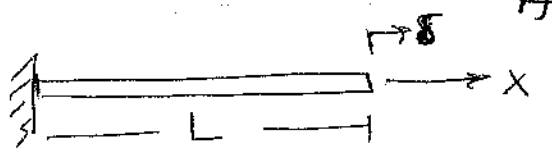
HW#1 Solutions

1.9)



Strain $\varepsilon_x = 1+2x^2$ Find tip displacement

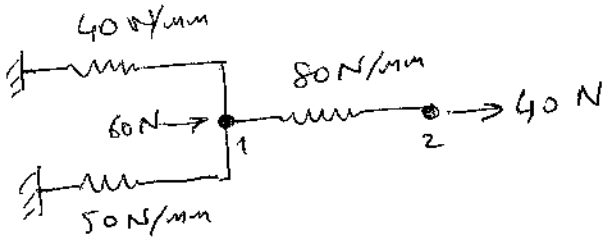
$$\varepsilon_x = \frac{du}{dx} \Rightarrow u = \int \varepsilon_x dx = \int (1+2x^2)\,dx$$

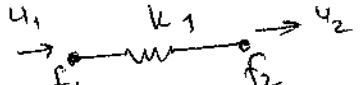
$$u = x + \frac{2}{3}x^3 + C$$

At $x=0 \Rightarrow u=0 \Rightarrow C=0 \Rightarrow u = x + \frac{2}{3}x^3$

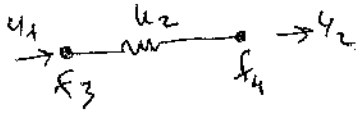
$$\text{at } x=L \quad u_L = L + \frac{2}{3}L^3$$

1.10)



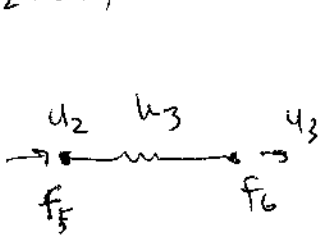


$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$



$$\begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} f_1+f_3 \\ f_2+f_4 \end{Bmatrix} = \begin{bmatrix} k_1+k_2 & -(k_1+k_2) \\ -(k_1+k_2) & k_1+k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$



$$\begin{Bmatrix} f_5 \\ f_6 \end{Bmatrix} = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} f_1+f_3 \\ f_2+f_4 \\ f_6 \end{Bmatrix} = \begin{bmatrix} k_1+k_2 & -(k_1+k_2) & 0 \\ -(k_1+k_2) & k_1+k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} f_1+f_3 \\ 60 \\ 40 \end{Bmatrix} = \begin{bmatrix} 40+50 & -(40+50) & 0 \\ -(40+50) & 40+50+80 & -80 \\ 0 & -80 & 80 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

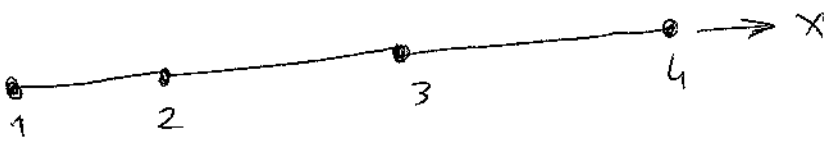
$$f_1+f_3 = -90u_2$$

$$\left.\begin{aligned} 60 &= 170u_2 - 80u_3 \\ 40 &= -80u_2 + 80u_3 \end{aligned}\right\} \quad 100 = 90u_2$$

$$u_2 = \frac{100}{90} = 1.11 \text{ mm}$$

$$60 = 170\frac{100}{90} - 80u_3 \Rightarrow u_3 = 1.61 \text{ mm}$$

3.3)



$$Q = [-0.2, 0, 0.6, -0.1] \qquad N_1 = 1 - \frac{x}{L} \quad N_2 = \frac{x}{L}$$

(i)

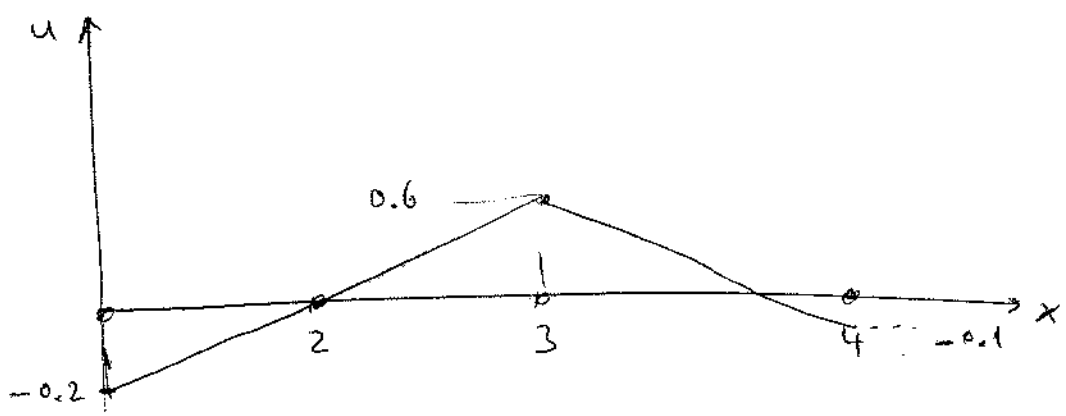
Between 1 & 2 $\quad u = N_1u_1 + N_2u_2$

$$u = \left(1 - \frac{x}{50}\right)(-0.2) = -0.2 + 0.004x$$

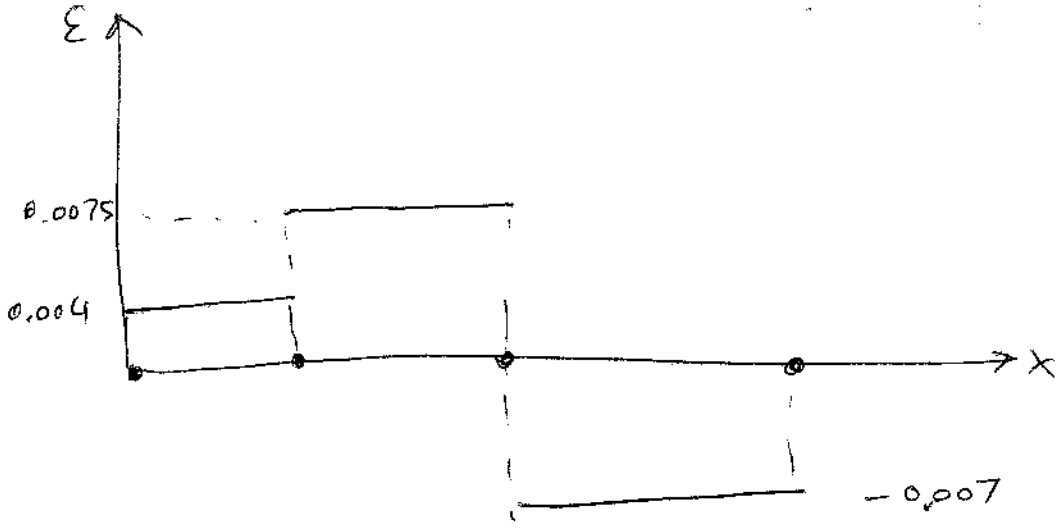
Between 2 & 3 $\quad u = \left(1 - \frac{x}{80}\right).0 + \frac{x}{80}(0.6) = 0.0075x$

