MAT 281E – Linear Algebra and Applications

Fall 2012

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Class Meets :	13.30 – 16.30, Friday EEB 5202
Office Hours :	10.00 – 12.00, Friday
Textbook :	G. Strang, 'Introduction to Linear Algebra', 4^{th} Edition, Wellesley Cambridge.
Grading :	2 Quizzes (15% each), Midterm Exam (30%), Final (40%).
Webpage :	`http://ninova.itu.edu.tr/Ders/1039/Sinif/6402`

Tentative Course Outline

- Solving Linear Equations via Elimination

  Linear system of equations, elimination, LU Decomposition, Inverses

- Vector Spaces

  The four fundamental subspaces, solving $Ax = b$*, rank, dimension.*

- Orthogonality

  Orthogonality, projection, least squares, Gram-Schmidt orthogonalization.

- Determinants

- Eigenvalues and Eigenvectors

  Eigenvalues, eigenvectors, diagonalization, application to difference equations, symmetric matrices, positive definite matrices, iterative splitting methods for solving linear systems, singular value decomposition.

MAT 281E – Homework 1

05.10.2012

1. Consider the linear system of equations

$$\begin{bmatrix} 2 & 3 \\ 3 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ b \end{bmatrix}.$$

   (a) Find a pair (a, b) so that the system has a unique solution.

   (b) Find a pair (a, b) so that the system has no solutions.

   (c) Find a pair (a, b) so that the system has infinitely many solutions.

2. Consider the linear system of equations

$$\begin{bmatrix} 3 & -3 & 2 & 0 \\ -3 & 3 & 1 & -3 \\ -15 & 10 & -10 & -3 \\ -12 & 17 & -2 & -4 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 6 \\ 0 \\ -19 \\ -21 \end{bmatrix}.$$

   Solve for $\mathbf{x}$ using Gaussian elimination and back-substitution.

3. Suppose A is an $n \times n$ matrix with inverse A^{-1}. Using A, we form the $2n \times 2n$ matrix C as,

$$C = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$$

   Is C invertible? If so, find the inverse. If not, explain why not.

4. (a) Consider the matrix $M = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$. Find M^{-1} using the Gauss-Jordan procedure.

   (b) Let A be an $n \times n$ matrix. Also, let I denote the $n \times n$ identity matrix. Using A and I we form the $2n \times 2n$ matrix C as,

$$C = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$$

   Is C invertible? If so, find the inverse. If not, explain why not.

MAT231E- HW1 Solutions

① Apply an elimination step to the augmented matrix:

$$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 3 & a & b \end{array}\right] \xrightarrow{(r3)-\frac{3}{2}(r1)} \begin{bmatrix} 2 & 3 & 4 \\ 0 & a-\frac{9}{2} & b-6 \end{bmatrix}$$

(a) Any (a,b) pair with $a \neq \frac{9}{2}$ works.

(b) Pick $a=\frac{9}{2}$, $b=0$ $\Rightarrow$ 2nd equation gives

$|0 \cdot y = -5$ $\Rightarrow$ no solution.

(c) Pick $a=\frac{9}{2}$, $b=6$. $\Rightarrow$ the solution set is $2x+3y=4$.

② Form the Augmented Matrix and apply elimination:

$$\begin{bmatrix} 3 & -3 & 2 & 0 & 6 \\ -3 & 3 & 1 & -3 & 0 \\ -15 & 10 & -10 & -3 & -19 \\ -12 & 17 & -2 & -4 & -21 \end{bmatrix} \xrightarrow{(r2)-(-r1)} \begin{bmatrix} 3 & -3 & 2 & 0 & 6 \\ 0 & 0 & 3 & -3 & 6 \\ -15 & 10 & -10 & -3 & -19 \\ -12 & 17 & -2 & -4 & -21 \end{bmatrix}$$

$$\xrightarrow{r3-(-5)r1} \begin{bmatrix} 3 & -3 & 2 & 0 & 6 \\ 0 & 0 & 3 & -3 & 6 \\ 0 & -5 & 0 & -3 & 11 \\ -12 & 17 & -2 & -4 & -21 \end{bmatrix} \xrightarrow{r4-(-4)r1} \begin{bmatrix} 3 & -3 & 2 & 0 & 6 \\ 0 & 0 & 3 & -3 & 6 \\ 0 & -5 & 0 & -3 & 11 \\ 0 & 5 & 6 & -4 & 3 \end{bmatrix}$$

