HW#1 Solutions

1.4. $Pr\{\text{two heads and one tail}\} = \dfrac{\#\text{ of outcomes favorable to two heads, one tail.}}{\#\text{ of all outcomes}}$

$$= \frac{\binom{3}{2}}{2^3} = \frac{3}{8}$$

1.7. $\Omega = (0,\infty) \times (0,\infty)$ is the combined space.

$\Omega = \{(X,Y) : X \in (0,\infty), Y \in (0,\infty)\}$

$E = \{(X,Y) : X > Y\}$ where X: height of woman
Y: height of man

1.14 a: two balls
b: two balls
ab: four balls.

$P(A) = \dfrac{\#\text{ of outcomes favorable to } a}{\text{total }\#\text{ of outcomes}}$

$= \dfrac{6}{8}$

$P(B) = \dfrac{6}{8}$

$P(AB) = \dfrac{4}{8}$

Note $P(A)P(B) = \dfrac{9}{16} \neq P(AB)$

Hence A&B are not independent

1.20

$$P(X=1 \mid Y=1) = \frac{P(Y=1 \mid X=1)\,P(X=1)}{P(Y=1)}$$

$$= \frac{P(Y=1 \mid X=1)\,P(X=1)}{\sum_{i=1}^{3} P(Y=1 \mid X=i)\,P(X=i)}$$

$$P(X=3) = 3\,P(X=1)$$

$$P(X=2) = 2\,P(X=1)$$

$$P(X=3) = \frac{1}{2}$$

$$P(X=2) = \frac{1}{3}$$

$$P(X=1) = \frac{1}{6}$$

$$P(X=1 \mid Y=1) = \frac{(1-\alpha)\frac{1}{6}}{(1-\alpha)\frac{1}{6} + \frac{\beta}{2}\frac{1}{3} + \frac{\gamma}{2}\frac{1}{2}}$$

$$= \frac{1-\alpha}{1-\alpha+\beta+3\gamma/2}$$

1.26 $R =$ "r balls appear in r preselected cells"

$$P(R) = \frac{k_R}{k}$$

$k_R \rightarrow$ # of combinations favorable to R

$k \rightarrow$ total number of combinations

$$k = \binom{n}{r}$$

$$k_R = \binom{r}{r} = 1$$

$$P(R) = \frac{1}{\frac{n!}{(n-r)!\,r!}} = \frac{(n-r)!\,r!}{n!}$$

1.21 $Pr\{\text{knows} \mid \text{correctly answers}\} = \dfrac{Pr\{\text{correctly answers} \mid \text{knows}\} Pr\{\text{knows}\}}{Pr\{\text{correctly answers}\}}$

$$= \frac{Pr\{\text{correctly answers} \mid \text{knows}\} Pr\{\text{knows}\}}{Pr\{\text{correctly answers} \mid \text{knows}\} Pr\{\text{knows}\} + Pr\{\text{correctly answers} \mid \text{doesn't know}\} \cdot Pr\{\text{doesn't know}\}}$$

$$= \frac{1 \cdot p}{1 \cdot p + \frac{1}{m}(1-p)}$$

$$= \frac{mp}{(m-1)p+1}$$

1.30 a)

$p = Pr\{\text{BM destroyed}\}$

$= Pr\{\{\text{SSK missile 1}\} \cup \{\text{SSK missile 2}\}\}$

$= 1 - Pr\{\{\overline{\text{SSK missile 1}}\} \cap \{\overline{\text{SSK missile 2}}\}\}$

$= 1 - Pr\{\overline{\text{SSK missile 1}}\} Pr\{\overline{\text{SSK missile 2}}\}$

$= 1 - (0.2)(0.2)$

$= 0.96$

$Pr\{\text{all BM's destroyed}\} = p^5 = (0.96)^5$

Note that $\{\text{SSK missile 1}\}$ & $\{\text{SSK missile 2}\}$ are stat. ind.

Therefore $\{\overline{\text{SSK missile 1}}\}$ & $\{\overline{\text{SSK missile 2}}\}$ are stat. ind.

b) Also each $\{\text{BM destroyed}\}$ is ind. of other $\{\text{BM destroyed}\}$

b). $\{\text{at least one BM gets through}\} = \overline{\{\text{all BM's destroyed}\}}$

$\Pr\{\text{at least one BM gets through}\} = 1 - \Pr\{\text{all BMs destroyed}\}$

$= 1 - 0.96$

$= 0.04$

c) $\Pr\{\text{exactly one BM gets through}\} = \binom{6}{1} p^5 (1-p)$

$= 6\,(0.96)^5\,(0.04)$

1.32 a). $\Pr\{\text{all 10 meet specs}\} = \dfrac{\#\text{ of outcomes in which all 10 meets specs.}}{\text{total }\#\text{ of outcomes}}$

$$= \frac{\binom{85}{10}}{\binom{100}{10}}$$

$$= \frac{\dfrac{85!}{75!\,10!}}{\dfrac{100!}{90!\,10!}} = \frac{85.84\ldots 76.75}{100.99\ldots 91.90}$$

b). $\Pr\{\text{2 or more discarded}\} = 1 - \Pr\{\text{1 discarded}\} - \Pr\{\text{no discarded}\}$

$$= 1 - \frac{\binom{5}{1}\binom{95}{9}}{\binom{100}{10}} - \frac{\binom{95}{10}}{\binom{100}{10}}$$

$$= 1 - \frac{\binom{5}{1}\binom{95}{9} + \binom{95}{10}}{\binom{100}{10}}$$

c) $\Pr\left\{\begin{array}{l}\text{8 meets spec,}\\ \text{1 needs reworking,}\\ \text{1 is discarded}\end{array}\right\} = \dfrac{\binom{85}{8}\cdot\binom{10}{1}\cdot\binom{5}{1}}{\binom{100}{10}}$

\# of outcomes favorable to event of interest

total number of outcomes.