3 & 4 $\quad u = \left(1 - \frac{x}{100}\right)0.6 + \frac{x}{100}(-0.1) = 0.6 - 0.007x$

(ii)



(ii) Strain $\varepsilon_x = \frac{du}{dx}$



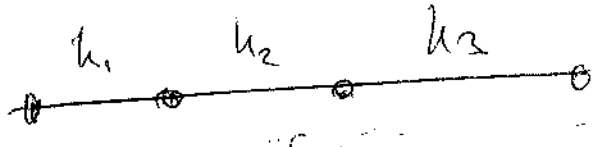
(iii) $[B] = \left[\frac{dN_1}{dx}, \frac{dN_2}{dx}\right]$

$N_1 = 1 - \frac{x}{L}$ $\quad \frac{dN_1}{dx} = -\frac{1}{L} = -\frac{1}{80}$

$N_2 = \frac{x}{L} = \frac{1}{L} = \frac{1}{80}$

$[B] = \left[-\frac{1}{80}, \frac{1}{80}\right]$

(iv) $U = \frac{1}{2} q^T k q$



$k_1 = \frac{E_1 A_1}{L_1} = \frac{1}{50}$

$k_2 = \frac{1}{80}$

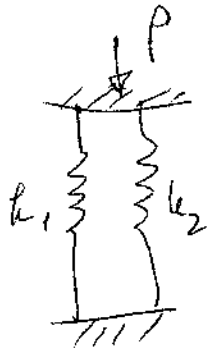
$k_3 = \frac{1}{80}$

Stiffness

$$k = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} = \begin{bmatrix} 0.02 & -0.02 & 0 & 0 \\ -0.02 & 0.0325 & -0.0125 & 0 \\ 0 & -0.0125 & 0.0225 & -0.01 \\ 0 & 0 & -0.01 & 0.01 \end{bmatrix}$$

$$U = \frac{1}{2}[-0.2, 0, 0.6, -0.1] \begin{bmatrix} 0.02 & -0.02 & 0 & 0 \\ -0.02 & 0.0325 & -0.0125 & 0 \\ 0 & -0.0125 & 0.0225 & -0.01 \\ 0 & 0 & -0.01 & 0.01 \end{bmatrix} \begin{bmatrix} -0.2 \\ 0 \\ 0.6 \\ -0.1 \end{bmatrix} = 0.0051 \text{ N/mm}$$

3.9)



Global stiffness

$$K = \begin{bmatrix} k_1+k_2 & -(k_1+k_2) \\ -(k_1+k_2) & k_1+k_2 \end{bmatrix}$$

$$k_1 = \frac{E_1 A_1}{L_1} = \frac{70\times10^3 \times 30\times60}{200} = 9\times 70\times10^3$$

$$k_2 = \frac{E_2 A_2}{L_2} = \frac{105\times10^3 \times 30\times60}{200} = 9\times 105\times10^3$$

$$385\ 10^3 \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = 10^3 \begin{bmatrix} 1575 & -1575 \\ -1575 & 1575 \end{bmatrix} \begin{Bmatrix} u_1 \\ 0 \end{Bmatrix}$$

$$385\times10^3 = 10^3 \times (1575\, u_1)$$

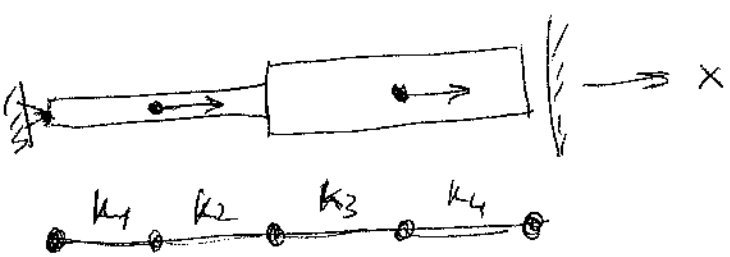
$$u_1 = 0.2444 \text{ mm}$$

$$\varepsilon = \frac{u_1}{L} = \frac{0.2444}{200} = 0.0012222$$

$$\sigma_1 = E_1 \varepsilon = 70\times10^3 \times 0.0012222 = 85.55 \text{ N/mm}^2$$

$$\sigma_2 = E_2 \varepsilon = 105\times10^3 \times 0.0012222 = 128.33 \text{ N/mm}^2$$

3.10)



Assemble the 4 element system.

$$K = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

$$k_1 = \frac{E A_1}{L_1} = \frac{200 \times 10^3 \times 250}{150}$$

$$k_1 = 333.33 \times 10^3$$

$$k_2 = k_1$$

$$k_3 = \frac{E A_2}{L_2} = \frac{200 \times 10^3 \times 400}{200}$$

$$k_3 = 400 \times 10^3$$

$$k_4 = k_3$$

$$\begin{Bmatrix} f_1 \\ 300 \times 10^3 \\ 0 \\ 600 \times 10^3 \\ f_R \end{Bmatrix} = 10^3 \begin{bmatrix} 333.33 & -333.33 & 0 & 0 & 0 \\ -333.33 & 666.66 & -333.3 & 0 & 0 \\ 0 & -333 & 733.33 & -400 & 0 \\ 0 & 0 & -400 & 800 & -400 \\ 0 & 0 & 0 & -400 & 400 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ u_4 \\ 3.5 \end{Bmatrix}$$

5 eq'ns 5 unknowns

$$f_1 = -672.727\ 10^3\ N$$

$$u_2 = 2.01818\ mm$$

$$u_3 = 3.13637\ mm$$

$$u_4 = 4.06818\ mm$$

$$f_R = -227.273 \times 10^3\ N$$

Nodal displacements

Element stresses

Element ①

$$\varepsilon = \frac{\Delta L}{L} = \frac{2.01818}{150} = 0.0134$$

$$\sigma = E\varepsilon = 200 \times 10^3 \times \varepsilon = 2691\ N/mm^2$$

Element ② $\varepsilon = \frac{3.13 - 2.018}{150} = 0.007$

$\sigma = 1491\ N/mm^2$

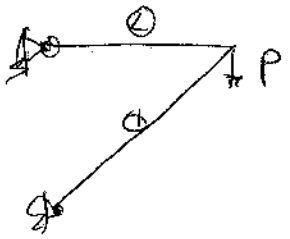
Element ③ $\varepsilon = \frac{4.06 - 3.13}{200} = 0.004$

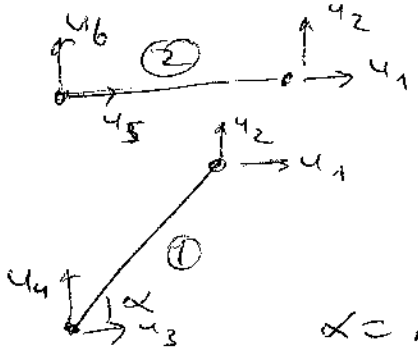
$\sigma = 932\ N/mm^2$

element ④ $\varepsilon = \frac{4.06 - 3.5}{200} = 0.002 \Rightarrow \sigma = 568\ N/mm^2$

