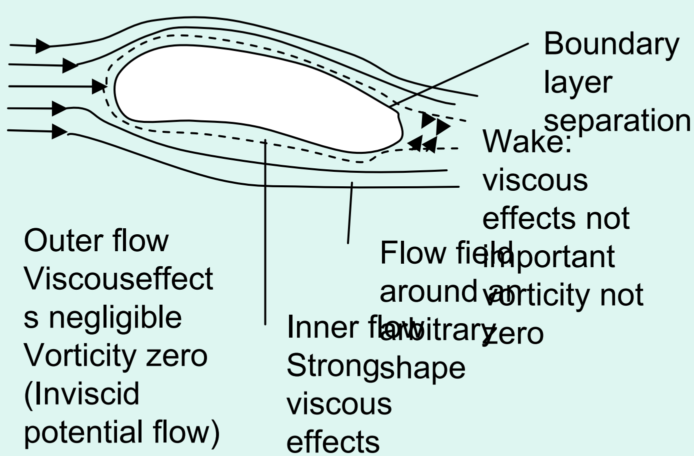
BOUNDARY LAYER THEORY

HIGH RENOLDS NUMBER FLOW ⟶ BOUNDARY LAYERS

$(Re \rightarrow \infty)$

BOUNDARY LAYER Thin region adjacent to surface of a body where viscous forces dominate over inertia forces

$$Re = \left(\frac{inertia\ forces}{viscous\ forces}\right) \qquad Re >> 1$$



$$\frac{d\theta}{dx} + (\delta^* + 2\theta)\frac{1}{\theta}\frac{dU}{dx} = \frac{\tau_0}{\rho U^2}$$

Steady ,incompressible 2-D flow with no body forces. Valid for laminar flow $\tau_0 \sim (\frac{\partial u}{dy})^n$

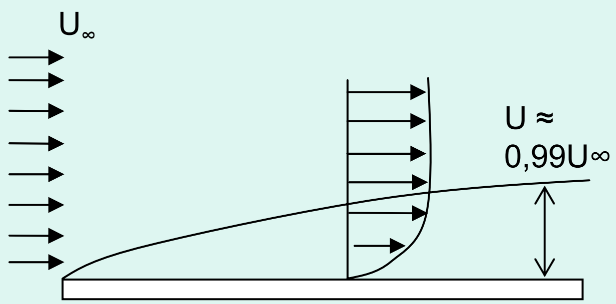
O.D.E for $\theta(x)$

To solve eq. we first "assume" an approximate velocity profile inside the B.L

Relate the wall shear stress to the velocity field

Typically the velocity profile is taken to be a polynomial in y, and the degree of fluid this polynominal determines the number of boundary conditions which may be satisfied

<u>EXAMPLE:</u> $\frac{u}{U} = a + b\eta + c\eta^2 = f(\eta)$ <u>LAMINAR FLOW OVER A FLAT PLATE:</u>



- Laminar boundary layer → predictable
- Turbulent boundary layer → poor predictability
- Controlling parameter ⇒ $\mathrm{Re} = \dfrac{UL}{\nu}$
- To get two boundary layer flows identical ⇒ match Re
  (dynamic similarity)
- Although boundary layer's and prediction are complicated,simplify the N-S equations to make job easier

High Reynolds Number Flow

2-D , planar flow

u* = $v^* = \dfrac{u, v}{U_\infty}$, x*, y*= $\dfrac{x, y}{L}$

Dimensionless gov. eqs. $\nabla . \vec{V} = 0$

X ; $$\frac{\partial u^*}{\partial t} + u^* \frac{\partial u^*}{\partial x} + v^* \frac{\partial u^*}{\partial y} = -\frac{\partial P^*}{\partial x} + \underbrace{\frac{1}{\text{Re}}\left(\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2}\right)}_{\text{viscous terms}}$$

$$P^* = \frac{P}{\rho U_\infty^2}$$

Y; $$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{1}{\text{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