1.34 We use the Poisson approximation to the binomial: Eq. (1.9-2).

$p = \frac{1}{1000} = 10^{-3}$, $n = 100$ $np = 10^{-1}$

P[at least one diamond found) = 1 − P[no diamonds found] $= 1 - (10^{-1})^0/0! \cdot e^{-0.1} = 1 - 0.9 \simeq 0.1$

1.38 A closed circuit can occur as

$$(A_2A_4 \cup A_3A_5)A_1A_6 = A_1A_2A_4A_6 \cup A_1A_3A_5A_6$$

$P(A \cup B) = P(A) + P(B) - P(AB)$ for any events A, B.

b) Thus P(at least one closed path)

$$= P(A_1 A_2 A_4 A_6) + P(A_1 A_3 A_5 A_6) - P(A_1 A_2 A_3 A_4 A_5 A_6)$$

$$= 2p^4 - p^6 = 2p^4\left(1 - \tfrac{1}{2}p^2\right).$$

1.44 The reasoning is false since it is now given that there is at least one bomb on board.

$$\Pr\{\text{two bombs or more} \mid \text{unarmed bomb}\} = \frac{\Pr\{\{\text{two or more bombs}\} \cap \{\text{unarmed bomb}\}\}}{\Pr\{\text{unarmed bomb}\}}$$

$$= \frac{\Pr\{\text{two or more bombs}\}}{\Pr\{\text{unarmed bomb}\}}$$

Note here that $\Pr\{\text{two or more bombs}\} = \Pr\{\text{unarmed bomb}\}\Pr\{\text{one or more armed bombs}\}$

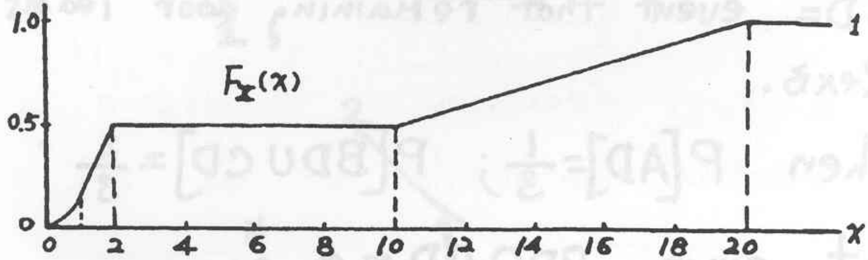
$$\Pr\{\text{two bombs or more} \mid \text{unarmed bomb}\} = \Pr\{\text{one or more armed bombs}\}$$

$$\geq \Pr\{\text{one armed bomb}\}$$

$$= 10^{-2}$$

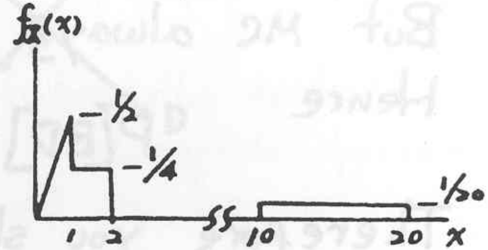
2.2

a)



b) By differentiating:

$$f_X(x) = \begin{cases} x/2 , & 0 \le x \le 1 \\ 1/4 , & 1 \le x \le 2 \\ 0 , & 2 \le x \le 10 \\ 1/20 , & 10 \le x \le 20 \\ 0 , & x \geq 20 \end{cases}$$



So $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{4}\,dx + \int_{10}^{20} \frac{1}{20}\,dx = 1.$

c) Let X be the waiting time. Then

1. $P[X \geq 10] = 1 - P[X \leq 10] = 1 - F_X(10) = 1/2$
2. $P[X \leq 5] = F_X(5) = 1/2$
3. $P[5 \leq X \leq 10] = \int_5^{10} f_X(x)\,dx = 0$
4. $P[X = 1] = 0$.