HW #2 Solutions

4.3)





$\alpha = \text{Arctan}\frac{500}{750}$

$\alpha = 33.7°$

For element ① stiffness

$$[K]^1 = \frac{E_1 A_1}{L_1}\begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha & -\cos^2\alpha & -\cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha & -\cos\alpha\sin\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\cos\alpha\sin\alpha & \cos^2\alpha & \cos\alpha\sin\alpha \\ -\cos\alpha\sin\alpha & -\sin^2\alpha & \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix}$$

$\alpha = 33.7°$ $\quad E = 200\times10^3$ MPa

$A_1 = 1250$

$L_1 = 901.388$

$$[K]^1 = \frac{200\times10^3\times1250}{901.388}\begin{bmatrix} 0.6923 & 0.4615 & -0.6923 & -0.4615 \\ 0.4615 & 0.3076 & -0.4615 & -0.3076 \\ -0.6923 & -0.4615 & 0.6923 & 0.4615 \\ -0.4615 & -0.3076 & 0.4615 & 0.3076 \end{bmatrix}$$

For element ②, stiffness becomes

$\alpha = 0°$, $A_2 = 1000$ $\quad L_2 = 750$

$$[K]^2 = \frac{200\times10^3\times1000}{750}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Expand the matrices

$$[K]^1 = \begin{bmatrix} x & x & x & x & 0 & 0 \\ x & x & x & x & 0 & 0 \\ x & x & x & x & 0 & 0 \\ x & x & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

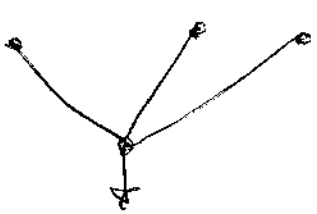
Displacement field

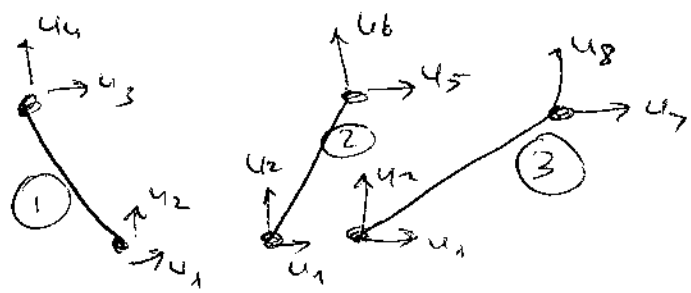
$$[K]^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

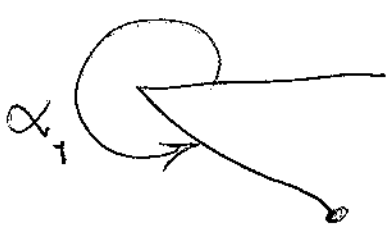
Add $[K]^1$ and $[K]^2$ to find $[K]^{global}$

$$[K]^{global} = \begin{bmatrix} 458678 & 128008 & -192012 & -128008 & -266667 & 0 \\ 128008 & 85338 & -128008 & -85338 & 0 & 0 \\ -192012 & -128008 & 192012 & 128008 & 0 & 0 \\ -128008 & -85338 & 128008 & 85338 & 0 & 0 \\ -266667 & 0 & 0 & 0 & 266667 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.7)







For element ① determine stiffness matrix

$$\alpha_1 = 2\pi - \text{Arctan}(600/450)$$

$$\alpha_1 = 5.35589 \text{ rad}$$

$$[K]^1 = \frac{EA_1}{L_1}\begin{bmatrix} 0.36 & -0.48 & -0.36 & 0.48 \\ & 0.64 & 0.48 & -0.64 \\ & & 0.36 & -0.48 \\ & & & 0.64 \end{bmatrix}$$

For element ②

$$\alpha_2 = \pi + \text{Arctan}(600/450)$$

$$\alpha_2 = 4.06889 \text{ rad}$$

$$[K]^2 = \frac{EA_2}{L_2}\begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ & 0.64 & -0.48 & -0.64 \\ & & 0.36 & 0.48 \\ & & & 0.64 \end{bmatrix}$$

For element ③

$$\alpha_3 = \pi + \text{Arctan}(600/800) = 3.78509 \text{ rad}$$

$$[K]^3 = \frac{EA_3}{L_3}\begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ & 0.36 & -0.48 & -0.36 \\ & & 0.64 & 0.48 \\ & & & 0.36 \end{bmatrix}$$

$E = 200 \times 10^3$ MPa $\quad A_1 = A_2 = A_3 = 250 \text{ mm}^2$

$$L_1 = L_2 = \sqrt{450^2 + 600^2} \qquad L_3 = \sqrt{800^2 + 600^2}$$

Expand element stiffness matrices and add them together to find global stiffness

$$[K]^{global} = \begin{bmatrix} 80000 & 24000 & -24000 & 32000 & -24000 & -32000 & -38000 & -24000 \\ & 103333 & 32000 & -42666.7 & -32000 & -42666.7 & -24000 & -18000 \\ & & 24000 & -32000 & 0 & 0 & 0 & 0 \\ & & & 42666.7 & 0 & 0 & 0 & 0 \\ & & & & 24000 & 32000 & 0 & 0 \\ & & & & & 42666.7 & 0 & 0 \\ & & & & & & 32000 & 24000 \\ & & & & & & & 18000 \end{bmatrix}$$

$$\{f\} = [K]\{u\}$$

$$\begin{Bmatrix} 0 \\ -18000 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{Bmatrix} = \begin{bmatrix} & & & & & & & \\ & & & K^{global} & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

