MAT 281E – Linear Algebra and Applications

Fall 2015

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Class Meets : 13.30 – 16.30, Friday
EEB 5301

Office Hours : 10.00 – 12.00, Monday

Textbook : G. Strang, 'Introduction to Linear Algebra', 4^{th} Edition, Wellesley Cambridge.

Grading : Homeworks (10%), 2 Midterms (25% each), Final (40%).

Webpage : http://ninova.itu.edu.tr/Ders/1039/Sinif/17825

Tentative Course Outline

- Solving Linear Equations via Elimination

  Linear system of equations, elimination, LU Decomposition, Inverses

- Vector Spaces

  The four fundamental subspaces, solving $Ax = b$, rank, dimension.

- Orthogonality

  Orthogonality, projection, least squares, Gram-Schmidt orthogonalization.

- Determinants

- Eigenvalues and Eigenvectors

  Eigenvalues, eigenvectors, diagonalization, application to difference equations, symmetric matrices, positive definite matrices, iterative splitting methods for solving linear systems, singular value decomposition.

MAT 281E – Homework 1

Due 09.10.2015

1. Consider an augmented matrix of the form

$$A = \begin{bmatrix} B & C \end{bmatrix},$$

where B and C are $n \times n$ matrices. Assume that B is an invertible matrix. Suppose that after some row operations on A, we obtain a new augmented matrix

$$A' = \begin{bmatrix} D & E \end{bmatrix},$$

Find an expression for E in terms of B, C and D.

Solution. Since A' is obtained from A by row operations, we can write $A' = RA$ for some matrix R. Observe that $D = FB$, where $F = DB^{-1}$. Thus, we have,

$$FA = \begin{bmatrix} FB & FC \end{bmatrix} = \begin{bmatrix} D & DB^{-1}C \end{bmatrix}.$$

Thus, we should have $E = DB^{-1}C$.

2. We are given the equation

$$x_1 \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 & 4 & 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 3 \end{bmatrix},$$

where x_1, x_2, x_3 are real numbers. Find x_1, x_2, x_3.

Solution. This is equivalent to solving

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}.$$

To obtain the solution, we form the augmented matrix and start elimination.

$$\begin{bmatrix} 1 & 2 & -1 & 5 \\ 2 & 4 & 3 & 0 \\ -1 & 1 & 0 & 3 \end{bmatrix} \xrightarrow{r_2 \leftarrow r_2 - 2r_1} \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 0 & 5 & -10 \\ -1 & 1 & 0 & 3 \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 + r_1} \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 0 & 5 & -10 \\ 0 & 3 & -1 & 8 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3}$$
$$\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 3 & -1 & 8 \\ 0 & 0 & 5 & -10 \end{bmatrix}$$

The coefficient matrix reached the upper triangular form so we can now stop elimination and start back-substituting. Note that the system we now have is,

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 5 \\ 3x_2 - x_3 &= 8 \\ 5x_3 &= -10 \end{aligned}$$

From the last equation, we obtain $x_3 = -2$. Substituting $x_3 = -2$ to the second equation, we find $x_2 = 2$. Finally, the first equation gives $x_1 = -1$.

3. Consider the linear system of equations

$$\begin{bmatrix} a & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ b \end{bmatrix}.$$

(a) Find a pair (a, b) so that the system has a unique solution.

(b) Find a pair (a, b) so that the system has infinitely many solutions.

(c) Find a pair (a, b) so that the system has no solutions.

Solution. Suppose we do elimination on the augmented matrix (assuming $a \neq 0$)

$$\begin{bmatrix} a & 2 & 1 \\ 4 & 3 & b \end{bmatrix} \xrightarrow{r_2 \leftarrow r_2 - (4/a) r_1} \begin{bmatrix} a & 2 & 1 \\ 0 & 3 - 8/a & b - 4/a \end{bmatrix}.$$

(a) As long as $3 \neq 8/a$, the system will have a unique solution. Take for instance $(a, b) = (1, 0)$.

(b) We obtain infinitely many solutions by setting the second equation to zero with the choice $(a, b) = (8/3, 3/2)$.

(c) To obtain an inconsistent system of solutions, we can set $(a, b) = (8/3, 1)$ − in this case, the second equation is $0 = -1/2$, which is not satisfied for any choice of x_1, x_2.

4. Suppose A is a 3×3 matrix whose rows are denoted by r_1, r_2, r_3, that is, $A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$. Also, let B be another 3×3 matrix given as

$$B = \begin{bmatrix} r_2 + 2\,r_3 \\ r_1 + r_2 \\ r_1 - 2\,r_2 \end{bmatrix}.$$

Suppose that for a specific vector c,

$$B \underbrace{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}_{c} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

Find $A\,c$.

Solution. Observe that we can express B in terms of A as,

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} A.$$

Thus, we have,

$$\underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix}}_{F} \underbrace{A\,c}_{z} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}}_{b}.$$

To obtain $A\,c = z$, we need to solve $F\,z = b$. Let us form the augmented matrix and do elimination.

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 3 \\ 1 & -2 & 0 & 6 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \\ 1 & -2 & 0 & 6 \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 - r_1} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & -3 & 0 & 3 \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 + 3r_2} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 6 & 6 \end{bmatrix}.$$

The new system of equations is,

$$\begin{aligned} z_1 + z_2 &= 3 \\ z_2 + 2z_3 &= 1 \\ 6z_3 &= 6 \end{aligned}$$

The last equation gives $z_3 = 1$. Plugging this in the second equation, we obtain $z_2 = -1$. Finally, plugging these in the first equation, we obtain $z_1 = 4$. Thus, $A\,c = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$.