(2.3)

$$P[X<a] = F_X(a) - P[X=a]$$

$$P[X\leq a] = F_X(a)$$

$$P[a\leq X<b] = F_X(b) - P[X=b] - F(a) + P[X=a]$$

$$P[a\leq X\leq b] = F_X(b) - F_X(a) + P[X=a]$$

$$P[a<X\leq b] = F_X(b) - F_X(a)$$

$$P[a<X<b] = F_X(b) - P[X=b] - F_X(a)$$

(2.5) $P[|X| \geq k\sigma] = 1 - P[-k\sigma \leq X \leq k\sigma]$

$$= 1 - \int_{-k\sigma}^{k\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx = 1 - \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} e^{-\frac{1}{2}x^2} dx .$$

Our table gives $\mathrm{erf}(x) = \frac{1}{\sqrt{2\pi}} \int^{x} e^{-\frac{1}{2}y^2} dy$. Hence

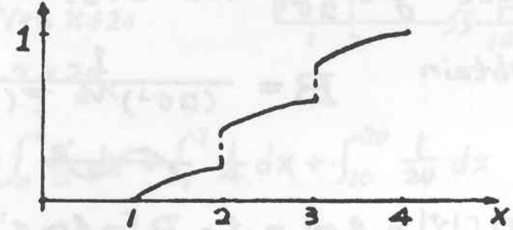
$$P[|X| \geq k\sigma] = 1 - 2\,\mathrm{erf}(x) .$$

For $k=1$, $P=0.3174$; for $k=2$, $P=0.0456$;
for $k=3$, $P=0.0026$ etc.

2.8 a) $A\int_1^4 e^{-x}dx + \frac{1}{4}\int_{-\infty}^{\infty} \delta(x-2)\,dx + \frac{1}{4}\int_{-\infty}^{\infty} \delta(x-3)\,dx = 1$

$A[e^{-1} - e^{-4}] + \frac{1}{4} + \frac{1}{4} = 1 \qquad A = 1.43$

b)



c) $f_X(x) = [1.43\, e^{-x} u(x-1) + \frac{1}{4}\delta(x-2) + \frac{1}{4}\delta(x-3)]\, u(4-x)$

$P[2 \le X < 3] = \int_2^3 1.43\, e^{-x} dx + \frac{1}{4}\int_{-\infty}^{\infty} \delta(x-2)\,dx$

d) $P[2 < X \le 3] = \int_2^3 1.43\, e^{-x} dx + \frac{1}{4}\int_{-\infty}^{\infty} \delta(x-3)\,dx$

e) $F_X(3) = P[2 < X \le 3] + \int_1^2 e^{-x} dx + \int_{-\infty}^{\infty} \delta(x-2)\,dx$

2.10 $P(X=k \mid X \text{ even}) = 0 \qquad k = 1, 3$

$$P(X=k \mid X \text{ even}) = \frac{P(X=k,\ X \text{ even})}{P(X \text{ even})} = \frac{P(X \text{ even} \mid X=k)\,P(X=k)}{P(X \text{ even})}$$

$$= 0 \qquad k = 1, 3$$

$$= \frac{P(X=k)}{P(X \text{ even})}$$

$$P(X \text{ even}) = P(X=0) + P(X=2) + P(X=4)$$

$$= \binom{4}{0}\left(\frac{1}{2}\right)^4 + \binom{4}{2}\left(\frac{1}{2}\right)^4 + \binom{4}{4}\left(\frac{1}{2}\right)^4 = \frac{1}{2}$$

$$P(X=0) = \binom{4}{0}\left(\frac{1}{2}\right)^4 = \frac{1}{16} \Rightarrow P(X=0 \mid X \text{ even}) = \frac{1}{8}$$

$$P(X=2) = \binom{4}{2}\left(\frac{1}{2}\right)^4 = \frac{6}{16} \Rightarrow P(X=2 \mid X \text{ even}) = \frac{6}{8}$$

$$P(X=4) = \binom{4}{4}\left(\frac{1}{2}\right)^4 = \frac{1}{16} \Rightarrow P(X=4 \mid X \text{ even}) = \frac{1}{8}$$

(2.11) # bulbs produced by $B = n_B$

" " " $A = n_A$

$$n_B = 3n_A \qquad n_A + 3n_A = n$$

$$4n_A = n$$

$$n_A = \frac{n}{4} \qquad n_B = \frac{3n}{4}$$

$$\therefore P(A) = n_A/n = 1/4 \qquad P(B) = 3/4$$

$$F_X(x|A) = (1 - e^{-x/5})u(x) \qquad F_X(x|B) = (1 - e^{-x/2})u(x)$$

$$F_X(x) = F_X(x|A)P(A) + F_X(x|B)P(B)$$

$$= (1 - e^{-x/5})\frac{1}{4} + (1 - e^{-x/2})\frac{3}{4}$$

$$F(2) = .560 \qquad F(5) = .85 \qquad F(7) = .92$$

P(burns at least 2 months) $= 1 - F(2) = 0.44$

P(burns at least 5 months) $= 1 - F(5) = 0.15$

P(burns at least 7 months) $= 1 - F(7) = 0.08$

2.14 Clearly X and Y are independent but how do we find the pdf's?

Write $f_X(x) = A e^{-\frac{1}{2}\left(\frac{x}{3}\right)^2} u(x)$.