$$\xrightarrow{(r2)\leftrightarrow(r3)} \begin{bmatrix} 3 & -3 & 2 & 0 & 6 \\ 0 & -5 & 0 & -3 & 11 \\ 0 & 0 & 3 & -3 & 6 \\ 0 & 5 & 6 & -4 & 3 \end{bmatrix} \xrightarrow{r4-(-1)r2} \begin{bmatrix} 3 & -3 & 2 & 0 & 6 \\ 0 & -5 & 0 & -3 & 11 \\ 0 & 0 & 3 & -3 & 6 \\ 0 & 0 & 6 & -7 & 14 \end{bmatrix}$$

$$\xrightarrow{r4-2r3}\begin{bmatrix}3&-3&2&0&6\\0&-5&0&-3&11\\0&0&3&-3&6\\0&0&0&-1&2\end{bmatrix}$$

Now do back-substitution.

The last equation gives: $-x_4 = 2 \Rightarrow x_4 = -2$

3rd equation : $3x_3 - 3x_4 = 6 \Rightarrow x_3 = 0$

2nd eqn : $-5x_2 - 3x_4 = 11 \Rightarrow x_2 = -1$

1st eqn : $3x_1 - 3x_2 + 2x_3 = 6 \Rightarrow x_1 = 1$

③ Let x be a length-n, non-zero column vector. Also, let

$$y = \begin{bmatrix}x\\-x\end{bmatrix} \Rightarrow Cy = \begin{bmatrix}A&A\\A&A\end{bmatrix}\begin{bmatrix}x\\-x\end{bmatrix} = \begin{bmatrix}Ax-Ax\\Ax-Ax\end{bmatrix} = 0.$$

$\Rightarrow$ C is not invertible.

④ (a)

$$\left[\underbrace{\begin{matrix}1&a\\0&1\end{matrix}}_{M}\;\middle|\;\underbrace{\begin{matrix}1&0\\0&1\end{matrix}}_{I}\right]\xrightarrow{r1-a\cdot r2}\left[\underbrace{\begin{matrix}1&0\\0&1\end{matrix}}_{I}\;\middle|\;\underbrace{\begin{matrix}1&-a\\0&1\end{matrix}}_{M^{-1}}\right]$$

(b) C "looks like" M. So try, $D = \begin{bmatrix}I&-A\\0&I\end{bmatrix}$.

$$\Rightarrow D\cdot C = \begin{bmatrix}I&-A\\0&I\end{bmatrix}\begin{bmatrix}I&A\\0&I\end{bmatrix} = \begin{bmatrix}I&A-A\\0&I\end{bmatrix} = \begin{bmatrix}I&0\\0&I\end{bmatrix}$$

$\Rightarrow$ D is the inverse of C.

MAT 281E – Homework 2

12.10.2012

1. A student was asked to compute the inverse of A by Gauss-Jordan elimination. She formed the augmented matrix and started elimination. Below is the augmented step she reached at an intermediate step.

   $$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & 6 & -4 & 1 & 0 \\ 0 & 0 & 7 & -4 & 1 & 1 \end{bmatrix}.$$

   Find A.

2. Suppose A and B are invertible matrices. Assume also that all of their rows sum to 1. Suppose we form a new matrix as $C = A - B$. Is C invertible? If so, find its inverse in terms of A^{-1} and B^{-1}. If not, explain why not.

3. Suppose $X\,A = B$, where

   $$A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ 1 & -4 \end{bmatrix}.$$

   Find X.

4. Find the LU decomposition of

   $$A = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}.$$

   Under what conditions (on a, b, c) is A invertible?

