PROJECT # 3

The one dimensional Euler equation is given by

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{1}$$

where

$$Q = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} \qquad F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix} \tag{2}$$

In here, ρ is the density, u is the velocity, E is the total energy per unit volume

$$E = \rho e + \frac{1}{2}\rho u^2 \tag{3}$$

and e is the internal energy per unit mass. The pressure term p is given by

$$p = (\gamma - 1)\left[E - \frac{1}{2}\rho u^2\right] \tag{4}$$

where $\gamma = 1.4$.

Consider one dimensional tube with $0 \leq x \leq 10$ and the following the initial conditions at $t = 0$:

$$\text{If} \quad x \leq 5 \begin{cases} \rho_1 = 1 \\ p_1 = 1 \\ u_1 = 0 \end{cases} \qquad \text{Else if} \begin{cases} \rho_2 = 0.125 \\ p_2 = 0.1 \\ u_2 = 0 \end{cases} \tag{5}$$

For the given shock tube problem given above obtain the solution of state variables at $t = 1.2$ using the following methods for the evaluation of the inviscid fluxes:

1. van Leer flux splitting method
2. AUSM flux splitting method
3. Roe method

For the present calculations use $\Delta x = 0.1$ and $\Delta x = 0.01$ with $\Delta t = 0.001$. Compare the numerical results with the analytical solution.

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One Dimensional Euler Equations: Quasi-Linear Form

$$\frac{\partial Q}{\partial t} + A\frac{\partial Q}{\partial x} = 0 \tag{6}$$

The Jacobian matrix A is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u & \gamma-1 \\ u\left[\frac{1}{2}(\gamma-1)u^2 - H\right] & H-(\gamma-1)u^2 & \gamma u \end{bmatrix} \tag{7}$$

where the total specific enthalpy is

$$H = \frac{E+p}{\rho} \tag{8}$$

The singular value decomposition of the A matrix is

$$A = T\Lambda T^{-1} \tag{9}$$

The eigenvalues are

$$\hat{\Lambda} = \begin{bmatrix} \hat{u}-\hat{c} & 0 & 0 \\ 0 & \hat{u} & 0 \\ 0 & 0 & \hat{u}+\hat{c} \end{bmatrix} \tag{10}$$

The right eigenvectors are

$$\hat{T} = \left[\begin{array}{c|c|c} 1 & 1 & 1 \\ \hat{u}-\hat{a} & \hat{u} & \hat{u}+\hat{a} \\ \hat{H}-\hat{a}\hat{u} & \frac{1}{2}\hat{u}^2 & \hat{H}+\hat{a}\hat{u} \end{array}\right] \tag{11}$$

The inverse of the right eigenvector is

$$\hat{T}^{-1} = \begin{bmatrix} \frac{\hat{u}}{4\hat{a}}(2+(\gamma-1)\frac{\hat{u}}{\hat{a}}) & -\frac{1}{2\hat{a}}(1+(\gamma-1)\frac{\hat{u}}{\hat{a}}) & \frac{\gamma-1}{2\hat{a}^2} \\ 1-\frac{\gamma-1}{2}\frac{\hat{u}^2}{\hat{a}^2} & (\gamma-1)\frac{\hat{u}}{\hat{a}^2} & -(\gamma-1)\frac{1}{\hat{a}^2} \\ -\frac{\hat{u}}{4\hat{a}}(2-(\gamma-1)\frac{\hat{u}}{\hat{a}}) & \frac{1}{2\hat{a}}(1-(\gamma-1)\frac{\hat{u}}{\hat{a}}) & \frac{\gamma-1}{2\hat{a}^2} \end{bmatrix} \tag{12}$$

Van Leer Flux Splitting Method

$$F(Q_L, Q_R) = F^+(Q_L) + F^-(Q_R) \tag{13}$$

$$F^{\pm}_{i+1/2} = \begin{bmatrix} \beta_1 \\ \beta_1\beta_2\frac{1}{\gamma} \\ \beta_1\beta_2^2\frac{1}{2(\gamma^2-1}) \end{bmatrix} \tag{14}$$

$$\begin{aligned} \beta_1 &= \pm\frac{1}{4}\rho a(M \pm 1)^2 && (15) \\ \beta_2 &= (\gamma-1)Ma \pm 2a && (16) \end{aligned}$$

Advection Upstream Splitting Method (AUSM)

$$F_{i+1/2} = M_{1/2} \begin{bmatrix} \rho a \\ \rho a u \\ \rho a H \end{bmatrix}_{L/R} + \begin{bmatrix} 0 \\ p_{1/2} \\ 0 \end{bmatrix} = F_c + F_p \tag{17}$$

$$L/R = \begin{cases} L & M_{1/2} \geq 0 \\ R & \text{otherwise} \end{cases} \tag{18}$$

$$M_{1/2} = M_L^+ + M_R^- \tag{19}$$

$$p_{1/2} = p_L^+ + p_R^- \tag{20}$$

$$M^{\pm} = \begin{cases} \pm\frac{1}{4}(M \pm 1)^2 & |M| \leq 1 \\ \frac{1}{2}(M \pm |M|) & |M| > 1 \end{cases} \tag{21}$$

$$p^{\pm} = \begin{cases} \frac{p}{4}(M \pm 1)^2(2 \mp M) & |M| \leq 1 \\ \frac{p}{2M}(M \pm |M|) & |M| > 1 \end{cases} \tag{22}$$

Roe Method

$$F(Q_L, Q_R) = \frac{1}{2}\left[F(Q_L) + F(Q_R) - \hat{T}|\hat{\Lambda}|\hat{T}^{-1}(Q_R - Q_L)\right] \tag{23}$$

The eigenvalues

$$\hat{\Lambda} = \begin{bmatrix} \hat{u} & 0 & 0 \\ 0 & \hat{u}+\hat{c} & 0 \\ 0 & 0 & \hat{u}-\hat{c} \end{bmatrix} \tag{24}$$

The right eigenvector

$$\hat{T} = \left[\begin{array}{c|c|c} 1 & 1 & 1 \\ \hat{u} & \hat{u}+\hat{a} & \hat{u}-\hat{a} \\ 0.5\hat{u}^2 & \hat{H}+\hat{a}\hat{u} & \hat{H}-\hat{a}\hat{u}) \end{array}\right] \tag{25}$$

The transformed variables $W = T^{-1}Q$

$$\Delta W = \hat{T}^{-1}\Delta Q = \begin{bmatrix} -\frac{\Delta p}{2\hat{a}^2} + \Delta\rho \\ \frac{\Delta p}{2\hat{a}^2} + \frac{1}{2\hat{a}}\left[\Delta(\rho u) - \hat{u}\Delta\rho\right] \\ \frac{\Delta p}{2\hat{a}^2} - \frac{1}{2\hat{a}}\left[\Delta(\rho u) - \hat{u}\Delta\rho\right] \end{bmatrix} \tag{26}$$

$$\hat{\rho} = \sqrt{\rho_L \rho_R} \tag{27}$$

$$\hat{u} = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{28}$$

$$\hat{H} = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{29}$$

$$\hat{a} = \sqrt{(\gamma - 1)\left[\hat{H} - 0.5\hat{u}^2\right]} \tag{30}$$

References

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