Then with $A = \frac{2}{3\sqrt{2\pi}}$, $\int_0^\infty f_X(x)\,dx = 1$

Now write $f_Y(y) = B e^{-\frac{1}{2}\left(\frac{y}{2}\right)^2} u(y)$

Then with $B = \frac{2}{2\sqrt{2\pi}}$, $\int_0^\infty f_Y(y)\,dy = 1$

But $f_X(x) f_Y(y) = AB\, e^{-\frac{1}{2}\left(\frac{x}{3}\right)^2} e^{-\frac{1}{2}\left(\frac{y}{2}\right)^2} \times u(x)u(y)$

and $AB = \frac{1}{3\pi}$ as given.

$$P[0 < X \leq 3] = 2 \cdot \frac{1}{3\sqrt{2\pi}} \int_0^3 e^{-\frac{1}{2}\left(\frac{x}{3}\right)^2} dx$$
$$= 2\,\mathrm{erf}(1)$$

Likewise:

$$P[0 < Y \leq 2] = 2 \cdot \frac{1}{2\sqrt{2\pi}} \int_0^2 e^{-\frac{1}{2}\left(\frac{y}{2}\right)^2} dy$$
$$= 2\,\mathrm{erf}(1)$$

Hence $P[0 < X \leq 3, 0 < Y \leq 2] = 4\,\mathrm{erf}^2(1)$

$= 0.466$

(2.16) We use Bayes formula for pdf's:

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)\, f_X(x)}{f_Y(y)}$$

$$f_X(x) = \frac{1}{2}\,\mathrm{rect}\left(\frac{x}{2}\right)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \int_{-\infty}^{\infty} f_{Y|X}(y|x)\, f_X(x)\,dx$$

$$= \int_{-1}^{1} \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(y-x)^2}{2\sigma^2}\right] dx$$

Let $\xi = \frac{x-y}{\sigma}$ $d\xi = \frac{dx}{\sigma}$, obtain

$$f_Y(y) = \frac{1}{2}\int_{\frac{-1-y}{\sigma}}^{\frac{1-y}{\sigma}} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\xi^2}\, d\xi$$

$$= \frac{1}{2}\left[\mathrm{erf}\left(\frac{1-y}{\sigma}\right) - \mathrm{erf}\left(\frac{-1-y}{\sigma}\right)\right].$$ But

$\mathrm{erf}(x) = -\,\mathrm{erf}(-x)$. Hence

$$f_Y(y) = \frac{1}{2}\left[\mathrm{erf}\left(\frac{1+y}{\sigma}\right) - \mathrm{erf}\left(\frac{y-1}{\sigma}\right)\right].$$

Then

$$f_{X|Y}(x|y) = \frac{\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(y-x)^2}{2\sigma^2}\right]\mathrm{rect}\left(\frac{x}{2}\right)}{\mathrm{erf}\left(\frac{1+y}{\sigma}\right) - \mathrm{erf}\left(\frac{y-1}{\sigma}\right)}$$

(2.20) This is a rather classic problem in detection theory.

$$P[A|M] = P[X \geq 0.5|M] = \frac{1}{\sqrt{2\pi}} \int_{0.5}^{\infty} e^{-\frac{1}{2}(x-1)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-0.5}^{\infty} e^{-\frac{1}{2}y^2} dy = \frac{1}{2} + \operatorname{erf}(0.5) = \underline{0.69}$$

$$P[A|M^c] = \frac{1}{\sqrt{2\pi}} \int_{0.5}^{\infty} e^{-\frac{1}{2}x^2} dx = \frac{1}{2} - \operatorname{erf}(0.5) = \underline{0.31}$$

$$P[A^c|M^c] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.5} e^{-\frac{1}{2}x^2} dx = \frac{1}{2} + \operatorname{erf}(0.5) = \underline{0.69}$$

$$P[A^c|M] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.5} e^{-\frac{1}{2}(x-1)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-0.5} e^{-\frac{1}{2}y^2} dy$$

$$= \frac{1}{2} - \operatorname{erf}(0.5) = \underline{0.31}$$

3.4

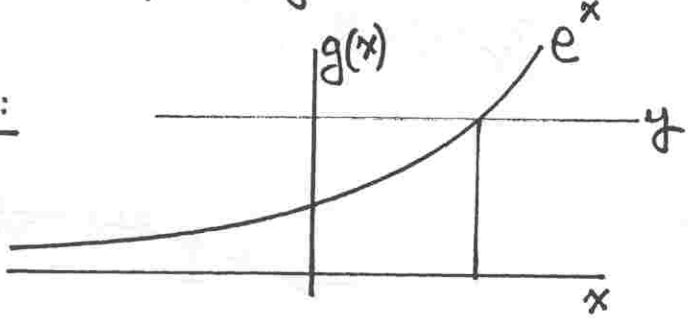
Method (i): $P[Y \le y] = P[e^X \le y] = P[X \le \ln y]$
$= F_X(\ln y)$

Hence $f_Y(y) = \frac{dF_X(\ln y)}{d(\ln y)} \frac{d(\ln y)}{dy} = f_X(\ln y) \cdot \frac{1}{y}$

for $y > 0$. Note for $y < 0$ $P[e^X \le y] = P[\phi] = 0$

Hence $f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma y} \exp\left[-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2\right] u(y).$

Method (ii):



Only solution to $y - g(x) = 0$ is $x = \ln y$ for $y > 0$;
for $y < 0$, no real solution to $y - g(x) = 0$; hence
for $y < 0$, pdf of $Y = 0$