MAT 281E – Homework 2

Due 16.10.2015

1. Find the LU decomposition of

$$A = \begin{bmatrix} -1 & 1 & -2 \\ -4 & 1 & -11 \\ -2 & -10 & -12 \end{bmatrix}.$$

Solution. Let us do elimination on A.

$$A \xrightarrow[E_1]{r_2 \leftarrow r_2 - 4r_1} \begin{bmatrix} -1 & 1 & -2 \\ 0 & -3 & -3 \\ -2 & -10 & -12 \end{bmatrix} \xrightarrow[E_2]{r_3 \leftarrow r_3 - 2r_1} \begin{bmatrix} -1 & 1 & -2 \\ 0 & -3 & -3 \\ 0 & -12 & -8 \end{bmatrix} \xrightarrow[E_3]{r_3 \leftarrow r_3 - 4r_2} \underbrace{\begin{bmatrix} -1 & 1 & -2 \\ 0 & -3 & -3 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

The elimination steps can be expressed in terms of matrices as, $E_3\, E_2\, E_1\, A = U$, where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}.$$

Observe also that these are easily inverted as,

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

Finally, notice that $A = L\,U$, for

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

Observe that L can be consrtucted by copying the non-zero diagonal entries of the matrices E_i^{-1} (please think about why – or see the book for a discussion on how L 'stores' the eliminaton steps).

2. Let A be a matrix given as

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 4 & 1 & 6 \end{bmatrix}.$$

Find a real number α, so that,

$$b = \begin{bmatrix} 4 \\ \alpha \\ 3 \end{bmatrix}$$

is in the column space of A.

Solution. In order for b to be in $C(A)$, we must be able to solve $A\,x = b$. Let us form the augmented matrix and do elimination

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 2 & 1 & 0 & \alpha \\ 4 & 1 & 6 & 3 \end{bmatrix} \xrightarrow{\substack{r_2 \leftarrow r_2 - 2r_1 \\ r_3 \leftarrow r_3 - 4r_1}} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -6 & \alpha - 8 \\ 0 & 1 & -6 & -13 \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 - r_2} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -6 & \alpha - 8 \\ 0 & 0 & 0 & -5 - \alpha \end{bmatrix}$$

Notice that the last equation is of the form $0 = -5 - \alpha$. Therefore, we must have $\alpha = -5$. Observe that with this choice we can find a solution.

3. Let V be the subspace of $\mathbb{R}^3$ spanned by the vectors v_1 and v_2, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Also, let S be the subspace of $\mathbb{R}^3$ spanned by

$$s_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Find a <u>non-zero</u> vector z that lies in both S and V.

Solution. We are asked to find α_1, α_2, β_1, β_2 such that

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

But we can rewrite this as

$$\underbrace{\begin{bmatrix} v_1 & v_2 & -s_1 & -s_2 \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus we need to find a non-zero vector from the null-space of A. Let us do elimination,

$$\begin{bmatrix} 1 & 1 & -2 & -1 \\ 1 & 1 & -1 & -2 \\ 1 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{r_2 \leftarrow r_2 - r_1} \begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 - r_1} \begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{r_3 \leftrightarrow r_2} \begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Here, there is only one free variable, namely β_2. Setting $\beta_2 = 1$, we find from the last row that $\beta_1 = 1$. Plugging these in the equation described by the second row, we find $\alpha_2 = 1$. Finally, the first row gives $\alpha_1 = 2$. Thus, the vector

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$$

lies in $S \cap V$.

4. Describe the null-space of

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & -2 & 1 & -1 \\ 2 & 4 & 3 & 1 \end{bmatrix}.$$

Solution. Let us do elimination on A.

$$A \xrightarrow{r_2 \leftarrow r_2 + r_1} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 3 & 1 \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 - 2r_1} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \underbrace{\begin{bmatrix} \boxed{1} & 2 & -1 & 1 \\ 0 & 0 & \boxed{5} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{R}.$$

In the last matrix, the pivots are framed. The columns that contain the pivots are the pivot columns and the other columns are called free columns. The variables that multiply the free columns (x_2 and x_4 in this case) are the free variables. In order to find the special solutions, we solve for the pivot variables x_1, x_3 under the choices $(x_2, x_4) = (1, 0)$ and $(x_2, x_4) = (0, 1)$. If we take $(x_2, x_4) = (1, 0)$, we must have $(x_1, x_3) = (-1/2, 0)$ in order for $Rx = 0$. Similarly, if $(x_2, x_4) = (0, 1)$, we must have $(x_1, x_3) = (-4/5, 1/5)$ in order for $Rx = 0$. Thus the two special solutions are,

$$s_1 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} -4/5 \\ 0 \\ 1/5 \\ 1 \end{bmatrix},$$

and $N(A)$ is the set of vectors of the form $\alpha_1 s_1 + \alpha_2 s_2$, where α_1, α_2 are real numbers.

MAT 281E – Homework 3

Due 23.10.2015

1. Consider the system of equations $A\,x = b$, where,

$$A = \begin{bmatrix} 1 & -1 & 3 & 0 & 1 \\ -1 & 1 & -2 & 1 & 0 \\ 2 & -2 & 5 & -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}.$$

(a) Find a basis for $N(A)$, the nullspace of A.

(b) Describe the solution set of $A\,x = b$.

Solution. We are asked to find the solution set in the question. So let us form the augmented matrix and do elimination.

$$\begin{bmatrix} 1 & -1 & 3 & 0 & 1 & -3 \\ -1 & 1 & -2 & 1 & 0 & 1 \\ 2 & -2 & 5 & -1 & 1 & -4 \end{bmatrix} \xrightarrow[r_3 \leftarrow r_3 - 2r_1]{r_2 \leftarrow r_2 + r_1} \begin{bmatrix} 1 & -1 & 3 & 0 & 1 & -3 \\ 0 & 0 & 1 & 1 & 1 & -2 \\ 0 & 0 & -1 & -1 & -1 & 2 \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 + r_1}$$
$$\begin{bmatrix} 1 & -1 & 3 & 0 & 1 & -3 \\ 0 & 0 & 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 \leftarrow r_1 - 3r_2} \begin{bmatrix} 1 & -1 & 0 & -3 & -2 & 3 \\ 0 & 0 & 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We reached the reduced row echelon form. Observe that the system of equations described by $Ax = b$ are equivalent to $C\,x = d$, where

$$C = \begin{bmatrix} 1 & -1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad d = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Note that here the first and third columns are pivot columns. Therefore the pivot variables are x_1 and x_3. x_2, x_4, x_5 are the free variables.