5. Write down a square matrix and find its LU decomposition. If invertible, find its inverse.

MAT281E - HW2 Solutions

① Let $\tilde{B}$ be the given matrix. From the description, we deduce that $\tilde{B} = E\,[A \;\; I]$, where E is a 3×3 matrix. In order to find A, we reduce the right half of $\tilde{B}$ to I, using <u>row</u> operations. This is equivalent to multiplying $\tilde{B}$ on the left by E^{-1}.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & 6 & -4 & 1 & 0 \\ 0 & 0 & 7 & -4 & 1 & 1 \end{bmatrix} \xrightarrow[r3+4(r1)]{r2+4(r1)} \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 4 & 8 & 7 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{r3-r2} \left[\begin{matrix} \underbrace{\begin{matrix} 1 & 2 & 0 \\ 4 & 5 & 6 \\ 0 & 3 & 1 \end{matrix}}_{=A} & \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \end{matrix}\right]$$

② If each row of A and B sum to 1, then,

$$A \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{v}, \quad B v = v. \quad \Rightarrow (A-B)v = 0.$$

Since $v \neq 0$, $A-B$ cannot be invertible.

③ Note that if $XA = B$, then $A^T X^T = B^T$. Let us solve for X^T. Form the augmented matrix as $\left[A^T \; B^T\right]$ and reduce the left half to I, using row operations.

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ -2 & 1 & -3 & -4 \end{bmatrix} \xrightarrow{r2+2r1} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & -1 & -2 \end{bmatrix}$$

$$\xrightarrow{r1-r2} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & -1 & -1 & -2 \end{pmatrix} \xrightarrow{-1(r2)} \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \; \underbrace{\begin{matrix} 2 & 3 \\ 1 & 2 \end{matrix}}_{= X^T = A^{-T} B^T}\right)$$

$$\Rightarrow X = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

④

$$\begin{tikzcd}
 & & & U \\
\begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix} \arrow[r] & \begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ a & b & c \end{bmatrix} \arrow[r] & \begin{pmatrix} a & a & a \\ a & b-a & b-a \\ 0 & b-a & c-a \end{pmatrix} \arrow[r] & \begin{bmatrix} a & a & a \\ a & b-a & b-a \\ 0 & 0 & c-b \end{bmatrix} \arrow[u, Rightarrow] \\
E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} & E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} &
\end{tikzcd}$$

$$\Rightarrow A = \underbrace{\left(E_1^{-1} E_2^{-1} E_3^{-1}\right)}_{L} U \longrightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

In order for A^{-1} to exist, U should be invertible

$\Rightarrow c \neq b, \; b \neq a, \; a \neq 0.$

MAT 281E – Homework 3

02.11.2012

1. Let A be a matrix with rank 1. We are given four vectors from the nullspace of A as,

$$\begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} b \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}.$$

   (a) Find a and b.

   (b) If the first column of A is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, determine A.

2. Let A be a matrix with rank 2. Also, let y be a vector in the column space of A. We are given three solutions from the solution set of $A\,x = y$ as,

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}.$$

   Find a and b.

3. Consider the planes P_1, P_2 in $\mathbb{R}^3$, described by the equations

$$\begin{aligned} (P_1)&\quad 2\,x_1 + x_2 - x_3 = 1, \\ (P_2)&\quad x_2 = 3x_3. \end{aligned}$$

   Let the set of intersection of these planes be called S. Notice that S is actually a line. Find a matrix A and a vector b such that the solution set of $A\,x = b$ is S.

4. Let S be the line in $\mathbb{R}^3$ described as the set of points of the form

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix},$$

   where $\alpha \in \mathbb{R}$. Find a matrix A and a vector b such that the solution set of $A\,x = b$ is S.

MAT281E HW3 Solutions

① (a) Notice that A should have 3 columns. Since rank $=1$, there are 2 free variables (3-rank $=2$). In fact the first two vectors are in the form of special solutions (they are lin. indep.). In that case, we should have,

$$\left.\begin{aligned} &1\cdot\begin{bmatrix}a\\1\\0\end{bmatrix}+2\cdot\begin{bmatrix}b\\0\\1\end{bmatrix}=\begin{bmatrix}5\\1\\2\end{bmatrix}, \Rightarrow a+2b=5 \\ &2\cdot\begin{bmatrix}a\\1\\0\end{bmatrix}+1\cdot\begin{bmatrix}b\\0\\1\end{bmatrix}=\begin{bmatrix}4\\2\\1\end{bmatrix}. \Rightarrow 2a+b=4 \end{aligned}\right\} \begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}$$