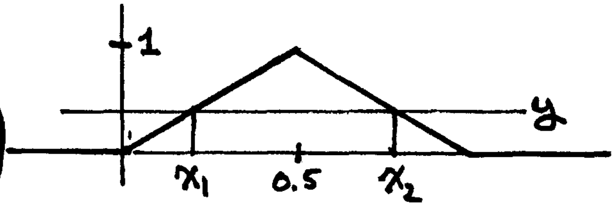
$$f_Y(y) = \sum_{i=1}^{r} \frac{f_X(x_i)}{|g'(x_i)|}$$

$$= f_X(\ln y) / e^x \big|_{x = \ln y}$$

$$= f_X(\ln y)/y \cdot u(y)$$

3.8

<u>Given:</u> $f_X(x) = \frac{1}{2}\,\text{rect}\left(\frac{x-1}{2}\right)$



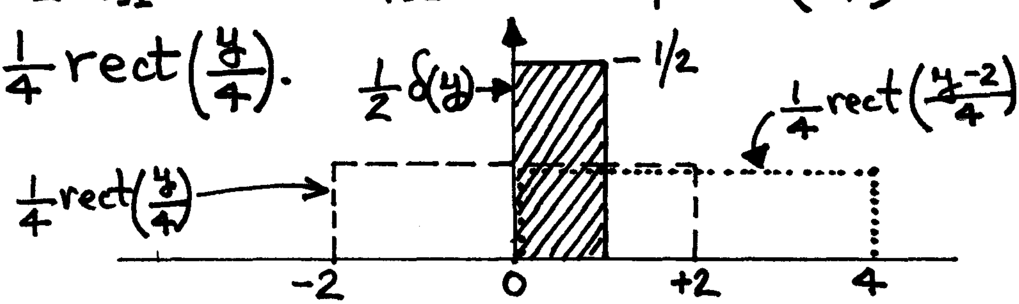
For $0 \le x < \frac{1}{2}$ $\quad g(x) = 2x$
For $\frac{1}{2} \le x < 1$ $\quad g(x) = 2-2x.$
Elsewhere $g(x) = 0.$

For $y > 1$, no real roots to $g(x) - y = 0$; $\therefore$ pdf is 0;
For $0 < y < 1$ $\quad g(x) - y = 0$ has 2 roots:
$2x - y = 0$ or $x_1 = y/2$ and $2 - 2x - y = 0$ or
$x_2 = 1 - y/2.$

For $y < 0$, no real roots to $g(x) - y = 0$; $\therefore$ pdf is 0;
For $y = 0$ $\quad P[Y=0] = P[X \le 0] + P[X \ge 1] = P[X \ge 1]$
since $X : U(0,2)$ i.e., is uniform in $(0,2)$.
But $P[X \ge] = \frac{1}{2}$. Therefore at $y = 0$, $f_Y(y)$
$= \frac{1}{2}\delta(y).$

At $y = 1$ $\quad P[Y=1] = P[X = \frac{1}{2}] = 0$ since X is cont.

Thus for $0 < y < 1$ $\quad f_Y(y) = \sum_{i=1}^{2} f_X(x_i)/|g'(x_i)|$
$= \frac{1}{2} f_X(y/2) + \frac{1}{2} f_X(1 - y/2) = \frac{1}{4}\,\text{rect}\left(\frac{y-2}{4}\right)$
$+ \frac{1}{4}\,\text{rect}\left(\frac{y}{4}\right).$



3.13 $\quad Z = X - Y \quad f_X(x) = f_Y(x) = \alpha e^{-\alpha x} u(x)$

Let $V = -Y$, $P[V \le v] = P[Y \ge -v]$

$$= 1 - F_Y(-v)$$

$$f_V(v) = \frac{dF_V}{dv} = f_Y(-v)$$

(cont'd)

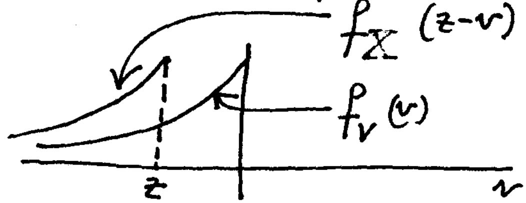
3.13 cont'd

Therefore $f_V(v) = \alpha e^{\alpha v} u(-v)$

$$f_Z(z) = \int_{-\infty}^{\infty} f_V(v) f_X(z-v)\, dv$$

$$= \int_{-\infty}^{z} e^{2\alpha v} dv \cdot e^{-\alpha z} \alpha^2 \quad \text{for } z<0$$

$$= \frac{\alpha}{2} e^{\alpha z}.$$



For $z>0$

$$f_Z(z) = \int_{-\infty}^{0} e^{2\alpha v} dv \; e^{-\alpha z} \alpha^2$$

$$= \frac{\alpha}{2} e^{-\alpha z}$$

Hence $f_Z(z) = \dfrac{\alpha}{2} e^{-\alpha |z|}$ all $|z|<\infty$.