(a) There are three special solutions for $C\,x = 0$, which is equivalent to $A\,x = 0$. Setting $(x_2, x_4, x_5) = (1, 0, 0)$ and solving for the pivot variables in the equation, we obtain $(x_1, x_3) = (1, 0)$. Thus,

$$s_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

To obtain s_2 we set $(x_2, x_4, x_5) = (0, 1, 0)$ and solve for the pivot variables. To obtain s_3 we set $(x_2, x_4, x_5) = (0, 0, 1)$ and solve for the pivot variables. We obtain,

$$s_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

$N(A)$ consists of vectors that can be expressed as $\alpha_1\,s_1 + \alpha_2\,s_2 + \alpha_3\,s_3$, where $\alpha_i \in \mathbb{R}$.

(b) We need to find a particular solution x_p satisfying $C\,x_p = d$. For that set the free variables to zero and solve for the pivot variables. That gives

$$x_p = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}.$$

The solution set of $Ax = b$, consists of all vectors of the form $x_p + \alpha_1\,s_1 + \alpha_2\,s_2 + \alpha_3\,s_3$, where $\alpha_i \in \mathbb{R}$.

2. We know that a plane in $\mathbb{R}^3$ is determined by three points on it. Suppose

$$p_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

lie on a plane P. Find a matrix A and a vector b such that the solution set of $A\,x = b$ is P.

Solution. Note that a plane is described by the equation $c^T\,x = b$, for a vector $c \in \mathbb{R}^3$ and $b \in \mathbb{R}$. It suffices to find two vectors orthogonal to c (will be clear when we discuss dimension). Notice that $A\,p_i = b$ for $i = 1, 2, 3$. Thus $A\,p_3 - A\,p_j = A(p_3 - p_j) = 0$ for $j = 1, 2$. Thus, $s_j = (p_3 - p_j) \in N(A)$ for $j = 1, 2$. These vectors are

$$s_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}. \tag{1}$$

To find a c, solve $\begin{bmatrix} s_1^T \\ s_2^T \end{bmatrix} c = 0$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \xrightarrow{r_2 \leftarrow r_2 - r_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix}$$

Note that c_3 is the only free variable. So there's a single special solution obtained by setting $c_3 = 1$. For this choice, we obtain $c_1 = -1$, $c_2 = -2$. Thus, $c = \begin{bmatrix} -1 & -2 & 1 \end{bmatrix}^T$. For this c, observe that $c^T\,p_1 = -1$. Thus, $A = \begin{bmatrix} -1 & -2 & 1 \end{bmatrix}$, $b = -1$ works.

3. Recall that we say x is orthogonal to y if $\langle x, y \rangle = 0$. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

Also, let $N(A)$ denote the null-space of A. Find two non-zero vectors z_1 and z_2 such that if $x \in N(A)$, then x is orthogonal to both z_1 and z_2. Here, I also ask that z_1 and z_2 have different directions, that is $z_1 \neq \alpha z_2$ for any $\alpha \in \mathbb{R}$.

Solution. Suppose $x \in N(A)$. This means $A\,x = 0$. But this means that $\langle r_i, x \rangle = 0$ for $i = 1, 2$, where r_i denotes the i^{th} row of A. Thus the two rows of A can be taken as z_1, z_2.

4. Find a non-zero vector x that is orthogonal to every vector in the column space of A, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 4 & 4 \end{bmatrix}.$$

Solution. Observe that if $x^T\,A = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, then $x^T\,A\,\alpha = 0$, for any column vector $\alpha \in \mathbb{R}^3$. Since any $v \in C(A)$ can be expressed as $A\,\alpha$ for some $\alpha \in \mathbb{R}^3$, find such an x is sufficient for our purpose. Instead of solving $x^t\,A = 0$, we can equivalently consider $A^T\,x = 0$. That is, obtain a vector from the nullspace of A^T. Let us do elimination.

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 4 \\ 3 & 1 & 4 \end{bmatrix} \xrightarrow{r_2 \leftarrow r_2 - 2r_1} [r_3 \leftarrow r_3 - 3r_1] \begin{bmatrix} 1 & 3 & 4 \\ 0 & -4 & -4 \\ 0 & -8 & -8 \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 - 2r_2} \begin{bmatrix} 1 & 3 & 4 \\ 0 & -4 & -4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Setting the free variable $x_3 = 1$, we find the pivot variables as $x_2 = -1$, $x_1 = -1$. Thus $x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ works.

MAT 281E – Homework 4

Due 20.11.2015

1. Suppose A, B are matrices of the same size and $Ax = Bx$ for all x. Show that $A = B$.

   Solution. Note that the stated condition implies that $(A - B)x = 0$ for all x. Now let $C = A - B$. Suppose $C_{i,j} \neq 0$ for some i, j. Now let x be a vector such that $x_j = 1$ and $x_k = 0$ for $k \neq j$. Then, if $y = C\,x$, we have $y_i = C_{i,j} \neq 0$. But this contradicts the assumption that $C\,x = 0$. Thus $C_{i,j} = 0$ for any i, j.

2. Suppose S, U are symmetric matrices and $x^T Sx = x^T Ux$ for all x. Show that $S = U$.

   Solution. The given condition implies that $x^T(S - U)x = 0$ for all x. Let $C = S - U$. Observe that because $S^T = S$ and $U^T = U$, we also have $C^T = C$, i.e., C is symmetric. Suppose $C_{i,j} \neq 0$. By symmetry, we also have $C_{j,i} \neq 0$. Now let x be a vector such that $x_i = 1$, $x_j = 1$ and $x_k = 0$ for $k \neq j$, $k \neq i$. For this x, we have $x^T Cx = C_{i,j} + Cj, i = 2C_{i,j} \neq 0$. But this contradicts the assumption that $x^T C\,x = 0$. Thus $C_{i,j} = 0$ for any i, j.

3. The claim above is no longer valid if S or U is not symmetric. To see this, find two square matrices A, B with $A \neq B$ such that $x^T Ax = x^T Bx$ for all x.

   Solution. Consider

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

   Then, for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we have $x^T\,A\,x = x^T\,Bx = x_1\,x_2$, but $A \neq B$.

4. Consider the plane described by the equation $x + y + z = 0$. Find the closest point of this plane to $v = \big(1, 2, 3\big)$.