8 eqns, 8 unknowns

Solving

$u_1 = 0.0561722$ mm

$u_2 = -0.187241$ mm

$f_3 = -7339.83$ N

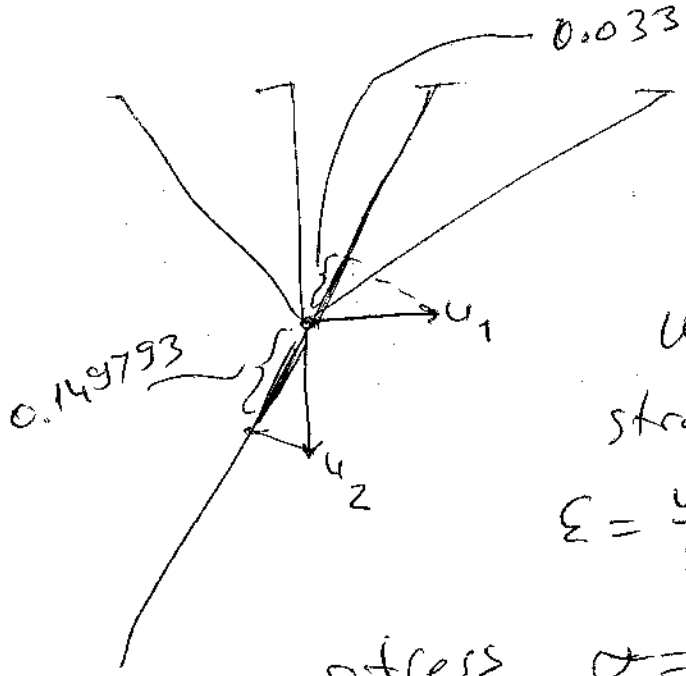
$f_4 = 9786.45$ N

$f_5 = -1342.14$ N

$f_6 = 6191.47$ N

$f_7 = 2696.26$ N

$f_8 = 2022.2$ N



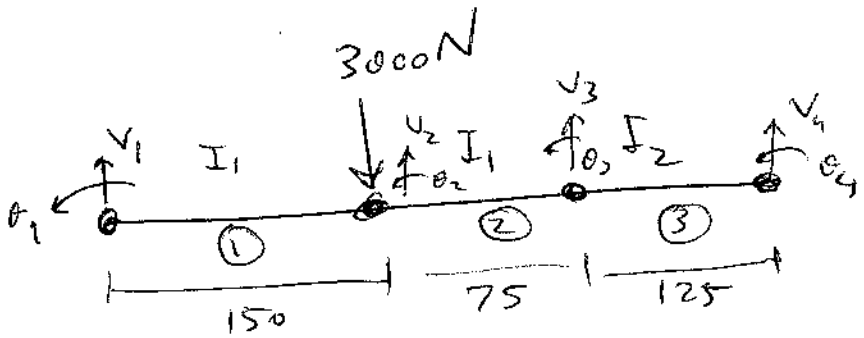
$u_{net} = 0.11609$

strain

$$\varepsilon = \frac{u}{L} = 0.000154787$$

stress $\sigma = E\varepsilon = 30.95$ MPa

8.1)



element ①

$$[K]^1 = \frac{EI_1}{L_1^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & & & 4L^2 \end{bmatrix}$$

$E = 200 \times 10^3$ MPa

$I_1 = 1.25 \times 10^5$ mm^4 $\quad L_1 = 150$ mm

$I_2 = 4 \times 10^6$ mm^4 $\quad L_2 = 75$ mm

$L_3 = 125$ mm

$$[K]^1 = \begin{bmatrix} 88888.9 & 6.66667\times10^6 & -88888.9 & 6.66667\times10^6 \\ & 6.66667\times10^8 & -6.66667\times10^6 & 3.3333\times10^8 \\ & & 88888.9 & -6.66667\times10^6 \\ & & & 6.66667\times10^8 \end{bmatrix}$$

$$[K]^2 = \begin{bmatrix} 711.111 & 2.66667\times10^{7} & -711111 & 2.66667\times10^{7} \\ & 1.3333\times10^{9} & -2.66667\times10^{7} & 6.66667\times10^{8} \\ & & 711111 & -2.66667\times10^{7} \\ & & & 1.33333\times10^{9} \end{bmatrix}$$

And for element (3)

$$[K]^3 = \begin{bmatrix} 49152 & 3072000 & -49152 & 3072000 \\ & 256\times10^{6} & -3072000 & 128\times10^{6} \\ & & 49152 & -3072000 \\ & & & 256\times10^{6} \end{bmatrix}$$

Expand them in the following form

$$[K]^1 = \begin{bmatrix} x & x & x & x & o & o & o & o \\ x & x & x & x & o & o & o & o \\ x & x & x & x & o & o & o & o \\ x & x & x & x & o & o & o & o \\ o & o & o & o & o & o & o & o \\ o & o & o & o & o & o & o & o \\ o & o & o & o & o & o & o & o \\ o & o & o & o & o & o & o & o \end{bmatrix}$$

$$[K]^2 = \begin{bmatrix} o & o & o & o & o & o & o & o \\ o & o & o & o & o & o & o & o \\ o & o & x & x & x & x & o & o \\ o & o & x & x & x & x & o & o \\ o & o & x & x & x & x & o & o \\ o & o & x & x & x & x & o & o \\ o & o & o & o & o & o & o & o \\ o & o & o & o & o & o & o & o \end{bmatrix}$$

$$[K]^{global} = \begin{bmatrix} 88888.9 & 6.66667\times10^{6} & -8888.9 & 6.66667\times10^{6} & 0 & 0 & 0 & 0 \\ & 6.66667\times10^{8} & -6.66667\times10^{6} & 3.33333\times10^{8} & 0 & 0 & 0 & 0 \\ & & 800000 & 2\times10^{7} & -711111 & 2.66667\times10^{7} & 0 & 0 \\ & & & 2\times10^{9} & -2.66667\times10^{7} & 6.66667\times10^{7} & 0 & 0 \\ & & & & 760263 & -2.35947\times10^{7} & -49152 & 3.072\times10^{3} \\ & & & & & 1.58933\times10^{9} & -3.072\times10^{3} & 128\times10^{6} \\ & & & & & & 49152 & -3.072\times10^{6} \\ & & & & & & & 256\times10^{6} \end{bmatrix}$$

(Symm)

Add them together

$$[K]^{3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & 0 & 0 & 0 & 0 \\ x & x & x & x & 0 & 0 & 0 & 0 \\ x & x & x & x & 0 & 0 & 0 & 0 \\ x & x & x & x & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\{f\} = [K]\{u\}$$

$$\begin{Bmatrix} f_1 \\ 0 \\ -300 \\ 0 \\ 0 \\ 0 \\ f_4 \\ 0 \end{Bmatrix} = \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & & K^{global} & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ 0 \\ \theta_4 \end{Bmatrix}$$