3.27

a) $Z = X + Y$ where $f_X(x) = \frac{1}{2}\,\mathrm{rect}\left(\frac{x}{2}\right)$

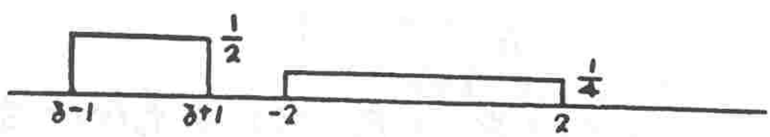
$f_Y(y) = \frac{1}{4}\,\mathrm{rect}\left(\frac{y}{4}\right)$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-\xi)\, f_Y(\xi)\, d\xi$$

Note: $f_X(z-\xi) = \frac{1}{2}$ $\quad |z-\xi| < 1 \Rightarrow f_X = \frac{1}{2}$ only for $\underline{z-1 < \xi < z+1}$

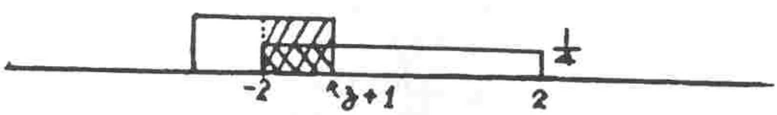
<u>region 1:</u> $z+1 < -2$ or $z < -3$

$f_Z(z) = 0$ (no overlap)



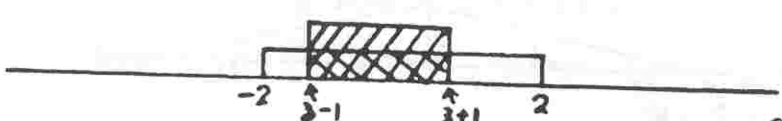
<u>region 2:</u> $-3 < z < -1$

$$f_Z(z) = \int_{-2}^{z+1} \frac{1}{8}\, d\xi = \frac{1}{8}(z+3)$$



<u>region 3:</u> $-2 < z-1$ but $z+1 < 2 \Rightarrow -1 < z < 1$

$$f_Z(z) = \int_{z-1}^{z+1} \frac{1}{8}\, d\xi = \frac{1}{4}$$



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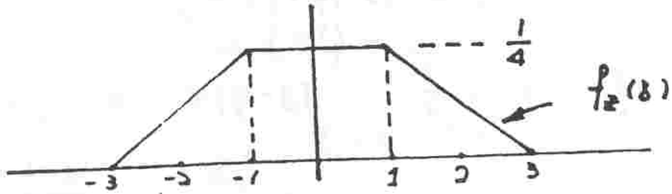
3.27 cont'd

region 4: $\quad z-1<2 \quad z+1>2 \Rightarrow 1<z<3$

$$f_Z(z) = \frac{1}{8}\int_{z-1}^{2} d\xi = (3-z)\frac{1}{8}$$

region 5: $\quad z>3 \qquad f_Z(z)=0$



Check: $\text{area} = 2\left[\frac{1}{2}(2)\frac{1}{4}\right] + 2\left[\frac{1}{4}\right] = \frac{1}{2}+\frac{1}{2} = 1$

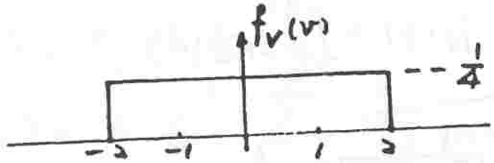
b) $Z = 2X - Y \qquad$ Let $V \triangleq 2X \qquad W \triangleq -Y$

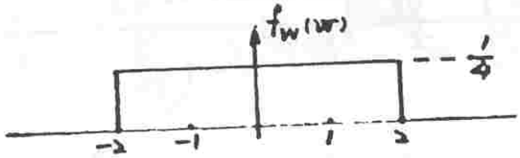
Then $f_Z(z) = \int_{-\infty}^{\infty} f_V(z-\xi)\, f_W(\xi)\, d\xi$

$$P(V\le v) = P(X \le v/2) = F_X(v/2) \Rightarrow f_V(v) = \frac{1}{2} f_X(v/2)$$

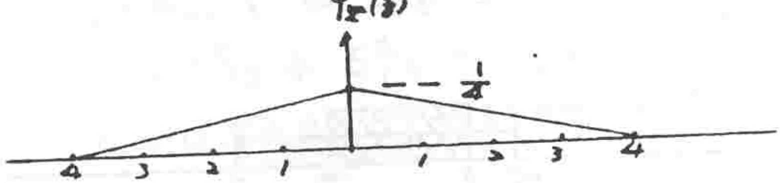
$$P(W\le w) = P(Y > -w) = 1 - F_Y(-w) \Rightarrow f_W(w) = f_Y(-w)$$

Hence





Now proceed as in (a) to obtain



3.28 $Z = X^2 + Y^2 \qquad W = X$

The real roots R_1, R_2 to $z = x^2 + y^2 \quad w = x$ are

R_1: $x = w$, $y = \sqrt{z - w^2} \qquad |w| < \sqrt{z} \qquad z > 0$

R_2: $x = w$, $y = -\sqrt{z - w^2}$

at both roots mag J is the same

$|J_1| = |J_2| = 2\sqrt{z^2 - w^2}$ where

$$J_{1,2} = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = 2y\Big|_{y = R_1, R_2}$$

Hence

$$f_{ZW}(z,w) = \frac{1}{2\sqrt{z - w^2}}\left[f_{XY}(w, \sqrt{z - w^2}) + f_{XY}(w, -\sqrt{z - w^2}) \right]$$

$$= \frac{1}{2\pi\sigma^2}\frac{1}{2\sqrt{z - w^2}} \exp[-z/2\sigma^2] \quad \begin{matrix} z > 0, \\ |w| < \sqrt{z} \end{matrix}$$

$$= 0 \quad , \text{ otherwise}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)\, dw = \frac{1}{2\sigma^2} e^{-z/2\sigma^2} u(z) \, .$$