   Solution. Notice that this plane is the null space of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. Thus we need to project v onto $N(A)$. An alternative is to project v onto $N(A)^{\perp}$ and subtract it from $N(A)$. For this problem, this latter approach is easier because $N(A)^{\perp} = C(A^T)$ and since A has only one row, $C(A^T)$ is spanned by $x = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$. The projection matrix onto $N(A)$ is therefore given as, $P = I - x(x^T\,x)^{-1}x^T$. Computing Pv, we find $Pv = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$.

5. Consider the plane described by the equation $x + y + z = 1$. Find the closest point of this plane to $v = \big(1, 2, 3\big)$.

   Solution. This plane is not a subspace. We need a description of the solution set of the plane equation

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1.$$

   Observe that a particular solution is $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. Also, the special solutions are given as,

$$s_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

   Thus the solution set is given as, the set of points of the form

$$\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{p} + \underbrace{\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.$$

   Thus we need to solve

$$\min_{\alpha} \|(v - p) - A\alpha\|.$$

Letting $b = v - p$, we know that the solution of this problem also solves

$$A^T A\alpha^* = A^T b.$$

Notice that

$$A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad A^T b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Solving for α^* (via elimination), we find $\alpha^* = \begin{bmatrix} 1/3 & 4/3 \end{bmatrix}^T$. Thus the closest point of the plane to v is $p + A\,\alpha^* = \begin{bmatrix} -2/3 & 1/3 & 4/3 \end{bmatrix}^T$.

6. Consider the lines $l_1 = \big(x,\, 2x,\, x+3,\, -x\big)$, $l_2 = \big(1-y,\, -2y,\, -1-y,\, 2\big)$ in $\mathbb{R}^4$. Find two points $p \in l_1$, $q \in l_2$ that minimize $\|p - q\|$.

   Solution. Notice that we are trying to solve

$$\min_{x,y} \left\| \left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right) - \left(\begin{bmatrix} -1 \\ -2 \\ -1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \right) \right\|.$$

   This can be equivalently written as,

$$\min_{x,y} \left\| \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\alpha} - \underbrace{\begin{bmatrix} 1 \\ 0 \\ -4 \\ 2 \end{bmatrix}}_{b} \right\|.$$

   The minimizing α^* satisfies, $A^T A\alpha^* = A^T b$. Notice that

$$A^T A = \begin{bmatrix} 7 & 6 \\ 6 & 6 \end{bmatrix}, \quad A^T b = \begin{bmatrix} -5 \\ -3 \end{bmatrix}. \tag{1}$$

   Thus we find $x^* = -2$, $y^* = 3/2$. Therefore the closest points of l_1 and l_2 are, $\big(-2,\, -4,\, 1,\, 2\big) \in l_1$ and $\big(-1/2,\, -3,\, -5/2,\, 2\big) \in l_2$.

7. Suppose that $N(A^T)$, the left null-space of A, is two dimensional and the projection of $\big(1, 2, 3\big)$ to $N(A^T)$ is $\big(0, 1, 1\big)$. Find a basis for $C(A)$, the column space of A.

   Solution. Observe that $N(A^T)$ is a subspace of $\mathbb{R}^3$. Since it is two dimensional, its complement $C(A)$ must be one dimensional. Thus it is sufficient to find a vector from $C(A)$ to find a basis. Recall that if Pv is the projection of v onto a space S, then $v - Pv$ is the projection onto $S^\perp$. Since $C(A) = N(A^T)^\perp$, it follows that $(1, 2, 3) - (0, 1, 1) = (1, 1, 2)$ is in $C(A)$ and forms a basis for $C(A)$.

MAT 281E – Homework 5

Due 27.11.2015

1. Show that if q_1, q_2, ..., q_k are orthogonal non-zero vectors, they are also linearly independent.

   Solution. Suppose

   $$\alpha_1 q_1 + \alpha_2 q_2 + \ldots + \alpha_k q_k = 0.$$

   Imagine we compute the inner product of both sides with q_j for some j. For $i \neq j$, since $\langle q_j, q_i \rangle = 0$, we have

   $$\begin{aligned} \langle q_j, (\alpha_1 q_1 + \alpha_2 q_2 + \ldots + \alpha_k q_k) \rangle &= \alpha_1 \langle q_j, q_1 \rangle + \ldots + \alpha_k \langle q_j, q_k \rangle \\ &= \alpha_j \langle q_j, q_j \rangle \\ &= \alpha_j \|q_j\|^2 \\ &= 0. \end{aligned}$$

   But since q_j's are non-zero vectors, $\|q_j\|^2 > 0$ and thus $\alpha_j = 0$. Since j was an arbitrary index, it follows that all α_i's must be zero. Thus q_i's are linearly independent.

2. Let S be the subspace of $\mathbb{R}^4$ described by the equation '$x_1 - x_2 + x_3 - 2\,x_4 = 0$'.

   (a) Find a basis for S.

   (b) Find an <u>orthonormal</u> basis for S.

   Solution. (a) Notice that $S = N(A)$, where $A = \begin{bmatrix} 1 & -1 & 1 & -2 \end{bmatrix}$. Thus the special solutions of $Ax = 0$ give a basis for S. The special solutions are,

   $$s_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

   (b) We apply the Gram-Schmidt procedure to s_1, s_2, s_3. We first set

   $$q_1 = \frac{s_1}{\|s_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

   Now let

   $$\tilde{q}_2 = s_2 - q_1\, q_1^T\, s_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}.$$

   Then set $q_2 = \tilde{q}_2 / \|\tilde{q}_2\| = \sqrt{2/3}\, \tilde{q}_2$. Finally,

   $$\tilde{q}_3 = s_3 - q_1\, q_1^T\, s_3 - q_2\, q_2^T\, s_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \\ 1 \end{bmatrix},$$

   and $q_3 = \tilde{q}_3 / \|\tilde{q}_3\| = \sqrt{3/7}\, \tilde{q}_3$.

3. Let A be a matrix whose columns are not linearly independent. Also, let b be a vector, that is not in $C(A)$ (i.e., the column space of A). Suppose also that z is another vector for which $A^T A\, z = A^T b$.

   (a) What is the closest vector in $C(A)$ to b? Express it in terms of A, b and/or z.

