TEL502E – Homework 2

Due 08.03.2016

1. Consider a disk with an unknown radius r. We are interested in the area of the disk. For this, we measure the radius n times but each measurement contains some error. Specifically, suppose that the measurements are of the form $X_i = r + Z_i$ for $i = 1, 2, \ldots, n$, where Z_i's are independent zero-mean Gaussian random variables with known variance σ^2 (this is not a very good model for the error in this example but it is convenient to work with).

   (a) Find a sufficient statistic for r.

   (b) A professor suggests that we use

   $$\hat{A} = \pi \left(\frac{1}{n} \sum_{i=1}^{n} X_i^2 \right)$$

   as an estimator of the area. Determine if $\hat{A}$ is biased or not.

   (c) Find the UMVUE for the area of the disk.

   Solution. (a) Notice that $X_i \sim \mathcal{N}(r, \sigma^2)$. Thanks to independence, we find the joint pdf as,

   $$f_X(x_1, \ldots, x_n; r) = \frac{1}{(2\pi)^{n/2}\, \sigma^n} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - r)^2 \right).$$

   Observe that we can write this pdf as,

   $$f_X(t; r) = \left[\frac{1}{(2\pi)^{n/2}\, \sigma^n} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2 \right) \right] \left[\exp\left(-\frac{1}{2\sigma^2} \left(nr^2 + 2r \sum_{i=1}^{n} x_i \right) \right) \right].$$

   Thus by the factorization theorem, $T = \sum_i X_i$ is a sufficient statistic.

   (b) Notice that $\mathbb{E}(X_i^2) = \mathrm{var}(X_i) + \big(\mathbb{E}(X_i)\big)^2 = \sigma^2 + r^2$. Using this, we find $\mathbb{E}(\hat{A}) = \pi(\sigma^2 + r^2)$. Therefore, $\hat{A}$ is not a biased estimator of the area $A = \pi\, r^2$.

   (c) We found in part (a) that $T = \sum_i X_i$ is a sufficient statistic for r. If we set $A = \pi r^2$, it can also be shown that T is a sufficient statistic for A (using the factorization theorem). Assuming completeness, the Rao-Blackwell theorem suggests that the UMVUE is therefore a function of T. Consider T^2. Observe that

   $$\mathbb{E}(T^2) = \sum_{i=1}^{n} \mathbb{E}(X_i^2) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathbb{E}(X_i\, X_j) = n\sigma^2 + nr^2 + n\,(n-1)\, r^2 = n\sigma^2 + n^2 r^2.$$

   Therefore, $\tilde{A} = \pi\,(T^2 - n\sigma^2)/n^2$ is an unbiased estimator of the area which is a function of the sufficient statistic for the area. Therefore it must be the UMVUE we are looking for, by the Rao-Blackwell theorem.

2. Suppose X_1, X_2 are independent random variables distributed as $\mathcal{N}(0, \theta)$, $\mathcal{N}(0, 2\theta)$, where θ is an unknown positive constant.

   (a) Find a sufficient statistic for θ.

   (b) Find the UMVUE for θ.

   Solution. (a) The joint pdf is,

   $$f_X(x_1, x_2) = \frac{1}{2\sqrt{2}\,\pi\,\theta} \exp\left(-\frac{2X_1^2 + X_2^2}{4\theta} \right).$$

   Thus, $T = 2X_1^2 + X_2^2$ is a sufficient statistic, by the factorization theorem.

   (b) Observe that $\mathbb{E}(T) = 4\theta$. Thus, assuming that T is complete, $\hat{\theta} = T/4$ is the UMVUE by the Rao-Blackwell theorem.

3. Suppose X_1, X_2, X_3 are independent random variables and the pdf of X_k is given as,

$$f_k(t) = \begin{cases} \frac{1}{k\theta} \exp\left(-\frac{t}{k\theta}\right), & \text{if } 0 \le t, \\ 0, & \text{if } t \le 0, \end{cases}$$

for $k = 1, 2, 3$, where θ is a positive unknown.

(a) Find a sufficient statistic for θ and compute its expected value.

(b) Find a function of the sufficient statistic which is unbiased as an estimator of θ.

(Note : $\int_0^\infty t\, e^{-t}\, dt = 1$.)

Solution. (a) Note that the joint pdf is given as,

$$f_X(x;\theta) = [u(x_1)\, u(x_2)\, u(x_3)] \left[\frac{1}{6\theta^3} \exp\left(-\frac{1}{6\theta} (6x_1 + 3x_2 + 2x_3) \right) \right],$$

where u denotes the unit step function. Thus $T = (6x_1 + 3x_2 + 2x_3)$ is a sufficient statistic for θ. Observe that

$$\mathbb{E}(X_k) = \int_0^\infty \frac{x}{k\theta} \exp\left(-\frac{x}{k\theta}\right) dt = k\theta \int_0^\infty s \exp(-s)\, ds = k\theta.$$

Therefore, $\mathbb{E}(T) = 18\theta$.

(b) It follows by the previous discussion that $\hat{\theta} = \theta/18$ is such an estimator.