Istanbul Technical University – Faculty of Aeronautics and Astronautics **14.10.2019**
2019-2020 Fall Term
12300 DNK201E Dynamics

Quiz – 1 - SOLUTION

Problem: A particle is moving in the horizontal plane along a curved path, which is given by the expressions $x(t)=t$ and $y(t)=t^2$, where x and y are in meters and t is in seconds. Find,

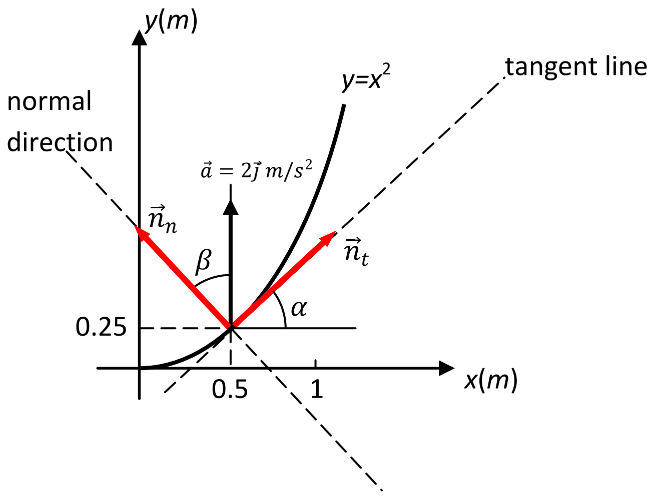
(a) the velocity vector as a function of time in Cartesian coordinates,
(b) the acceleration vector as a function of time in Cartesian coordinates,
(c) the tangent and normal components of the acceleration vector at the instant t=0.5 s,
(d) the radial and transverse components of the acceleration vector at the instant t=0.5 s, and
(e) the tangent and normal components of the velocity vector at the instant t=0.5 s.

Solution:

(a) $v_x = \dot{x} = \frac{dx}{dt} = 1\, m/s, \quad v_y = \dot{y} = \frac{dy}{dt} = 2t\, m/s \; \boxed{\rightarrow \vec{v} = v_x\vec{\imath} + v_y\vec{\jmath} = \vec{\imath} + 2t\vec{\jmath}\, m/s}$

(b) $a_x = \dot{v}_x = \ddot{x} = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = 0, \quad a_y = \dot{v}_y = \ddot{y} = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = 2\, m/s \; \boxed{\rightarrow \vec{a} = a_x\vec{\imath} + a_y\vec{\jmath} = 2\vec{\jmath}\, m/s^2}$

(c) Relation between the x and y coordinates of the particle can be found by eliminating time parameter in the y expression by substituting the value t=x for t: $y = x^2$. The path of the particle's motion and the tangent and normal directions at the instant t=0.5 s ()are given in the figure below:



The slope of the tangent line: $y'(x) = \frac{dy}{dx} = \frac{d}{dx}(x^2) = 2x \quad \rightarrow y'(x = 0.5) = (2)(0.5) = 1$

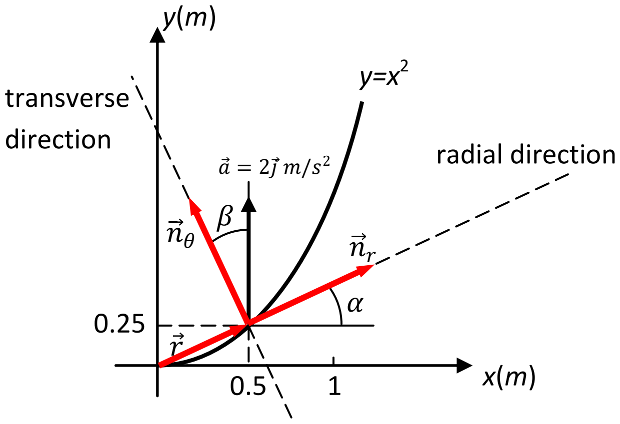
$\rightarrow \tan\alpha = 1 \quad \Rightarrow \quad \alpha = 45°, \; \beta = 45°$

Now the components of the acceleration vector in normal and tangential direction can be computed:

$$\boxed{a_t = |\vec{a}|\cos 45° = (2)\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} = 1.4142\, m/s^2}$$

$$\boxed{a_n = |\vec{a}|\cos 45° = (2)\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} = 1.4142\, m/s^2}$$

(d) In order to calculate the polar components of the acceleration vector the following diagram will be used:



The angular orientation of the position vector $\vec{r}$ can be easily calculated as shown below:

$\vec{r} = x\vec{\imath} + y\vec{\jmath}$ at t=0.5 s: $\vec{r} = 0.5 + 0.25\vec{\jmath}\, m$ $\tan\alpha = \frac{y}{x} = \frac{0.25}{0.5} = 0.5 \rightarrow \alpha = 26.56° = \beta$

The radial and transverse components of the acceleration vector become like this:

$$\boxed{a_r = |\vec{a}|\sin 26.56° = (2)(0.447) = 0.894\, m/s^2}$$

$$\boxed{a_\theta = |\vec{a}|\cos 26.56° = (2)(0.894) = 1.788\, m/s^2}$$

(e) Velocity vector is always tangent to the path of the particle therefore only tangent component of velocity vector will be non-zero; the normal component of velocity vector is always equal to zero.

$\vec{v}(t = 0.5) = \vec{\imath} + 2(0.5)\vec{\jmath} = \vec{\imath} + \vec{\jmath}\ \ m/s \rightarrow |\vec{v}| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}\ \ m/s$

Since normal component of the velocity vector is zero the magnitude of the tangential component must be equal to the magnitude of the velocity vector, therefore

$\boxed{v_t = \sqrt{2}\ \frac{m}{s} \text{ and } v_n = 0}$.