   (b) Let P be the projection matrix onto $C(A)$. Also, let Q be the projection matrix onto the left nullspace of A, i.e. $N(A^T)$. Express '$(P - Q)\, b$' in terms of A, b and/or z.

Solution. (a) Recall that if $b = b_1 + b_2$ where $b_1 \in C(A)$ and $b_2 \in C(A)^\perp = N(A^T)$, then b_1 is the closest vector in $C(A)$ to b. Since $b_1 \in C(A)$, we can find a vector u such that $A\,u = b$. But we know that such a u is found by solving $A^T A\,u = A^T b$. Therefore, Az is closest vector in $C(A)$ to b.

(b) According to the discussion above, $P\,b = b_1$, $Q\,b = b_2 = b - b_1$. Using $b_1 = Az$, we thus have $(P - Q)b = Az - (b - Az) = 2Az - b$.

4. Let S be a plane in $\mathbb{R}^3$. Also, let the projection matrix onto S be given as

$$P = \begin{bmatrix} 5/6 & -2/6 & 1/6 \\ -2/6 & 2/6 & 2/6 \\ 1/6 & 2/6 & 5/6 \end{bmatrix}.$$

Find a set of coefficients a_1, a_2, a_3, such that the solution set of the equation

$$a_1\,x_1 + a_2\,x_2 + a_3\,x_3 = 0$$

is equivalent to S.

Solution. A plane in $\mathbb{R}^3$ is 2-dimensional. Therefore S is 2-dimensional. Thus $S^\perp$ is 1-dimensional and a basis for $S^\perp$ contains a single vector and thus it's sufficient to find a single non-zero vector in $S^\perp$ to obtain a basis. Let $x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. Then $z = (x - Px) \in S^\perp$ and z is a basis for $S^\perp$. Notice $z = \begin{bmatrix} 1/6 & 2/6 & -1/6 \end{bmatrix}^T$. Thus, $S = N(A)$, where $A = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$.

MAT 281E – Homework 6

Due 04.12.2015

1. Suppose we are given

$$\mathbf{a}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix}1\\-1\\1\end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix}1\\1\\3\end{bmatrix}$$

that span $\mathbb{R}^3$.

Also, let

$$A = \underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{\mathbf{u}} \underbrace{\begin{bmatrix}1 & 1 & 0\end{bmatrix}}_{\mathbf{a}_1^T}.$$

(a) Apply the Gram-Schmidt procedure to the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ to find three vectors $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ which form an orthonormal basis for $\mathbb{R}^3$.

(b) What are the dimensions of $N(A)$ and $C(A)$?

(c) Find three eigenvectors, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, of A that span $\mathbb{R}^3$. What are the associated eigenvalues?

Solution. (a) We start by setting

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}.$$

Consider now the second vector, we first define

$$\tilde{q}_2 = a_2 - q_1\, q_1^T\, a_2 = \begin{bmatrix}1\\-1\\1\end{bmatrix} - 0\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix},$$

and then set

$$q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}.$$

Finally for the third vector, we first define

$$\tilde{q}_3 = a_3 - q_1\, q_1^T\, a_3 - q_2\, q_2^T\, a_3 = \begin{bmatrix}1\\1\\3\end{bmatrix} - 1\begin{bmatrix}1\\1\\0\end{bmatrix} - 1\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}-1\\1\\2\end{bmatrix},$$

and then set

$$q_3 = \frac{\tilde{q}_3}{\|\tilde{q}_3\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\1\\2\end{bmatrix}.$$

(b) Observe that the columns of $A$are multiples of u. Therefore, $C(A)$ is spanned by u, so that dim$C(A) = 1$. Since dim$C(A)$ = rank = dim$C(A^T)$and dim$C(A^T)$ + dim$N(A)$ = number of columns, it follows that dim$N(A) = 2$. The same conclusion may also be deduced by observing that $N(A)$is the set of vectors orthogonal to a_1.

(c) First observe that since q_2 and q_3 are orthogonal to a_1 they are eigenvectors of $A$with eigenvalue 0 (that is, $A\, q_2 = 0q_2$, $A\, q_3 = 0q_3$). Finally observe that since $C(A)$is spanned by u, for any x, we have $A\, x = c\, u$, where c is some scalar. Thus if we take $x = u$, we obtain, $A\, u = \lambda\, u$, where $\lambda = a_1^T\, u = 3$. We know from part (a) that $q_2$and $q_3$are linearly independent. Check that since $q_1^T\, u \neq 0$, u is not in the span of $q_2$and q_3 (why?). Thus, u, q_2, q_3 span $\mathbb{R}^3$ and they are eigenvectors of A.

2. Let

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Find an orthogonal matrix Q and an upper triangular matrix R such that $A = Q\,R$.

Solution. Let us denote the i^{th} column of A by a_i. We will apply the Gram-Schmidt procedure on A. We start with

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For the second vector, we set

$$\tilde{q}_2 = a_2 - q_1\, q_1^T\, a_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} - \frac{-1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 4/3 \\ -2/3 \end{bmatrix},$$

and

$$q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}.$$

For the last vector, we define

$$\tilde{q}_3 = a_3 - q_1\, q_1^T\, a_3 - q_2\, q_2^T\, a_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-2}{6} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

and set

$$q_3 = \frac{\tilde{q}_3}{\|\tilde{q}_3\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Now observe that

$$\begin{aligned} a_1 &= \sqrt{3} q_1, \\ a_2 &= q_1\, q_1^T\, a_2 + q_2\, q_2^T\, a_2 = \frac{-1}{\sqrt{3}} q_1 + \frac{4}{\sqrt{6}} q_2, \\ a_3 &= q_1\, q_1^T\, a_3 + q_2\, q_2^T\, a_3 + q_3\, q_3^T\, a_3 = \frac{5}{\sqrt{3}} q_1 + \frac{-2}{\sqrt{6}} q_2 + \sqrt{2} q_3. \end{aligned}$$

Therefore,

$$A = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} \sqrt{3} & -1/\sqrt{3} & 5/\sqrt{3} \\ 0 & 4/\sqrt{6} & -2/\sqrt{6} \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

3. Let A be a matrix with eigenvalues 1, 2, and associated eigenvectors

$$\mathbf{e}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Also, let I denote the 2×2 identity matrix. Compute $(A - I)^{10}$.