Let us solve for a, b:

$$\begin{bmatrix}1&2&5\\2&1&4\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&5\\0&-3&-6\end{bmatrix}\Rightarrow \begin{aligned}&b=2,\\ &a+2\cdot2=5 \Rightarrow a=1.\end{aligned}$$

(b) $A=\begin{bmatrix}2&c_2&c_3\\3&&\end{bmatrix}$. Since $A\begin{bmatrix}a\\1\\0\end{bmatrix}=0 \Rightarrow c_2=-a\cdot\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}-2\\-3\end{bmatrix}$

$$A\cdot\begin{bmatrix}b\\0\\1\end{bmatrix}=0 \Rightarrow c_3=-b\cdot\begin{bmatrix}2\\3\end{bmatrix}$$

$$=\begin{bmatrix}-4\\-6\end{bmatrix} \Rightarrow A=\begin{bmatrix}2&-2&-4\\3&-3&-6\end{bmatrix}.$$

② A has 3 columns. Since rank $= 2$, $\dim N(A) = 1$.

In this case, $N(A)$ consists of vectors of the form $\alpha \cdot s$ where s is a special solution.

The solution set of $Ax = b$ is of the form $p + \alpha \cdot s$, where p is a particular solution.

Therefore, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a multiple of s.

It should be that $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$.

$\Rightarrow$ From the first line, we obtain $\beta = -1 \Rightarrow a = 0,\ b = -1$.

③ $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & -3 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

④ It suffices to find A with rank$=2$ s.t. $A \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 0$, and set $b = A \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Take $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, set $b = A \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

MAT 281E – Homework 4

30.11.2012

1. Let ℓ be the intersection of the planes

   $$\begin{aligned} &(P_1) \quad x_1 - x_2 + x_3 = 1, \\ &(P_2) \quad x1 + x_2 - 2x_3 = 2. \end{aligned}$$

   Find the closest point of ℓ to $y = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

2. Let S be the set of vectors described as

   $$\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},$$

   where α is a real number. Find the closest point of S to $y = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

3. Let A be a matrix given as

   $$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 0 \end{bmatrix}.$$

   (a) Find the projection matrix that projects any vector onto $C(A)$, the column space of A.

   (b) Find the projection matrix that projects any vector onto $N(A)$, the nullspace of A.

4. Suppose S and V are subspaces of $\mathbb{R}^n$ and $S \subset V$. Show that $V^\perp \subset S^\perp$.

5. Consider a linear system of equations as $A\,x = b$. Let S denote the (non-empty) solution set of this system. Also, let P denote the projection operator onto $N(A)$. Finally, suppose $x_0 \in S$ – i.e., x_0 is a solution of the given system. Find a function $f(z)$, in terms of P and x_0, that maps the vector z to the vector in S that is closest to z.

MAT 281E - HW4 Solution

① ℓ is the solution set of $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Let us find an explicit description of ℓ:

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & 1 \end{bmatrix}$$

$\Rightarrow$ special solution: $\begin{bmatrix} 1/2 \\ 3/2 \\ 1 \end{bmatrix}$ particular soln: $\begin{bmatrix} 3/2 \\ 1/2 \\ 0 \end{bmatrix}$

$\Rightarrow \ell$ consist of $\begin{bmatrix} 3/2 \\ 1/2 \\ 0 \end{bmatrix} + \alpha \cdot \begin{bmatrix} 1/2 \\ 3/2 \\ 1 \end{bmatrix}$, where $\alpha \in \mathbb{R}$.

Our goal is to minimize (over α)

$$\left\| \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 3/2 \\ 1/2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1/2 \\ 3/2 \\ 1 \end{bmatrix} \right) \right\|^2 = \left\| \begin{bmatrix} 1/2 \\ -1/2 \\ -1 \end{bmatrix} - \alpha \cdot \begin{bmatrix} 1/2 \\ 3/2 \\ 1 \end{bmatrix} \right\|^2$$

$$\Rightarrow \hat{\alpha} = \frac{\left(\frac{1}{2} \cdot \frac{1}{2} - \frac{3}{2} \cdot \frac{1}{2} - 1 \cdot 1 \right)}{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 1^2}$$

(the α value that minimizes the cost function)