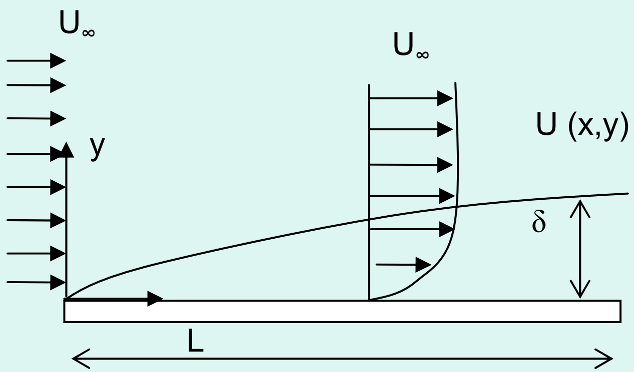
“Naïve” way of solving problem for

$$\text{Re} \to \infty \quad \Rightarrow \quad \frac{1}{\text{Re}} \to 0$$

If you drop the viscous term ⟶ Euler’s eqs. (inviscid fluid)

- We can not satisfy all the boundary B.C.s because order of eqs. Reduces by 1

Inside B-L can not get rid of viscous terms



$$\delta^* = \frac{\delta}{L} \langle \frac{1}{100}$$

Derivation of B-L eqs. From the N-S eqs

- Physically based argument :determine the order of terms in N-S
- Limiting procedure as $Re \longrightarrow \infty$ eqs. and throw out small terms

Assumption 1

$$\delta^* = \frac{\delta}{L} \langle\langle 1$$

Term		**Order**
$\frac{\partial u^*}{\partial x^*}$	→	$\frac{(1)}{(1)} = 1$
$\frac{\partial v^*}{\partial y^*}$	→	$\frac{\delta^*}{\delta^*} = 1$
v^*	→	δ^*
$\frac{\partial v^*}{\partial x^*}$	→	$\frac{\delta^*}{1} = \delta^*$
$\frac{\partial^2 u^*}{\partial y^{*2}}$	→	$\frac{1}{\delta^{*2}}$
$\frac{du^*}{dt^*}$	→	$u^* \frac{\partial u^*}{\partial x^*} = 1$

$$\frac{\partial u^*}{\partial t} + u^*\frac{\partial u^*}{\partial x} + v^*\frac{\partial u^*}{\partial y} = -\frac{\partial P^*}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2}\right)$$

(1) (1) $\delta^*\frac{1}{\delta^*}=1$ $\frac{(1)}{(1)}=1$ δ^{*2} $\frac{(1)}{(1)^2}$ $\frac{(1)}{(\delta^*)^2}$

Neglect since of order

$\frac{(1)}{(\delta^*)^2}$ >>>1

Also for y –direction

$$\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{\partial P^*}{\partial y^*} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right)$$

* $(1)\frac{(\delta^*)}{(1)}$ $(\delta^*)\frac{(\delta^*)}{(\delta^*)}$ $(\delta^{*2})\left\{\frac{\delta^*}{(1)^2}+\frac{\delta^*}{(\delta^*)^2}\right\}$

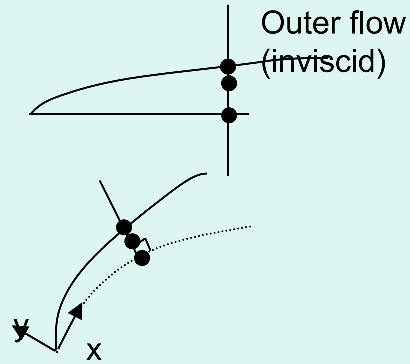
$\mho(\delta^*)$ $\mho(\delta^*)$ $\mho(\delta^*)$ $\mho(\delta^*)$

$$\frac{\partial P^*}{\partial y^*} \Rightarrow \mho(\delta^*) \Rightarrow \text{small relative to } \frac{\partial P^*}{\partial x^*} \Rightarrow \mho(1)$$

To good approximation $P \cong P(x)$ $\Rightarrow$ pressure at the edge of B-L. is equal to pressure on boundary layer.

- Time – dependant $\Rightarrow$ $P \cong P(x,t)$ $\longleftarrow$ known from the other flow
- Pressure at all points is the same
- Only need to consider x-direction B-L. eqs.