Solution. Observe that

$$\begin{aligned} (A - I)e_1 &= Ae_1 - e_1 = 0 \\ (A - I)e_2 &= Ae_2 - e_2 = (2 - 1)e_2. \end{aligned}$$

Thus e_i's are eigenvectors of $A - I$also (any vector is an eigenvector of I). Thus we can write,

$$A - I = \underbrace{\begin{bmatrix} e_1 & e_2 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\Lambda} E^{-1}.$$

Thus, $(A - I)^{10} = E\,\Lambda^{10}\,E^{-1}$. But observe that $\Lambda^1 0 = \Lambda$. Therefore, $A - I = A - I$. To find A, we invert $E$and plug it in

$$A = E \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} E^{-1},$$

which I leave to you.

4. Let A be a matrix with eigenvalues 1, 1/2, and associated eigenvectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(a) Write down the eigenvalues and eigenvectors of A^{10}.

(b) Compute $A^{10}\,v$, where $v = \begin{bmatrix} (2^9 - 1) \\ (2^{10} - 1) \end{bmatrix}$.

(c) Find A.

Solution. (a) First note that

$$A = \underbrace{\begin{bmatrix} e_1 & e_2 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}}_{\Lambda} E^{-1}.$$

Observe that $A^{10} = E\,\Lambda^{10}\,E^{-1}$. Since Λ^{10} is also diagonal, its diagonal entries (which are 1^{10} and $(1/2)^{10}$) must be the eigenvalues and the columns of E, namely e_1, e_2 must be the eigenvectors.

(b) Notice that

$$v = 2^9\,e_2 - e_1.$$

Thus, by the discussion in part (a), we have,

$$A\,v = -A\,e_1 + 2^9\,A\,e_2 = -e_1 + 2^9\,\frac{1}{2^{10}} e_2 = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}.$$

(c) We find $E^{-1}$by Gasus-Jordan elimination as,

$$E^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Plugging this in the expression for A, we find

$$A = E\,\Lambda\,E^{-1} = \begin{bmatrix} 3/2 & -1/2 \\ 1 & 0 \end{bmatrix}$$

5. Let S be a subspace of $\mathbb{R}^3$. Also, let P_S be the non-zero projection matrix for S. Suppose that,

$$P_S \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \quad P_S \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Find a basis for $S^\perp$, the orthogonal complement of S.

(b) Find a basis for S.

(c) Find three linearly independent eigenvectors and the associated eigenvalues for P_S.

Solution. (a) Note that $\dim(S) + \dim(S^\perp) = 3$ since both are subspaces of $\mathbb{R}^3$. Observe first that $v = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ is in S, so that $\dim(S) \geq 1$. Therefore, $\dim(S^\perp) \leq 2$. Observe also that

$$u_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - P_S \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in S^\perp, \text{ and } u_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - P_S \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \in S^\perp.$$

Further, observe that u_1 $u_2$are linearly independent. Since $\dim(S^\perp) \leq 2$, it follows that $u_1$and u_2 actually form a basis for $S^\perp$.

(b) Form part (a), we find that $\dim(S) = 1$. Thus $v \in S$ is a basis for S.

(c) Observe that $P_S\, v = v$. Thus $v$is an eigenvector of P_S with eigenvalue 1. Observe also that $P_S\, u_1 = 0u_1$, $P_S u_2 = 0u_2$. Thus, u_1 and u_2 are eigenvectors with eigenvalue 0. Since v, u_1, u_2 are linearly independent (why?), we are done.

6. Suppose A is an $n \times n$ matrix and it has an eigenvalue equal to λ. Let B be the $2n \times 2n$ matrix defined as

$$B = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}.$$

Show that λ and $-\lambda$ are eigenvalues of B.

Solution. Suppose $A\,x = \lambda\,x$ (such an $x$has to exist). Consider now the length-$2n$vector $y$formed as $y = \begin{bmatrix} x \\ x \end{bmatrix}$. Observe that

$$B\,y = \begin{bmatrix} Ax \\ Ax \end{bmatrix} = \lambda \begin{bmatrix} x \\ x \end{bmatrix} = \lambda y.$$

Thus λ is an eigenvalue of B.

Consider now the length-$2n$vector $z$formed as $y = \begin{bmatrix} x \\ -x \end{bmatrix}$. Observe that

$$B\,z = \begin{bmatrix} -Ax \\ Ax \end{bmatrix} = \lambda \begin{bmatrix} -x \\ x \end{bmatrix} = -\lambda z.$$

Thus $-\lambda$ is an eigenvalue of B.

MAT 281E – Homework 7

Due 30.12.2015

Decide whether the following are true or false. Give brief explanations to justify your answer. Assume that the matrix A has real-valued entries.

1. For any A, we can find invertible E and a diagonal Λ such that $A = E\,\Lambda\,E^{-1}$.

   Answer. False. Think of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Both eigenvalues of this matrix are zero but the only eigenvector is of the form $\begin{bmatrix} c \\ 0 \end{bmatrix}$ for $c \neq 0$.

2. If $A = E\,\Lambda\,E^{-1}$, for a diagonal Λ, then E contains the eigenvectors of A and Λ contains the eigenvalues of A.

   Answer. True. If $A\,E = E\,\Lambda$, then $A\,e_i = \lambda_i\,e_i$, where e_i represents column i of $E$and λ_i is the entry at the i^{th} diagonal entry of Λ.

3. An $n \times n$ matrix always has n distinct eigenvalues.

   Answer. False. I has only 1 as an eigenvalue.

4. If A is invertible, then all of its eigenvalues are non-zero.

   Answer. True. If this were not true, we could have $A\,x = 0$ for a non-zero x, and A would not be invertible in that case.

5. If all eigenvalues of A are non-zero, then A is invertible.

   Answer. True. Remember that $\det A$ is the product of the eigenvalues. So, if the eigenvalue are non-zero, $\det A$is non-zero, and therefore $A$is invertible. Notice that this is the converse of the previous statement and an argument based on the determinant can also be used for showing that statement.