$\Rightarrow p = \begin{bmatrix} 3/2 \\ 1/2 \\ 0 \end{bmatrix} + \hat{\alpha} \cdot \begin{bmatrix} 1/2 \\ 3/2 \\ 1 \end{bmatrix}$ $(\in \ell)$ is the solution

② We need to find the α value which minimizes

$$\left\| \begin{bmatrix}1\\2\\0\end{bmatrix} - \left(\begin{bmatrix}1\\-2\\-1\end{bmatrix} + \alpha\begin{bmatrix}1\\1\\2\end{bmatrix}\right) \right\|^2 = \left\| \begin{bmatrix}0\\4\\1\end{bmatrix} - \alpha \cdot \begin{bmatrix}1\\1\\2\end{bmatrix} \right\|^2$$

$$\Rightarrow \hat{\alpha} = \frac{1\cdot 0 + 1\cdot 4 + 2\cdot 1}{1^2 + 1^2 + 2^2} = 1.$$

$$\Rightarrow p = \begin{bmatrix}1\\-2\\1\end{bmatrix} + \hat{\alpha}\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}2\\-1\\3\end{bmatrix}$$ is the closest point in S to y.

③ (a) One way to find this matrix is to find a basis for $C(A)$ and then obtain the projection matrix. Recall that the pivot columns of A form a basis for $C(A)$.

$$\begin{bmatrix}1&2&1\\2&4&2\\1&1&0\end{bmatrix} \longrightarrow \begin{bmatrix}1&2&1\\0&0&0\\0&-1&1\end{bmatrix}$$ $\Rightarrow$ The first two columns of A are the pivot columns.

$$\Rightarrow \text{Let } D = \begin{bmatrix}1&2\\2&4\\1&1\end{bmatrix}.$$

$$P = D\,(D^T D)^{-1} D^T$$

③ (b) Let us first find a basis for $N(A)$:

Using the elimination from (a):

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \right\}$$ forms a basis for $N(A)$.

$$\Rightarrow P_{N(A)} = \frac{1}{11} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 1 & 1 \end{bmatrix}$$

④ Let x be an arbitrary element of $V^{\perp}$. If we can show that $x \in S^{\perp}$ then it follows that $V^{\perp} \subset S^{\perp}$.

Since $x \in V^{\perp}$, we have $<x, y>$ for any $y \in V$. But since $V \supset S$, we also have that $<x, t>$ for any $t \in S$. Therefore $x \in S^{\perp}$.

⑤ Notice that $S - x_0 = N(A)$.

So if we translate every vector by $-x_0$, the closest point can be found by projecting onto $N(A)$. If we shift back by $+x_0$, the resulting point will be the closest point in S to z.

Therefore $f(z) = P(z - x_0) + x_0$.

MAT 281E – Homework 5

28.12.2012

1. Let A be a 3×3 matrix with three eigenvalues, λ_1, λ_2, λ_3. Suppose also that the eigenvalues are distinct (i.e., $\lambda_1 \neq \lambda_2$, $\lambda_2 \neq \lambda_3$, $\lambda_1 \neq \lambda_3$). Also, let x_1, x_2, x_3 be the associated eigenvectors. Show that x_1, x_2, x_3 are linearly independent.

2. Consider the matrix

$$A_2 = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}.$$

In order to find the eigenvalues for this matrix, we know that we can look for the roots of $|A - \lambda I|$. For matrices in this special form (notice that incidence matrices derived from difference equations are of this form), an alternative is the following. Suppose $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ is an eigenvector with eigenvalue λ. In that case, we should have,

$$A_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a\,x_1 + b\,x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} \lambda\,x_1 \\ \lambda\,x_2 \end{bmatrix}$$

The second line in the last equality says that $x_1 = \lambda\,x_2$. Plugging this into the first equality,

$$a\,x_1 + b\,x_2 = \lambda\,x_1,$$

we obtain,

$$a\,\lambda\,x_2 + b\,x_2 = \lambda^2\,x_2.$$

Dividing by x_2, we get

$$\lambda^2 - a\,\lambda - b = 0.$$

   (a) Suppose we solve for the roots of the last equation to obtain an eigenvalue λ^*. Find an eigenvector associated with this eigenvalue.

   (b) Consider the matrix

$$A_3 = \begin{bmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

   By generalizing the procedure outlined above for the 2×2 case, find a polynomial, whose roots give the eigenvalues of A_3.