Prandtl (1904)



2-D planar

1) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

2) $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

Governing

eqs.for B.L

B-L.eqs.
still non-linear
but parabolic type

unknows u,v (x,y,t)

$P \cong P(x,t)$ ⟶ known from the potential flow

Need B.C.s & I.C.(time dependant)

- 2-D, steady

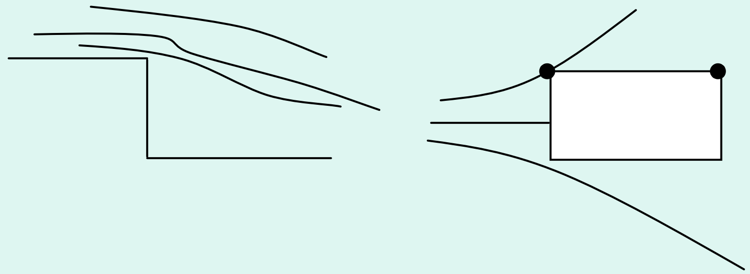
BCs

- u=ν =0 at y=0
- u=u(y) at x=0
- u= U_∞(x) y $\longrightarrow$ ∞ (y $\longrightarrow$ δ) $\longleftrightarrow$ marching condition

- B-L. eqs. can be solved exactly for several cases
- Can approximate solution for other cases

Limitation of B.L eqs.: where they fail?

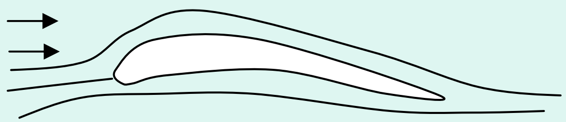
(1) Abrupt chances

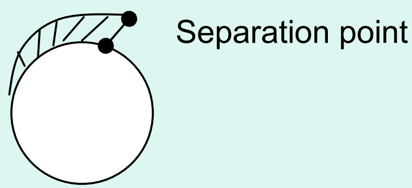


(2) Eqs. are not applicable near the leading edge

L is small ⟶ $\delta^* = \frac{\delta}{L} \langle\langle 1$ invalid

(3) Where the flow separates not valid beyond the separation point





Bernouilli eqs. ρ =constant

$$\frac{P}{\rho} + \frac{V^2}{2} = \text{constant} \quad \Longrightarrow \quad \frac{1}{\rho}\frac{dP}{dx} + \frac{1}{2}2U\frac{dV}{dx} = 0$$

Valid along the streamlines

$$-\frac{1}{\rho}\frac{dP}{dx}=U\frac{dU}{dx}$$

substitute the B.L eqs u,v can be found

U → known

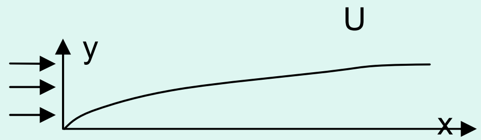
SIMILARITY SOLUTION TO B.L. EQS

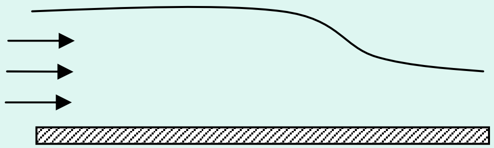
Example 1

Flow over a semi-infinite flat plate

Zero pressure gradient ⇒ $\frac{dp}{dx}=0$ p = constant

Steady ,laminar & U=constant ($\frac{dp}{dx}=0$)





- Bernouilli eqs. outsideB.L $\Longrightarrow$ $$p + \frac{1}{2}\rho U^2 = cons.$$

U=constant , $\frac{dp}{dx} = 0$

Governing (B.L. eqs.) become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (2)$$

B.C.

- y=0 u= v =0 (no-slip) & y $\rightarrow$ ∞ , u $\rightarrow$ U
- x=0 u=U

Blasuis(1908) :

1.Introduce the stream function ψ (x,y)

- Recall ; $u = \dfrac{\partial \psi}{\partial y}$ $v = -\dfrac{\partial \psi}{\partial x}$

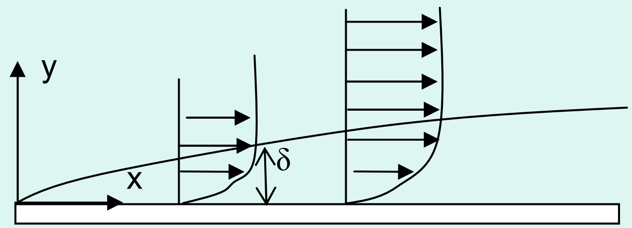
note that ψ satisfies cont. eqs. substitute intoB.L. mom. Eqs

$$\frac{\partial \psi}{\partial y} \cdot \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3} \qquad (2')$$