3.29 Try the transformation (actually the inverse)

$$x = \hat{a}z + \hat{b}w$$
$$y = \hat{c}z + \hat{d}w$$

Then $x^2 + xy + y^2 = (\hat{a}z + \hat{b}w)^2 + (\hat{c}z + \hat{d}w)^2$
$+ (\hat{a}z + \hat{b}w)(\hat{c}z + \hat{d}w)$

We want the cross terms to vanish; this will allow $f_{zw} = f_z f_w$. The cross terms are

$$(2\hat{a}\hat{b} + 2\hat{d}\hat{c} + \hat{a}\hat{d} + \hat{b}\hat{c})zw = 0$$

choose $\hat{a} = \hat{c}$, $\hat{b} = -\hat{d}$. Then $x = \hat{a}z + \hat{b}w$

$y = \hat{a}z - \hat{b}w$

3.29 Cont'd

$x^2+y^2-2\rho xy = 3\hat{a}^2 z^2 + \hat{b}^2 w^2$. Hence

$$Q(x,y) = \frac{1}{2}\left\{\left[\frac{\hat{z}}{\sigma/2\hat{a}}\right]^2 + \left[\frac{\hat{w}}{\sigma\sqrt{3}/2\hat{b}}\right]^2\right\}$$

$$|J| = \mathrm{mag}\begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = \frac{1}{2|\hat{a}\hat{b}|}$$

where

$$g(x,y) = \frac{x+y}{2\hat{a}} \qquad h(x,y) = \frac{x-y}{2\hat{b}}$$

With $\sigma_1 \triangleq \sigma/2\hat{a}$ $\qquad \sigma_2 \triangleq \sqrt{3}\,\sigma/2\hat{b}$

Hence

$$f_{ZW}(z,w) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\frac{z^2}{\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2}\frac{w^2}{\sigma_2^2}}$$

$$= f_Z(z)\, f_W(w)$$

3.32 $z = x\cos\theta + y\sin\theta$ (1)

$w = x\sin\theta - y\cos\theta$ (2)

Multiply (1) by $\cos\theta$; (2) by $\sin\theta$ & add

$$x = z\cos\theta + w\sin\theta$$

Now, multiply (1) by $\sin\theta$; (2) by $\cos\theta$ & subtract

$$y = z\sin\theta - w\cos\theta$$

$$J = \begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{vmatrix} = -1 \qquad |J| = 1$$

only root is

$$x = z\cos\theta + w\sin\theta$$

$$y = z\sin\theta - w\cos\theta$$

$$f_{ZW}(z,w) = \frac{1}{2\pi} e^{-\frac{1}{2}[z^2\cos^2\theta + w^2\sin^2\theta + 2zw\cos\theta\sin\theta + z^2\sin^2\theta + w^2\cos^2\theta - 2zw\cos\theta\sin\theta]}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}[z^2+w^2]}$$

no change!

4.3 A binomial random variable representing n trials may be defined as the sum of n Bernoulli random variables, one for each trial : $X = \sum_{k=1}^{n} X_k$

Since the expected value of a Bernoulli r.v. is simply $E[X_k] = p.1 + (1 - p).0 = p$

the expected value of the binomial r.v. must be $E[X] = E[\sum_{k=0}^{n} X_k] = \sum_{k=0}^{n} E[X_k] = np$

4.11 $n_A = 3n_B \quad n_B = 2n_c \quad n_A + n_B + n_c = n$

$\therefore 6n_c + 2n_c + n_c = n \quad 9n_c = n \quad P(C) = 1/9$

$P(B) = 2P(C) = 2/9$ & $P(A) = 3P(B) = 6/9$

$$E(X) = E(X|A)P(A) + E(X|B)P(B) + E(X|C)P(C)$$

$$= \frac{6}{9}\left[\frac{1}{5}\int_0^\infty x e^{-x/5} dx\right] + \frac{2}{9}\left[\frac{1}{6.5}\int_0^\infty x e^{-x/6.5} dx\right]$$

$$+ \frac{1}{9}\left[\frac{1}{10}\int_0^\infty x e^{-x/10} dx\right]$$

$\simeq 5.9$ years

(4.12) $E(Y) = \int_{-\infty}^{\infty} E[Y \mid \theta]\, f(\theta)\, d\theta$

$E(Y \mid \theta) = \theta$ by inspection

$E(Y) = \frac{1}{2\pi} \int_0^{2\pi} \theta\, d\theta = \pi.$

4.16 $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2(1-\rho^2)^{1/2}} \exp\left[-\frac{x^2+y^2-2\rho xy}{2\sigma^2(1-\rho^2)}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right)$$

Y is Gaussian random variable with distribution $N(0, \sigma^2)$, so $E[Y] = 0$ *

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{f_{XY}(x,y)}{\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy}$$

$$= \frac{\frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[-\frac{x^2+y^2-2\rho xy}{2\sigma^2(1-\rho^2)}\right]}{\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{x^2}{2\sigma^2}\right]}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{1-\rho^2}} \exp\left[-\frac{(y-\rho x)^2}{2\sigma^2(1-\rho^2)}\right]$$

$$E[Y|X=x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\,dy$$

$$= \int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi}\,\sigma\sqrt{1-\rho^2}} \exp\left[-\frac{(y-\rho x)^2}{2\sigma^2(1-\rho^2)}\right] dy$$

$$= \rho X$$ *

This result shows that although $E(Y)=0$, the expected value of Y given $X = \rho X$. The best <u>predictor</u> Y_p of Y is $Y_p = \rho X$. Of course this result only makes sense if we can observe X.