3. Consider the linear difference equation

$$y_n = \frac{1}{2} y_{n-1} + \frac{1}{4} y_{n-2} + \frac{1}{4} y_{n-3},$$

which is valid for $n \geq 3$.

   (a) Define the vector $u_n = \begin{bmatrix} y_n & y_{n-1} & y_{n-2} \end{bmatrix}^T$. Find the incidence matrix A which links u_n to u_{n-1} through $u_n = A\,U_{n-1}$.

   (b) Observe that if $y_n = \alpha$ for all $n \geq 0$ (i.e., if y_n is constant for all n), then it satisfies the difference equation above. What does this imply about the incidence matrix?

MAT231E - HW5 Solutions

① We first show that x_1 and x_2 are linearly independent, that is, $x_1 \neq \alpha x_2$, where $\alpha \in \mathbb{R}$.

To see this, suppose $x_1 = \alpha x_2$. Multiplying both sides of this equation by A on the left, we obtain $Ax_1 = \lambda_1 x_1 = \alpha \lambda_1 x_2 = A \cdot (\alpha x_2)$

$$= \alpha \lambda_2 x_2.$$

But since $\lambda_1 \neq \lambda_2$, this cannot happen. $\Rightarrow x_1 \neq \alpha x_2$.
($\alpha = 0$ is not allowed because $x_1 \neq 0$.)

Now let us show that x_1, x_2, x_3 are lin. indep.
Suppose not. That is, suppose "$x_3 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$" (*)
for some $\alpha_1, \alpha_2 \in \mathbb{R}$. Notice that $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, because $x_3 \neq \alpha x_1$, $x_3 \neq \alpha x_2$, by the reasoning above.

Multiplying (*) by A on the left, we obtain

$$\lambda_3 x_3 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \alpha_1 \\ \lambda_2 \alpha_2 \end{bmatrix}.$$

Therefore from (*), we obtain: $\begin{bmatrix} \lambda_1 \alpha_1 \\ \lambda_2 \alpha_2 \end{bmatrix} = \begin{bmatrix} \lambda_3 \alpha_1 \\ \lambda_3 \alpha_2 \end{bmatrix}$.

This cannot be $\Rightarrow x_1, x_2, x_3$ are lin. indep.

② (a) Suppose $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the corresponding eigenvector.

Then, $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a x_1 + b x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} \lambda^* x_1 \\ \lambda^* x_2 \end{bmatrix}$

$\Rightarrow x_1 = \lambda^* x_2$ $\Rightarrow$ Since $\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector with the same eigenvalue, take $\alpha = \frac{1}{x_2}$ (Why is $x_2 \neq 0$?)

$\Rightarrow$ $\begin{bmatrix} \lambda^* \\ 1 \end{bmatrix}$ is such an eigenvector.

(b) $A_3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a x_1 + b x_2 + c x_3 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix}$

$\Rightarrow x_2 = \lambda x_3, \quad x_1 = \lambda x_2 = \lambda(\lambda x_3)$.

$\Rightarrow \lambda x_1 = \lambda^3 x_3 = a \cdot \lambda^2 x_3 + b \lambda x_3 + c x_3$

$\Rightarrow$ Divide by x_3 (why is $x_3 \neq 0$?) to obtain

$\lambda^3 - a\lambda^2 - b\lambda - c = 0$. $\Rightarrow$ roots give the eigenvalues.

③ (a) $u_n = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} u_{n-1}$.

(b) '1' is an eigenvalue of A with eigenvector $\begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix}$, (or $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$).

MAT 281E, Fall 2012, Quiz 1

Student Name : ______________________

Student Num. : ______________________

1. Consider the linear system of equations :

$$\underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & -2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}}_{\mathbf{b}}$$

   Solve for $\mathbf{x}$ using elimination and back-substitution. Use the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$. Show your steps clearly.

2. Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 7 \\ 0 & 12 & 8 \end{bmatrix}.$$

   Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

MAT 281E – Linear Algebra and Applications

Midterm Examination

23.11.2012

Student Name : ____________________

Student Num. : ____________________

5 Questions, 120 Minutes

(25 pts) 1. We are given the equation

$$x_1 \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 & 1 & 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \end{bmatrix},$$

where x_1, x_2, x_3 are real numbers. Find x_1, x_2, x_3.