- Now, assume that we have a similarity "stretching" variable, which has all velocity profiles on plate scaling on δ .

i.e

$$\frac{u}{U_\infty} = f\left(\frac{y}{\delta}\right)$$



$$\delta = g(U_\infty, x, \nu)$$ → dimensional analysis

$$\frac{\delta}{x} = g\left(\frac{U_\infty x}{\nu}\right) = g(\mathrm{Re}) \Rightarrow \frac{1}{\mathrm{Re}} \sim \mho(\delta^2) \qquad \delta \sim \sqrt{\nu}$$

$$\Rightarrow \delta \sim \sqrt{\frac{\nu x}{U_\infty}} \qquad \frac{m^2}{s}\cdot\frac{m}{m}.s = [m] \qquad \frac{\delta}{x} \sim \frac{1}{\sqrt{\mathrm{Re}_x}}$$ both $\mho(\delta)$

Viscous dif. Depth

$$\mathrm{Re} = \frac{U_\infty x}{\nu} \qquad \delta \approx 5\sqrt{\frac{\nu x}{U_\infty}}$$

Let $\eta = \dfrac{y}{\delta}$ [-] similarity variable

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \Rightarrow \frac{u}{U} = f(\eta)$$

Use similarity profile assumption to turn

2 P.D.E $\Rightarrow$ 1 O.D.E

$$u = \left.\frac{\partial \psi}{\partial y}\right|_{x = fixed} \qquad \psi = \int_0^y u\,dy = \int_0^y Uf(\eta)\,dy = \int_0^\eta Uf(\eta)\sqrt{\frac{\nu x}{U}}\,d\eta$$

$$\psi = \sqrt{U\nu x}\int_0^{\eta} f(\eta)d\eta = \sqrt{U\nu x}F(\eta)$$

$$\underbrace{\int_0^{\eta} f(\eta)d\eta}_{F(\eta)}$$

$$\boxed{\psi = \sqrt{U\nu x}F(\eta)}$$

$$\psi = \sqrt{U\nu x}F(\eta) \qquad \eta = y\sqrt{\frac{U_\infty}{\nu x}}$$

$$\psi - \psi_0 = \int_0^y u\,dy \qquad d\psi = \frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial x}dx$$

- Now, substitute ψ into P.D.E for ψ (x,y) to get O.D.E for F(η)

$$\frac{\partial \psi}{\partial x} = \frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}F + \sqrt{U_\infty \nu x}F' \frac{\partial \eta}{\partial x} \qquad F' = \frac{dF}{d\eta} \qquad F'' = \frac{d^2F}{d\eta^2}$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{2} y \sqrt{\frac{U_\infty}{\nu x}} \frac{1}{x} = -\frac{1}{2x}\eta \qquad \frac{\partial \psi}{\partial x} = \frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}(F - \eta F') \qquad \frac{\partial \psi}{\partial y} = \sqrt{U_\infty \nu x}F' \sqrt{\frac{U_\infty}{\nu x}} = U_\infty F'$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = -\frac{U_\infty}{2x}\eta F'' \qquad \frac{\partial^2 \psi}{\partial y^2} = U_\infty \sqrt{\frac{U_\infty}{\nu x}} F'' \qquad \frac{\partial^3 \psi}{\partial y^3} = \frac{U_\infty^2}{\nu x} F'''$$

Substituting into eq. (2')

$$U_\infty F'(-\frac{U_\infty}{2x}\eta F''') - \left[\frac{1}{2}(\frac{U_\infty \nu}{x})^{1/2}(F - \eta F')\right]\left[U_\infty (\frac{U_\infty}{\nu x})^{1/2} F''\right] = \nu \frac{U_\infty^2}{\nu x} F'''$$

or

$$-\frac{U_\infty^2}{2x}\eta F'' F' - \frac{1}{2}\frac{U_\infty^2}{x}F''F + \frac{1}{2}\frac{U_\infty^2}{x}\eta F'' F' = \frac{U_\infty^2}{x}F'''$$

$$F''' + \frac{1}{2}FF'' = 0$$

blasius eq.   3rd order , non linear ODE

Note: $F''' + FF'' = 0$ for $\eta = y\sqrt{\frac{U_\infty}{2\nu x}}$ BVP

BC's are

At y=0 u=v=0 ⇨ $\eta = 0$

BC 1) $u\big|_{y=0} = \frac{\partial \psi}{\partial y}\Big|_{y=0} = 0$ ⇨ $U_\infty F'\big|_{\eta=0} = 0$ **<u>F'(0)=0</u>**