4.18 a) $P(X=-1) = P(\xi=-1) = \frac{1}{5}$

$P(Y=1) = P(\xi=-1) + P(\xi=1) = \frac{2}{5}$

$P(X=-1, Y=1) = P(\xi=-1) = \frac{1}{5}$

$P(X=-1)P(Y=1) = \frac{1}{5}\cdot\frac{2}{5} = \frac{2}{25} \neq P(X=-1, Y=1)$

Hence X and Y are dependent r.v's.

b) $E[X] = \Sigma\, x\, P(x)$

$$= \sum_{i=1}^{5} \xi_i P(\xi_i) = \frac{1}{5}\left(-1 - \frac{1}{2} + 0 + \frac{1}{2} + 1\right)$$

$$= 0$$

$E[Y] = \Sigma\, y\, P(y)$

$$= \sum_{i=1}^{5} \xi_i^2 P(\xi) = \frac{1}{5}\left(1 + \frac{1}{4} + 0 + \frac{1}{4} + 1\right)$$

$$= \frac{1}{5}\cdot\frac{5}{2} = \frac{1}{2}$$

$E[XY] = \Sigma\, x y\, P(x, y)$

$$= \sum_{i=1}^{5} \xi^3 P(\xi) = \frac{1}{5}\left(-1 - \frac{1}{8} + 0 + \frac{1}{8} + 1\right)$$

$$= 0$$

$E[X]E[Y] = 0\cdot\frac{1}{2} = 0 = E[XY]$

Hence X and Y are uncorrelated r.v's.

4.28

$$\Theta_X(t) = \int_{-\infty}^{\infty} f_X(x) e^{tx} dx$$

$$= \lambda \int_0^{\infty} e^{-\lambda x} e^{tx} dx = \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx$$

(will converge if $Re(t) < \lambda$) $= \frac{\lambda}{\lambda - t}$

$$P[X \geq a] \leq e^{-at} \frac{\lambda}{\lambda - t} \triangleq h(t)$$

For a minimum $dh/dt = 0$.

$$\frac{dh}{dt} = \frac{\lambda}{\lambda - t}(-a) e^{-at} + e^{-at} \lambda (-) (\lambda - t)^{-2} (-) = 0$$

$\Rightarrow t = \lambda - \frac{1}{a}$ (after some trivial algebra)

Hence $$P[X \geq a] \leq e^{-a(\lambda - \frac{1}{a})} \frac{\lambda}{\lambda - (\lambda - \frac{1}{a})}$$

$$= a\lambda \exp[-a\lambda + 1]$$

So Chernoff bound is $a\lambda \exp[-a\lambda + 1]$.

4.34 $E[X_n^2] = K$

$$E[X_nX_{n-1}] = \overline{(bX_{n-1} + Z_n)(X_{n-1})} = bK$$

$$E[X_nX_{n-2}] = \overline{(bX_{n-1} + Z_n)(X_{n-2})} = \overline{\{b[bX_{n-2} + Z_{n-1}] + Z_n\}(X_{n-2})}$$

$$= b^2K$$

$$\vdots$$

$$E[X_nX_{n-N}] = b^{|N|}K \qquad |b| < 1$$

4.36 In general $P(Z=n)=\sum_{k=-\infty}^{\infty} P(X=k)\,P(Y=n-k)$

$$\theta_Z(t)=\sum_{n=-\infty}^{\infty} e^{tn} P(Z=n)=\sum_{k=-\infty}^{\infty} P(X=k)\sum_{n=-\infty}^{\infty} P(Y=n-k)\,e^{tn}$$

Let $m=n-k$; then

4.36 contd.

$$\theta_Z(t) = \sum_{k=-\infty}^{\infty} P_X[k] e^{tk} \cdot \sum_{m=-\infty}^{\infty} P_Y[m] e^{tm}$$

$$= \theta_X(t) \cdot \theta_Y(t)$$

$$\theta_X(t) = \sum_{n=-\infty}^{\infty} e^{tn} \frac{1}{n!} e^{-a} a^n = e^{-a[1-e^t]}$$

Thus if X: Poisson(a); Y: Poisson(b)

then

$$\theta_Z(t) = e^{-a[1-e^{-t}]} \cdot e^{-b[1-e^{-t}]}$$

$$= e^{-(a+b)[1-e^{-t}]}.$$

This implies that $f_Z(z) = $ Poisson$(a+b)$

$$f_Z(z) = \frac{1}{5!} e^{-5} 5^n \quad n = 0, 1, 2, \ldots$$

for $a=2$ and $b=3$.