(25 pts) 2. Consider the system of equations $A\,x = b$, where,

$$A = \begin{bmatrix} 1 & -1 & 3 & 0 & 1 \\ -1 & 1 & -2 & 1 & -1 \\ 2 & -2 & 5 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}.$$

(a) Find a basis for $N(A)$, the nullspace of A.

(b) What are the dimensions of the row space (i.e. $C(A^T)$) and the left-nullspace of A (i.e. $N(A^T)$)?

(c) Describe the solution set of $A\,x = b$.

(20 pts) 3. Consider the plane P, in $\mathbb{R}^3$, described by the equation

$$x_1 - 2\,x_2 + 3\,x_3 = 0.$$

Note that this is a subspace of $\mathbb{R}^3$. Notice also that

$$z = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

is in the plane P. Find another vector y, in the plane P, that is orthogonal to z.

(15 pts) 4. Let A be a matrix given as

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 4 & 1 & 6 \end{bmatrix}.$$

Find a real number α, so that,

$$\begin{bmatrix} 4 \\ \alpha \\ 3 \end{bmatrix}$$

is in the column space of A.

(15 pts) 5. Let A be a matrix given as

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \\ 1 & 3 & 5 \end{bmatrix}.$$

Find a vector x that is <u>not</u> in the column space of A.

MAT 281E, Fall 2012, Quiz 2

Student Name : ____________________

Student Num. : ____________________

1. Let S be the set of vectors described as

$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

   where α_1 and α_2 are real numbers. Find the closest point of S to $y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

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2. Let A be a matrix whose columns are not linearly independent. Also, let b be a vector, that is not in $C(A)$ (i.e., the column space of A). Suppose also that z is another vector for which $A^T A z = A^T b$.

   (a) What is the closest vector in $C(A)$ to b? Express it in terms of A, b and/or z.

   (b) Let P be the projection matrix onto $C(A)$. Also, let Q be the projection matrix onto the left nullspace of A, i.e. $N(A^T)$. Express '$(P - Q)\, b$' in terms of A, b and/or z.

   Please briefly explain your answers for full credit.

MAT 281E – Linear Algebra and Applications

Final Examination

09.01.2013

Student Name : ____________________

Student Num. : ____________________

5 Questions, 120 Minutes

Please Show Your Work!

(25 pts) 1. Consider the system of equations

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 3 & -1 & 2 \\ 2 & 1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}}_{\mathbf{b}}.$$

(a) Describe the solution set of $A\,\mathbf{x} = \mathbf{b}$.

(b) Write down a basis for the nullspace of A.

(c) What is the rank of A? What are the dimensions of the four fundamental subspaces, $N(A)$, $C(A)$, $N(A^T)$, $C(A^T)$?

(20 pts) 2. Let V be the subspace of $\mathbb{R}^3$ spanned by the vectors v_1 and v_2, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Find a vector that is orthogonal to both v_1 and v_2.

(b) Let $V^{\perp}$ denote the set of all vectors that are orthogonal to V. Notice that $V^{\perp}$ is a subspace. Find a basis for $V^{\perp}$.

(c) Let S be the subspace of $\mathbb{R}^3$ spanned by

$$s_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Find a non-zero vector z that lies in both S and V.

(15 pts) 3. Let P be a plane in $\mathbb{R}^3$. Suppose we are given three points on P as

$$p_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}.$$

Find the point on P that is closest to the origin.

(20 pts) 4. Consider the vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

(a) Find the projection of v_2 onto the space spanned by v_1.

(b) Find two orthogonal, unit norm vectors z_1, z_2 that span the same space as v_1 and v_2.

(c) Find three orthogonal, unit norm vectors z_1, z_2, z_3 that span the same space as v_1, v_2 and v_3.

(20 pts) 5. Let A be a matrix with eigenvalues 1, 1/2, and associated eigenvectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(a) Write down the eigenvalues and eigenvectors of A^{10}.

(b) Compute $A^{10}\,v$, where $v = \begin{bmatrix} (2^9 - 1) \\ (2^{10} - 1) \end{bmatrix}$.

(c) Find A.