6. For an $n \times n$ matrix, we can always find n eigenvectors that span $\mathbb{R}^n$.

   Answer. False. Think again of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This matrix has a single eigenvector. (It has a 'generalized eigenvector', which we did not discuss in class – that's related with the Jordan form.)

7. If A is a singular $n \times n$ matrix, we cannot find n linearly independent eigenvectors.

   Answer. False. Think of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The vectors $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors of A and they are linearly independent.

8. If A is symmetric, we can find n linearly independent eigenvectors.

   Answer. True. By the spectral theorem, $A = Q\,\Lambda\,Q^T$ for a diagonal Λ and orthogonal Q. The columns of Qcontain eigenvectors of $A$and they are linearly independent as a consequence of orthogonality.

9. If we can find n orthogonal eigenvectors for an $n \times n$ matrix A, then A is symmetric.

   Answer. True. If we can find n orthogonal eigenvectors for A, namely $q_1, \ldots, q_n$, then we can form $Q = \begin{bmatrix} q_1 & \ldots & q_n \end{bmatrix}$and write $AQ = Q\,\Lambda$ for a diagonal Λ. But this means $A = Q\,\Lambda\,Q^T$ and so A is symmetric.

10. If A is symmetric, then any two linearly independent eigenvectors of A have to be orthogonal.

    Answer. False. Think of I. $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent eigenvectors of I but they are not orthogonal. (Spectral theorem implies that we can find an orthogonal set of eigenvectors but this does not exclude the possibility that there might be two linearly independent eigenvectors that are not orthogonal.)

11. If A is real, then its eigenvalues have to be real-valued.

    Answer. False. Think of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The eigenvalues of A are $\pm i$.

12. If A is symmetric, then its eigenvalues have to be real-valued.

    Answer. True. Recall the spectral theorem.

13. If A is real, then its eigenvectors have to be real-valued.

    Answer. False. Think of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. $e = \begin{bmatrix} i \\ -i \end{bmatrix}$ is an eigenvector of A.

14. If A is symmetric, then its eigenvectors have to be real-valued.

    Answer. False. Think of I. $e = \begin{bmatrix} i \\ i \end{bmatrix}$ is an eigenvector. (Spectral theorem implies that we can find real-valued eigenvectors but this does not exclude the possibility that a complex-valued vector is an eigenvector.)

15. If A and B are diagonalizable and have the same eigenvectors, then $AB = BA$.

    Answer. True. If A and B are diagonalizable and have the same eigenvectors, then we can write $A = E\,\Lambda_A\,E^{-1}$, $B = E\,\Lambda_B\,E^{-1}$for diagonal matrices $\Lambda_A$and Λ_B. So,

    $$A\,B = E\,\Lambda_A\,E^{-1}E\,\Lambda_B\,E^{-1} = E\,\Lambda_A\,\Lambda_B\,E^{-1} = E\,\Lambda_B\,\Lambda_A\,E^{-1} = E\,\Lambda_B\,E^{-1}E\,\Lambda_A\,E^{-1} = B\,A.$$

16. If A and B are symmetric, then $AB = BA$.

    Answer. False. Take any non-trivial symmetric matrices and try the two products above.

17. If A and B are symmetric, then AB is also symmetric.

    Answer. False. Again, try this on your own.

18. For any square A, we can find an orthogonal Q and an upper-triangular U such that $A = Q\,U\,Q^T$.

    Answer. True. This is the Schur decomposition discussed in class.

19. For any square A, we can find an orthogonal Q and an lower-triangular L such that $A = Q\,L\,Q^T$.

    Answer. True. This follows from the Schur decomposition. Let $B = A^T$. Then, by the Schur decomposition, we can find orthogonal Q and upper triangular Usuch that $B = Q\,U\,Q^T$. But then $A = Q\,U^T\,Q^T$. Since U^T is lower-triangular, the claim follows.

20. Any matrix has at least one eigenvalue.

    Answer. True. The characteristic polynomial has at least a single root by the 'fundamental theorem of algebra'.

21. If $e_1, \ldots, e_k$ are eigenvectors of A associated with different eigenvalues, then $e_1, \ldots, e_k$ are linearly independent.

    Answer. True. I'll demonstrate this for $k = 2$. Let $E = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$ and let Λbe the diagonal matrix holding the associated eigenvectors so that $A\,E = E\,\Lambda$. Now suppose $E\,\alpha = 0$ for a non-zero vector α. But this means, $E\,\Lambda\,\alpha = 0$ also. Thus, we can write

    $$0 = E \begin{bmatrix} \alpha_1 & \lambda_1,\,\alpha_1 \\ \alpha_2 & \lambda_2,\,\alpha_2 \end{bmatrix} = E \begin{bmatrix} \alpha_1 & \lambda_1\,\alpha_1 \\ \alpha_2 & \lambda_2\,\alpha_2 \end{bmatrix} = E \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{bmatrix}.$$

    But by the assumption that λ_i are distinct, it follows that the last matrix is invertible. Therefore,

    $$E \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1\,e_1 & \alpha_2\,e_2 \end{bmatrix} = 0.$$

    Since e_i's are non-zero vectors, the last equality cannot hold. Thus we cannot find a non-zero vector α such that $E\,\alpha = 0$. This means that the columns of E are linearly independent.

    For the case of arbitrary k, the proof of which I leave to you, a keyword that might be useful is 'Vandermonde matrices'.

22. If an $n \times n$ matrix A has n distinct eigenvalues, then it is diagonalizable.

    Answer. True. If $\lambda_1, \dots, \lambda_n$ are the distinct eigenvalues, then we obtain different eigenvectors by solving $(A - \lambda_i)x = 0$. By the argument for the previous statement these eigenvectors are linearly independent, and thus follows the claim.

23. The sum of the diagonal entries of A is equal to the sum of the eigenvalues.

    Answer. True. Recall that the trace of A is equal to the sum of the eigenvalues as discussed in class.

24. If the diagonal entries of A are non-zero then the eigenvalues are also non-zero.

    Answer. False. Think of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. This matrix has non-zeros on its diagonal but it is not invertible so it has a zero eigenvalue.