Solving yields

$f_1 = 171.429$ N

$f_2 = 128.571$ N

$\theta_1 = -0.000114614$ rad

$v_2 = -0.013335$ mm

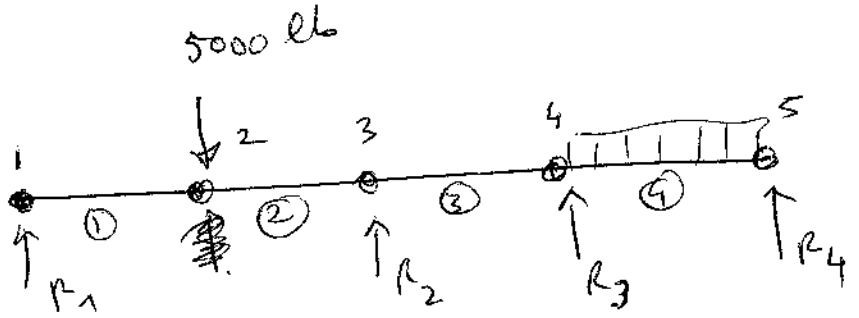
$\theta_2 = -0.000374713$ rad

$v_3 = -0.0136141$ mm

$\theta_3 = 0.0000252073$ rad

$\theta_4 = 0.000150765$ rad

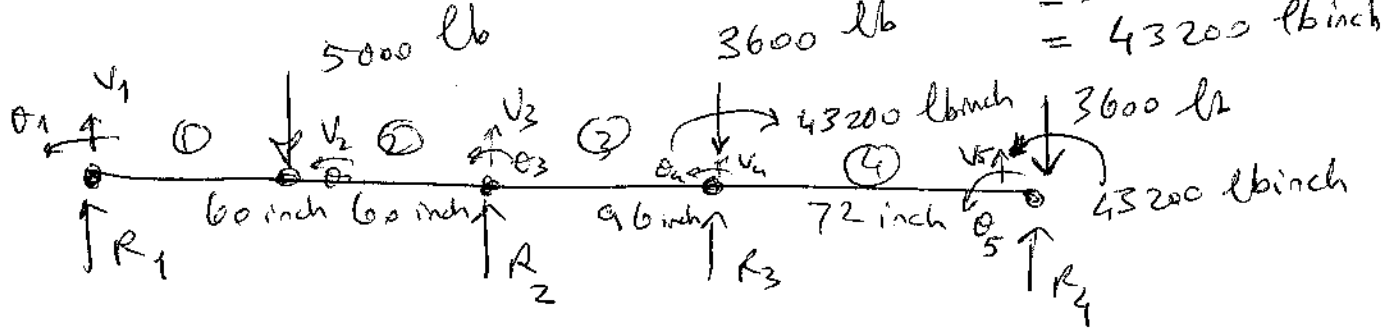
8.2)



Find equivalent nodal loads

$$\{f\}_{eq} = \int_0^6 \frac{-1200}{L^3} \begin{Bmatrix} L^3 - 3LX^2 + 2X^3 \\ L^2X - 2LX^2 + LX^3 \\ 3LX^2 - 2X^3 \\ -L^2X^2 + LX^3 \end{Bmatrix} dx = \begin{Bmatrix} -3600 \\ -3600 \\ -3600 \\ 3600 \end{Bmatrix}$$

3600 lbf
$= 3600 \times 12$
$= 43200$ lbinch



$E = 30 \times 10^6$ psi
$I = 305$ in^4

$$[K]^1 = \begin{bmatrix} 508333.33 & 15.25\times10^6 & -508333.33 & 15.25\times10^6 \\ & 610\times10^6 & -15.25\times10^6 & 305\times10^6 \\ & & 508333.33 & -15.25\times10^6 \\ & & & 610\times10^6 \end{bmatrix}$$

$$[K]^2 = \begin{bmatrix} 508333.33 & 15.25\times10^6 & -508333.33 & 15.25\times10^6 \\ & 610\times10^6 & -15.25\times10^6 & 305\times10^6 \\ & & 508333.33 & -15.25\times10^6 \\ & & & 610\times10^6 \end{bmatrix}$$

$$[K]^3 = \begin{bmatrix} 124105 & 5.95703\times10^6 & -124105 & 5.95703\times10^6 \\ & 3.8125\times10^8 & -5.95703\times10^6 & 1.90625\times10^8 \\ & & 124105 & -5.95703\times10^6 \\ & & & 3.8125\times10^8 \end{bmatrix}$$

$$[K]^4 = \begin{bmatrix} 294174 & 1.05903\times10^7 & -294174 & 1.05903\times10^7 \\ & 5.08333\times10^8 & -1.05903\times10^7 & 2.54167\times10^8 \\ & & 294174 & -1.0593\times10^7 \\ & & & 5.08333\times10^8 \end{bmatrix}$$

Assembly is obtained in a similar manner.

Displacement vector

$$\begin{Bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \\ V_3 \\ \theta_3 \\ V_4 \\ \theta_4 \\ V_5 \\ \theta_5 \end{Bmatrix}$$

B.C's

$V_1 = 0$

$V_3 = 0$

$V_4 = 0$

$V_5 = 0$

$$
[K]^{global} = \begin{bmatrix}
508333 & 1.525\times10^{7} & -508333 & 1.525\times10^{7} & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 6.1\times10^{8} & -1.525\times10^{7} & 3.05\times10^{8} & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1.01667\times10^{6} & 0 & -508333 & 1.525\times10^{7} & 0 & 0 & 0 & 0 \\
 & & & 1.22\times10^{9} & -1.525\times10^{7} & 3.05\times10^{8} & 0 & 0 & 0 & 0 \\
 & & & & 632438 & -9.29297\times10^{6} & -12405 & 5.95703\times10^{6} & 0 & 0 \\
 & & & & & 9.9125\times10^{8} & -5.95703\times10^{6} & 1.90625\times10^{8} & 0 & 0 \\
 & & & & & & 418279 & 4.63328\times10^{6} & -294174 & 1.05903\times10^{7} \\
 & & & & & & & 8.89583\times10^{8} & -1.05903\times10^{7} & 2.54167\times10^{8} \\
 & & & & & & & & 294174 & -1.05903\times10^{7} \\
 & & & & & & & & & 5.08333\times10^{8}
\end{bmatrix}
$$

(symm)

$$\{F\} = [K]\{u\}$$

$$\begin{Bmatrix} R_1 \\ 0 \\ -5000 \\ 0 \\ R_2 \\ 0 \\ R_3 - 3600 \\ -43200 \\ R_4 - 3600 \\ 43200 \end{Bmatrix} = \begin{bmatrix} & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & K_{global} & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_1 \\ v_2 \\ \theta_2 \\ 0 \\ \theta_3 \\ 0 \\ \theta_4 \\ 0 \\ \theta_5 \end{Bmatrix}$$