<u>Alternative solution of (c), and (d):</u>

- Tangential and normal components of acceleration vector:

$\vec{a} = a_t\vec{n}_t + a_n\vec{n}_n$

$$a_t = \frac{dv}{dt} = \dot{v}, \quad |\vec{v}| = v = \sqrt{{v_x}^2 + {v_y}^2} = \sqrt{(1)^2 + (2t)^2} = \sqrt{1 + 4t^2} \rightarrow \dot{v} = \frac{4t}{\sqrt{1 + 4t^2}}$$

$$\boxed{a_t(t = 0.5s) = \frac{(4)(0.5)}{\sqrt{1 + 4(0.5)^2}} = \sqrt{2}\ \ m/s^2}$$

$a_n = v^2/\rho$

radius of curvature of the path of the particle at instant t =0.5 s (x=0.5 m):

$$\rho = \frac{\left[1+\left(y'(x)\right)^2\right]^{3/2}}{|y''(x)|} = \frac{[1+(2x)^2]^{3/2}}{2} \rightarrow \rho(x=0.5\,m) = \frac{[1+(2\cdot 0.5)^2]^{3/2}}{2} = \sqrt{2}\,m$$

$$\boxed{a_n = \frac{\left(\sqrt{2}\right)^2}{\sqrt{2}} = \sqrt{2}\,m/s^2}$$

- Radial and transverse components of the acceleration vector at instant t =0.5 s:

$\vec{a} = a_r\vec{n}_r + a_\theta\vec{n}_\theta, \quad a_r = \ddot{r} - r\dot{\theta}^2, \quad a_r = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

To calculate the quantities above the radial distance of the particle to the origin and the angular orientation together with their first and second derivatives as a function of time must be known. Below these expressions are derived and their numerical values are calculated for *t*=0.5:

$r = \sqrt{x^2+y^2} = \sqrt{t^2+t^4} \quad \rightarrow \quad r(0.5s) = \sqrt{(0.5)^2+(0.5)^4} = 0.559\,m$

$$\dot{r} = \frac{t+2t^3}{\sqrt{t^2+t^4}} \quad \rightarrow \quad \dot{r}(0.5s) = \frac{0.5+2\cdot 0.5^3}{\sqrt{0.5^2+0.5^4}} = 1.341\,m/s$$

$$\ddot{r} = \frac{1+6t^2}{\sqrt{t^2+t^4}} - \frac{[t+2t^3]^2}{[t^2+t^4]^{3/2}} \quad \rightarrow \ddot{r}(0.5s) = \frac{1+6\cdot 0.5^2}{\sqrt{0.5^2+0.5^4}} - \frac{[0.5+2\cdot 0.5^3]^2}{[0.5^2+0.5^4]^{\frac{3}{2}}} = 1.253\,m/s^2$$

$\tan\theta = \frac{y}{x} = \frac{t^2}{t} = t \quad \rightarrow \theta = \tan^{-1} t \;\rightarrow\; \theta(0.5s) = \tan^{-1} 0.5 = 26.56\,rad$

$\dot{\theta} = \frac{1}{1+t^2} \;\rightarrow \dot{\theta}(0.5s) = \frac{1}{1+0.5^2} = 0.8\,rad/s$

$\ddot{\theta}(0.5s) = -\frac{2t}{(1+t^2)^2} \;\rightarrow\; \ddot{\theta}(0.5s) = -\frac{(2)(0.5)}{(1+0.5^2)^2} = -0.64\,rad/s^2$

$$a_r = \ddot{r} - r\dot{\theta}^2 \boxed{\rightarrow a_r(0.5s) = 1.253 - (0.559)(0.8)^2 = 0.895\,m/s^2}$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \;\boxed{\rightarrow \quad a_\theta(0.5s) = (0.559)(-0.64) + (2)(1.341)(0.8) = 1.787\;m/s^2}$$

As can be seen the analytical solution yields same results as the graphical solution.

Additional information: Curve length

It is possible to define a coordinate that is equal to the length of the path traced out by the particle choosing a point on the curve as the origin. Here, it will be assumed that the origin of the Cartesian coordinate system is also the point where the coordinate along the curve is equal to zero.

$$ds = [1 + (y'(x))^2]^{1/2} dx \quad \rightarrow \quad s = \int_0^x (1 + 4x^2)^{1/2}\, dx$$

After variable substitutions and some mathematical manipulations the arclength is found as follows:

$$s(x) = \frac{1}{4}\left[\sinh^{-1}(2x) + 2x(1 + 4x^2)^{1/2}\right]$$

Since x(t)=t, replacing x by t in the equation above s(t) is obtained, thus

$$s(t) = \tfrac{1}{4}\left[\sinh^{-1}(2t) + 2t(1 + 4t^2)^{1/2}\right].$$

Time rate of change of this last equations is equal to the speed of the particle. Indeed,

$$v = \frac{ds}{dt} = \frac{1}{4}\left[\frac{2}{\sqrt{1 + 4t^2}} + 2(1 + 4t^2)^{1/2} + \frac{8t^2}{\sqrt{1 + 4t^2}}\right] = \frac{1}{4}\left[\frac{4 + 16t^2}{\sqrt{1 + 4t^2}}\right] = \sqrt{1 + 4t^2}$$

is the speed of the particle at a specific instant t, which is equal to the magnitude of the velocity vector as was found in the alternative solution of question (c).