25. The product of the diagonal entries of A is equal to the product of the eigenvalues.

    Answer. False. Think again of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. This matrix has a zero eigenvalue, so the product of the eigenvalues is zero, wheras the product of the diagonal entries is unity.

MAT 281E – Linear Algebra and Applications

Midterm Examination – I

06.11.2015

(20 pts) 1. Consider the matrices A, B below, related to each other by an elimination operation.

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}}_{A} \xrightarrow{r_3 \leftarrow r_3 - 2r_2} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{bmatrix}}_{B}$$

(a) Find two matrices C_1, C_2 such that $C_1 \, A \, C_2 = B$.

(b) Find two matrices D_1, D_2 such that $A = D_1 \, B \, D_2$.

(30 pts) 2. Consider the system of linear equations

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 1 & -2 \\ 2 & 3 & -1 & 1 & -3 \\ 0 & -1 & 1 & -1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 3 \\ 7 \\ -3 \end{bmatrix}}_{b}.$$

(a) Find a particular solution that solves this system of linear equations.

(b) Describe $N(A)$, the nullspace of A (that is, find the special solutions).

(c) What is the rank of A?

(d) Describe the whole solution set of the system of linear equations $A \, x = b$.

(25 pts) 3. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 4 \end{bmatrix}.$$

(a) Find the rank of A. Find also the dimension of $N(A)$ (the nullspace of A).

(b) Find two non-zero orthogonal vectors v_1, v_2 that are both in $N(A)$.

(25 pts) 4. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 5 & 5 \end{bmatrix}.$$

(a) Find the rank of A.

(b) What is the dimension of $C(A)$?

(c) What is the maximum number of linearly independent non-zero vectors you can find in $C(A)$?

(d) Find a basis for $C(A)$, the column space of A.

(e) Find a basis for $C(A^T)$, the row space of A.

(f) Find a $(3 \times r)$ matrix B and an $(r \times 3)$ matrix C such that $A = B \, C$.
(Hint : Express the either the columns or rows of A in terms of the bases you found in part (d) or part (e).)

MAT 281E – Linear Algebra and Applications

Midterm Examination – II

10.12.2015

Student Name : ____________________

Student Num. : ____________________

4 Questions, 100 Minutes

Please Show Your Work for Full Credit!

(25 pts) 1. Suppose we are given a matrix A and a vector b as

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix}.$$

(a) Find a vector x that minimizes $\|Ax - b\|$.

(b) Find the projection of b onto the column space of A, that is $C(A)$.

(c) Find the projection of b onto the left null-space of A, that is $N(A^T)$.

(25 pts) 2. Consider the subspace S of $\mathbb{R}^3$ defined as the solution set of

$$x_1 + 2\,x_2 + 3\,x_3 = 0.$$

(a) Find the projection matrix for $S^\perp$, where $S^\perp$ is the orthogonal complement of S.

(b) Find the projection matrix for S.

(c) Let

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find $p \in S$, and $q \in S^\perp$ such that $v = p + q$.

(25 pts) 3. Consider the vectors

$$a_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Apply the Gram-Schmidt process to a_1, a_2, a_3 to obtain three orthogonal unit vectors q_1, q_2, q_3 that span the same space as a_1, a_2, a_3.

(25 pts) 4. Consider the matrix

$$A = \begin{bmatrix} 9 & -4 \\ 20 & -9 \end{bmatrix}.$$

(a) Find the eigenvalues and the eigenvectors of A.

(b) Find an invertible matrix E and a diagonal matrix D such that $A = E\,D\,E^{-1}$.

MAT 281E – Linear Algebra and Applications

Final Examination

13.01.2016

Student Name : ______________________

Student Num. : ______________________

5 Questions, 120 Minutes

Please Show Your Work for Full Credit!

(20 pts) 1. Consider the system of equations

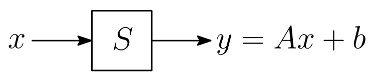
$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}}_{b}.$$

(a) Describe the solution set of $A\,x = b$.

(b) Write down a basis for $N(A)$, the nullspace of A.

(c) What is the rank of A? What are the dimensions of the four fundamental subspaces, $N(A)$, $C(A)$, $N(A^T)$, $C(A^T)$?

(20 pts) 2. We are given a system S that takes a vector $x \in \mathbb{R}^2$ as input and outputs a vector $y \in \mathbb{R}^2$ according to the equation $y = A\,x + b$, as shown below.



We do not know A and b and we would like to determine them. For this, we input different vectors x_1, x_2, x_3 to S and observe the outputs y_1, y_2, y_3. Here y_i denotes the output for x_i. These vectors are given as,

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad y_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 4 \\ 9 \end{bmatrix}, \quad y_3 = \begin{bmatrix} 12 \\ 27 \end{bmatrix}.$$

(a) Find $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \in \mathbb{R}$ such that $x_3 = \alpha_1\,x_1 + \alpha_2\,x_2$.

(b) Determine A and b.

(20 pts) 3. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

(a) Find a vector x that satisfies $A^T\,A\,x = A^T\,b$.

(b) Find a vector x that minimizes $\|A\,x - 2\,b\|$. (Pay attention to the factor 2 in front of b).

(c) Find the projection of $2\,b$ onto $C(A)$ (the column space of A).

(d) Find the projection of $2\,b$ onto $N(A^T)$ (the left nullspace of A).

(20 pts) 4. Let

$$q_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad q_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Observe that q_1 and q_2 are orthogonal unit vectors. Find two more vectors q_3, q_4 so that q_1, q_2, q_3, q_4 is an orthonormal basis for $\mathbb{R}^4$ (that is, q_i's are orthogonal unit vectors).

(20 pts) 5. Consider the matrix

$$A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}.$$

(a) Find an invertible matrix E and a diagonal matrix D such that $E^{-1}\,A\,E = D$.

(b) Let B be a matrix that satisfies $G^{-1}\,B\,G = F$, where

$$G = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad F = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}.$$

Also, let Z be the 2×2 zero matrix, that is, $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Finally, let C be the 4×4 matrix formed as,

$$C = \begin{bmatrix} A & Z \\ Z & B \end{bmatrix}.$$

Write down the eigenvalues and the corresponding eigenvectors of C.