$R_1 = 1998.77$ lb

$R_2 = 3517.49$ lb

$R_3 = 3230.77$ lb

$R_4 = 3452.97$ lb

$\theta_1 = -0.000360337$ rad

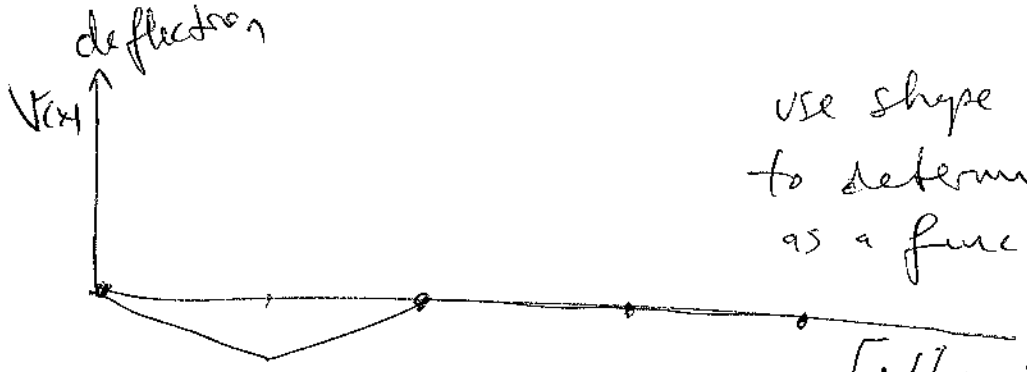
$v_2 = -0.013756$ inch

$\theta_2 = 0.0000328675$ rad

$\theta_3 = 0.000228864$ rad

$\theta_4 = -0.000142199$ rad

$\theta_5 = 0.000156083$ rad



use shape functions to determine $v(x)$ as a function of x

$\{u(x)\} = [N]\{u\}_{nodal}$

Shape functions $N_1 = \frac{1}{L^3}(L^3 - 3Lx^2 + 2x^3)$

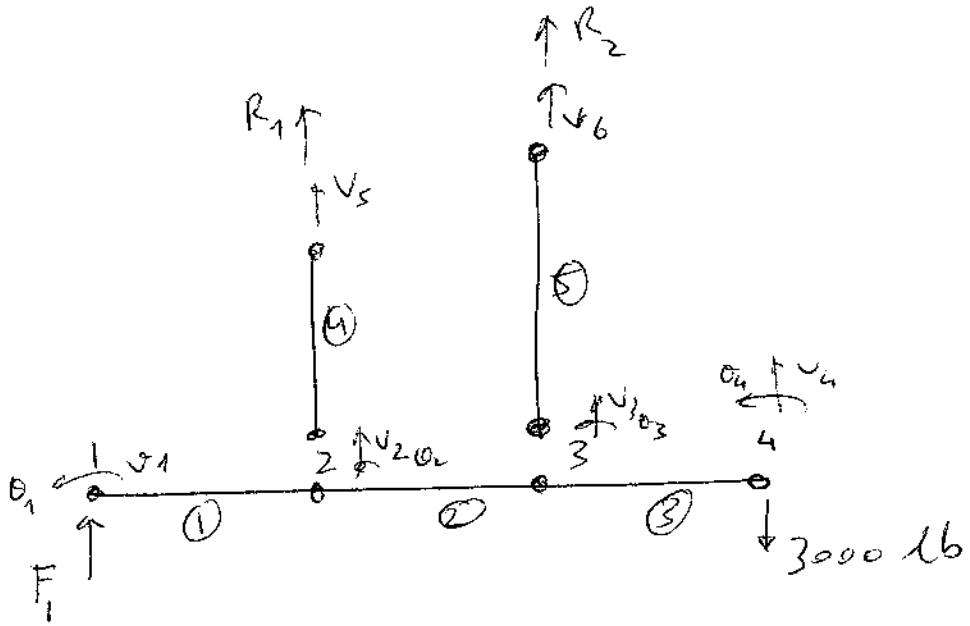
$N_3 = \frac{1}{L^3}(3Lx^2 - 2x^3)$

$v(x) = N_1 v_1 + N_2 v_2$

$L = 60$ inch $\quad v = \frac{1}{60^3}(60^3 - 180x^2 + 2x^3)\cdot 0 + \frac{1}{60^3}(180x^2 - 2x^3)(-0.013756)$

$v(x) = -0.000011463\,3x^2 + 1.2737\times10^{-7}x^3$ - - etc

8.5)



Element ①

$$[K]^1 = \begin{bmatrix} 3.6\times10^6 & 7.2\times10^6 & -3.6\times10^6 & 7.2\times10^6 \\ & 1.92\times10^7 & -7.2\times10^6 & 9.6\times10^6 \\ & & 3.6\times10^6 & -7.2\times10^6 \\ & & & 1.92\times10^7 \end{bmatrix}$$

$[K]^2 = [K]^1$

$$[K]^3 = \begin{bmatrix} 1.06667\times10^6 & 3.2\times10^6 & -1.06667\times10^6 & 3.2\times10^6 \\ & 1.28\times10^7 & -3.2\times10^6 & 6.4\times10^6 \\ & & 1.06667\times10^6 & -3.2\times10^6 \\ & & & 1.28\times10^7 \end{bmatrix}$$

$$[K]^4 = \begin{bmatrix} 2\times10^5 & -2\times10^5 \\ & 2\times10^5 \end{bmatrix} \qquad [K]^5 = \begin{bmatrix} 1.2\times10^5 & -1.2\times10^5 \\ & 1.2\times10^5 \end{bmatrix}$$

Embed $[K]^4$ as

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\times10^5 & 0 & 0 & 0 & 0 & 0 & -2\times10^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 & 0 \\ & & & & & & & 0 & 0 & 0 \\ & & & & & & & & 2\times10^5 & 0 \\ & & & & & & & & & 0 \end{bmatrix}$$

$[K]^5$ similarly done

Note that displacement vector

$$\begin{bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \\ V_3 \\ \theta_3 \\ V_4 \\ \theta_4 \\ V_5 \\ V_6 \end{bmatrix}$$

B.C's

$V_1 = 0$

$V_5 = 0$

$V_6 = 0$

$$
[K]^g = \begin{pmatrix}
3.6\times10^6 & 7.2\times10^6 & -3.6\times10^6 & 7.2\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1.92\times10^7 & -7.2\times10^6 & 9.6\times10^6 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 7.4\times10^6 & 0 & -3.6\times10^6 & 7.2\times10^6 & 0 & 0 & -2\times10^5 & 0 \\
 & & & 3.84\times10^7 & -7.2\times10^6 & 9.6\times10^6 & 0 & 0 & 0 & 0 \\
 & & & & 4.78667\times10^6 & -4.10^6 & -1.06667\times10^6 & 3.2\times10^6 & 0 & -120\times10^3 \\
 & & & & & 3.2\times10^7 & -3.2\times10^6 & 6.4\times10^6 & 0 & 0 \\
 & & & & & & 1.06667\times10^6 & -3.2\times10^6 & 0 & 0 \\
 & & & & & & & 1.28\times10^7 & 0 & 0 \\
 & & & & & & & & 2\times10^5 & 0 \\
 & & & & & & & & & 12\times10^4
\end{pmatrix}
$$

$$
\begin{Bmatrix} F_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -3000 \\ 0 \\ R_1 \\ R_2 \end{Bmatrix} = [K] \begin{Bmatrix} 0 \\ \theta_1 \\ V_2 \\ \theta_2 \\ V_3 \\ \theta_3 \\ V_4 \\ \theta_4 \\ 0 \\ 0 \end{Bmatrix}
$$

$F_1 = -3436.36$ lb

$R_1 = -2372.73$ lb

$R_2 = -4063.64$ lb

$\theta_1 = -0.00248864$ rad

$V_2 = -0.0118636$ inch

$\theta_2 = -0.00392045$

$V_3 = -0.0338636$ inch

$\theta_3 = -0.00722727$ rad

$V_4 = -0.0884773$ inch

$\theta_4 = -0.0100398$ rad

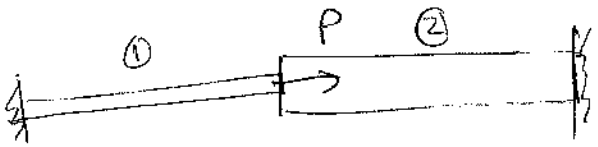
Stresses

$$\sigma_1 = E\frac{v_2}{\ell_1} = E\ \frac{0.0118636}{12} = 29659 \text{ psi}$$

$$\sigma_2 = E\frac{v_3}{\ell_2} = E\ \frac{0.0338636}{20} = 507954 \text{ psi}$$

