m}}{1-\rho^2}\right)$$

so $$K_X[m,n] = \rho^{n-m}\sigma_W^2\left(\frac{1-\rho^{2m}}{1-\rho^2}\right) \text{ for } m\leq n$$

In general, for $m,n\geq 0$, we have

$$K_X[m,n] = \frac{\sigma_W^2}{1-\rho^2}\left[\rho^{|m-n|}-\rho^{m+n}\right].$$

c) Let $|\rho|<1$, Then as $m,n\to\infty$, $K_X \to \frac{\sigma_W^2}{1-\rho^2}\rho^{|m-n|}$ and K_X becomes asymptotically just a function of $(m-n)$, i.e. $K_X[m-n]$ a one-parameter function.

6.23

a) $\mu_X[n] = E[A\cos\omega n + B\sin\omega n]$
$= E[A]\cdot\cos\omega n + E[B]\cdot\sin\omega n$
$= 0 \cdot$ " $+ 0 \cdot$ "
$= 0$, a constant.

$E[X[n+m]X[n]] =$

$$= E[(A\cos\omega(n+m) + B\sin\omega(n+m))(A\cos\omega n + B\sin\omega n)]$$
$$= \sigma^2(\cos\omega(n+m)\cos\omega n + \sin\omega(n+m)\sin\omega n)$$
$$= \sigma^2\cos(\omega(n+m) - \omega n)$$
$$= \sigma^2\cos\omega m = R_X[m]. \quad \therefore \text{ WSS}.$$

b) Consider $E[X^3[n]] = E[(A\cos\omega n + B\sin\omega n)^3]$
$= E[A^3]\cdot\cos^3\omega n + E[A^2B](\cdot)$
$+ E[AB^2](\cdot) + E[B^3]\cdot\sin^3\omega n$

Now $E[A^2B] = E[A^2]\cdot E[B] = \sigma^2\cdot 0 = 0 = E[AB^2]$

so

$$E[X^3[n]] = \xi_3 \cdot \underbrace{(\cos^3\omega n + \sin^3\omega n)}_{\neq \text{ constant}}$$

third-order moment $(\neq 0)$

Hence $X[n]$ cannot be stationary in the strict sense.