HW #3 SOLUTIONS

1.13



$u_1 = a_1 + a_2 x$

$u_2 = a_3 + a_4 x$

$$U^1 = \frac{1}{2}\int_0^{200} E_1 A_1 (u_1')^2 dx = \frac{1}{2}\int_0^{200} E_1 A_1 a_2^2 dx = 100 E_1 A_1 a_2^2$$

$$U^2 = \frac{1}{2}\int_{200}^{500} E_2 A_2 (u_2')^2 dx = \frac{1}{2}\int_{200}^{500} E_2 A_2 a_4^2 dx = 150 E_2 A_2 a_4^2$$

$u_1\big|_{x=0} = 0 \Rightarrow a_1 = 0 \qquad u_1 = a_2 x$

$u_1\big|_{x=200} = u_2\big|_{x=200} \Rightarrow 200 a_2 = a_3 + 200 a_4$

$\Rightarrow a_3 = 200(a_2 - a_4)$

External work $W^1 = P u_1\big|_{x=200} = 200 P a_2$.

$$W^2 = P\,u_2\Big|_{x=200} = P(a_3 + a_4 x)\Big|_{x=200} = P(a_3 + 200\,a_4)$$

$$a_3 = -500\,a_4$$

$$W^2 = P(-500\,a_4 + 200\,a_4) = -300\,a_4 P$$

$$\Pi = U^1 + U^2 - W^1 - W^2$$

$$\Pi = 100\,E_1 A_1 a_2^2 + 150\,E_2 A_2 a_4^2 - 200\,P\,a_2 + 300\,P\,a_4$$

$$\left.\begin{aligned} \frac{\partial \Pi}{\partial a_2} &= 200\,E_1 A_1 a_2 - 200P = 0 \\ \frac{\partial \Pi}{\partial a_4} &= 300\,E_2 A_2 a_4 + 300P = 0 \end{aligned}\right\} \Longrightarrow \quad a_2 = \frac{P}{E_1 A_1}$$

$$a_2 = \frac{10000}{70000 \times 900}$$

$$a_2 = 0.0015873$$

$$a_4 = \frac{-P}{E_2 A_2} = \frac{-10000}{200000 \times 1200} = -0.0000416667$$

$$\Rightarrow a_3 = -500\,a_4 = 0.0400794$$

$$u_1 = 0.0015873\,x \qquad 0 \le x \le 200$$

$$u_2 = 0.0400794 - 0.0000416667\,x \qquad 200 \le x \le 500$$

Strength of materials, elementary solution

$$EA\,u_1'' = 0$$

$$u_1 = a + bx$$

$$u_1|_{x=0} = 0 \Rightarrow a = 0$$

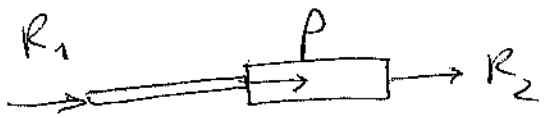
$$u_1 = bx$$

$$EA\,u_2'' = 0$$

$$u_2 = cx + d$$

$$u_2|_{x=500} \Rightarrow 0 = 500c + d$$

$$u_2 = c(x - 500)$$



Continuity of displacements

$$u_1\Big|_{X=200} = u_2\Big|_{X=200}$$

$$200b = c(200-500)$$

$$2b = -3c$$

Force equilibrium $P = -R_1 - R_2 \quad (\Sigma F_x = 0)$

$$P = -E_1 A_1 \underbrace{u_1'}_{\text{strain}} - E A_2 \underbrace{u_2'}_{\text{strain}}$$

$$P = -E_1 A_1 b - E_2 A_2 c$$

$$P = -E_1 A_1 \left(-\frac{3c}{2}\right) - E_2 A_2 c$$

$$\Rightarrow \quad c = -0.0000687285$$

$$b = 0.0001030 93$$

$$u_1 = 0.0001030 93\, X$$

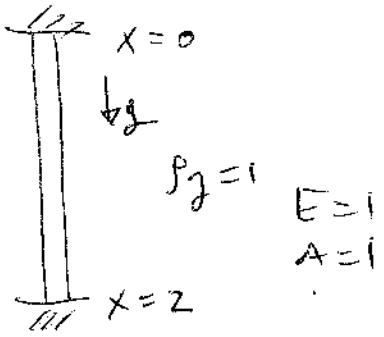
$$u_2 = -0.0000\ 687285\,(X-500)$$

$$u_1' = \varepsilon_1 = 0.0001030 93$$

$$\sigma_1 = E\varepsilon_1 = 7.21649 \text{ MPa}$$

$$\sigma_2 = 13.7457 \text{ MPa}$$

1.14



Differential equation

$$EAu'' = \rho g A = \rho g$$

($A=1$)

propose a solution

$$u^{app} = cx^2$$

Diff. eq'n $\quad EAu'' - \rho g = 0$

$$EA(2c) - \rho g = 0$$

Galerkin

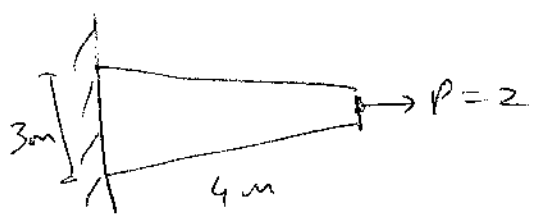
$$\int_0^2 (2EAc - \rho g)\, x^2 dx = 0$$

$$c = \frac{\rho g}{2EA} = \frac{\rho g}{2} = \frac{1}{2}$$

$$u^{app} = \frac{x^2}{2}$$

at midpoint $x=1$ $\quad u^{app} = 0.5$

3.12



Area distribution

$$A = 0.6 - 0.1x \qquad 0 \le x \le 4$$

$$u = a_0 + a_1 x + a_2 x^2 \qquad u' = a_1 + 2a_2 x$$

a) Use Rayleigh-Ritz

$$\pi = \frac{1}{2}\int_0^4 EAu'^2 dx - \int fu\,dx - Pu\Big|_{x=4}$$

$f = Ax^2$ body force

$$f = (0.6 - 0.1x)x^2$$

$$\Pi = \frac{1}{2} E \int_0^4 (0.6-0.1x)(a_1+2a_2x)^2\,dx - \int_0^4 (0.6-0.1x)x^2(a_0+a_1x+a_2x^2)\,dx - \underline{6.4a_0+17.92a_1+54.61a_2}$$

$$1.6a_1^2+10.6667\,a_1a_2+25.6a_2^2 \qquad -P(a_0+a_1(4)+a_2(4^2))$$

$$\Pi = \frac{1}{2}50\left(1.6a_1^2+10.6667\,a_1a_2+25.6a_2^2\right)-2(a_0+4a_1+16a_2)$$
$$-6.4a_0-17.92a_1-54.6133a_2$$

$$\frac{\partial\Pi}{\partial a_0}=0 \quad\Rightarrow\quad a_0=0 \qquad \text{Indeed } u\Big|_{x=0}=0 \Rightarrow a_0=0$$

$$\frac{\partial\Pi}{\partial a_1} = -25.92+25(3.2a_1+10.6667a_2)=0$$

$$\frac{\partial\Pi}{\partial a_2} = -86.6133+25(10.6667a_1+51.2a_2)=0$$

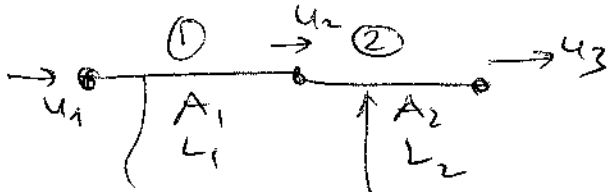
$$a_1 = 6.322184$$
$$a_2 = 0.000544687$$

$$u = 0.322184x+0.000544687x^2$$

Strain $\varepsilon = u' = 0.322184+0.00108937x$

Stress $\sigma = E\varepsilon = 50\varepsilon = 16.1092+0.0544687x$

b) FEM solution



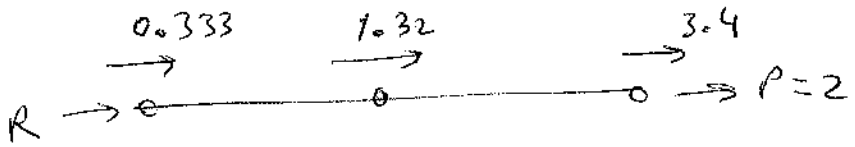
$A_{av}=0.5$ $A_{av}=0.3$ (average area)

equivalent nodal load $f=Ax^2$

$$\{f\} = \int_0^2 \begin{bmatrix} 1-0.5x \\ 0.5x \end{bmatrix} \{0.5x^2\}\,dx = \begin{Bmatrix} 0.333 \\ 1 \end{Bmatrix}$$

element ②

$$\{F\} = \int_2^4 \begin{Bmatrix} 1-0.5\xi \\ 0.5\xi \end{Bmatrix} \{0.3(\xi+2)^2\} \, d\xi = \begin{bmatrix} 0.32 \\ 3.4 \end{bmatrix}$$



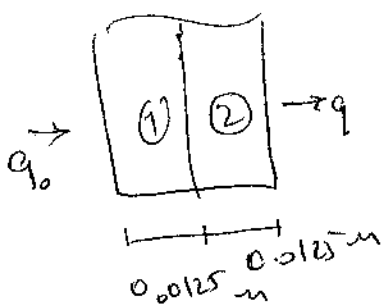
$$\begin{Bmatrix} R+0.333 \\ 1.32 \\ 5.4 \end{Bmatrix} = \begin{bmatrix} K \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$R = -7.05333$$

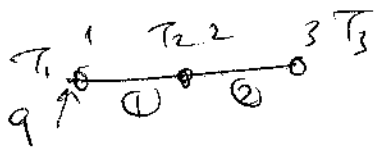
$$u_2 = 0.5376$$

$$u_3 = 1.2576$$

10.2)



$k = 1 \text{ W/m}^2\text{C}$



Stiffness $[K]^1 = \frac{k_1 A_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1 \times 1}{0.0125} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 80 & -80 \\ -80 & 80 \end{bmatrix}$

For element ② $[K]^2 = \frac{k_2 A_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1 \times 1}{0.0125} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 80 & -80 \\ -80 & 80 \end{bmatrix}$

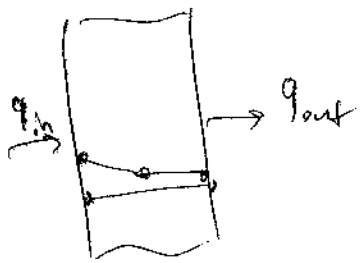
Assembly $[K]^{\text{global}} = \begin{bmatrix} 80 & -80 & 0 \\ -80 & 160 & -80 \\ 0 & -80 & 80 \end{bmatrix}$

$$\begin{bmatrix} 80 & -80 & 0 \\ -80 & 160 & -80 \\ 0 & -80 & 80 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ 10 \end{Bmatrix} = \begin{Bmatrix} 300 \\ 0 \\ q_b \end{Bmatrix}$$

Solving $T_1 = 17.5\ ^\circ C$

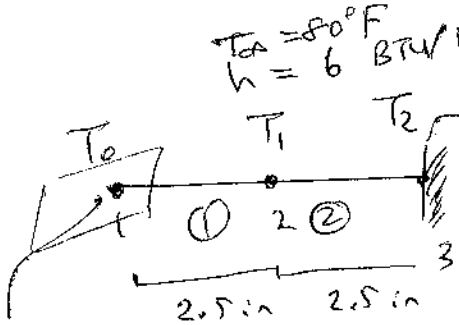
$T_2 = 13.75\ ^\circ C$

$q_b = -300\ W/m^2$



Thin plate useful for fin design

10.4)



$d = \frac{5}{16}$ in $= 0.0260417$ ft

$A = \frac{\pi d^2}{4} =$

$A = 0.000532632\ ft^2$

$L_1 = L_2 = 0.208333$ ft

$K = 24.8$ BTU/h ft °F

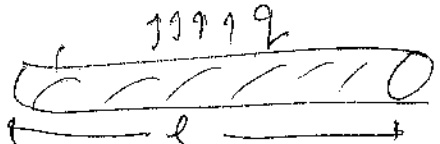
$$[K]^1 = \frac{K_1 A_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{24.8 \times 0.000532632}{0.208333} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$[K]^2 = [K]^1$

Assembly

$$[K] = \begin{bmatrix} 0.0634046 & -0.0634046 & 0 \\ -0.0634046 & 1.126809 & -0.0634046 \\ 0 & -0.0634046 & 0.0634046 \end{bmatrix}$$

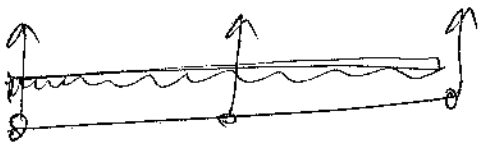
Load vector



Heat flows through outer face of cylinder

area $A = 2\pi r l = 2\pi \frac{0.0260417}{2} \times (2 \times 0.208333)$

$= 0.0340885\ ft^2$



Total loss $= hAT_\infty$
$= 6A80^\circ = 16.362568$

Find equivalent loses at nodes (local vector)

$$f_{eq} = \int_0^L \begin{bmatrix} 1-\frac{x}{L} \\ x/L \end{bmatrix} \{16.3625\}\, dx = \begin{Bmatrix} 4.09061 \\ 4.09061 \end{Bmatrix}$$

Similarly for element ② $\begin{Bmatrix} 4.09061 \\ 4.09061 \end{Bmatrix}$

equivalent local $\{f_e\} = \begin{Bmatrix} 4.09061 \\ 8.18123 \\ 4.09061 \end{Bmatrix}$

B.C's $\{f_b\} = \begin{Bmatrix} Aq_1 \\ 0 \\ 0 \end{Bmatrix}$ — insulated

$$[K]\{T\} = \{f_e\} + \{f_b\}$$

$T_1 = 150^\circ F$ is given

$$\begin{pmatrix} 1 & 0 & 0 \\ -0.0634046 & 1.126809 & -0.0634046 \\ 0 & -0.0634046 & +0.0634046 \end{pmatrix} \begin{Bmatrix} 150 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 150+Aq_1 \\ 8.18123 \\ 4.09061 \end{Bmatrix}$$

$$T_2 = 99^\circ F$$
$$T_3 = 89^\circ F$$