BC 2) $\nu\big|_{y=0} = 0$ ⇨ $-\frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}(F - \eta F') = 0$ **<u>F(0)=0</u>**

BC 3) (x,y ⟶ ∞) ⟶ U_∞

$\frac{\partial \psi}{\partial y}\Big|_{y\to\infty} \to U_\infty$ $U_\infty F'\big|_{\eta\to\infty} = U_\infty$ $F'(\eta \to \infty)$ ⟶ 1 $F'(\infty) = 1$

F(η) dimensionless function

Or At x=0 $u = U_\infty$ ⇨ $U_\infty F'\big|_{\substack{x=0 \\ \eta\to\infty}} = U_\infty$

F'(∞)=1 same with BC 3) Matching B.C

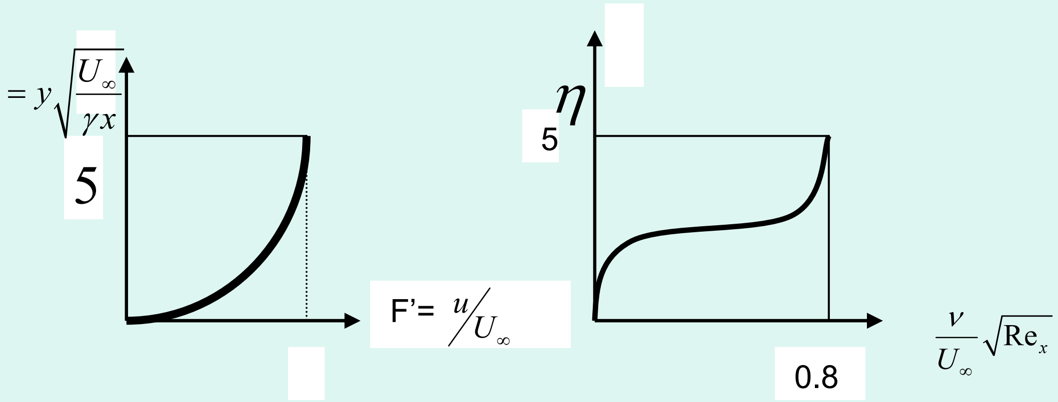
- Solution to blasius eg a)power series
  b)runge-kutta
- results tabulated form for F,F',F'',etc

p.g 121

$\eta = y\sqrt{\dfrac{U_\infty}{\nu x}}$	F	$F' = \dfrac{u}{U_\infty}$	F''
0	0	0	0.33206
•	•	•	•
•	•	•	•
•	•	•	•
5.0	3.28329	0.99155	0.01591

F''= 0.33206 ← From the solution

- Velocity profile



$$v = -\frac{\partial \psi}{\partial x} = \frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}(\eta F' - F)$$

$$\frac{v}{U_\infty} = \frac{1}{2}\mathrm{Re}_x^{-1/2}\left[\eta F' - F\right]$$

$$\eta \to \infty \qquad v_\infty = \frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}(5x1 - 3.28)$$

$$\frac{v_\infty}{U_\infty} = 0.86\frac{1}{\sqrt{\mathrm{Re}_x}}$$

Shear stress distribution along the flat plate

$$\tau = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau(x, y)$$

$$\frac{\partial u^*}{\partial y^*} \gg \frac{\partial v^*}{\partial x^*} \qquad \tau \cong \mu \frac{\partial u}{\partial y}$$

$$For \quad \mathrm{Re}_x = 10^4 \quad \Rightarrow \quad \frac{\nu_\infty}{U_\infty} = 0.00865 \approx \frac{1}{100}$$

$$\text{For} \quad \mathrm{Re}_x = 10^6 \quad \Rightarrow \quad \frac{\nu_\infty}{U_\infty} = 0.000865 \approx \frac{1}{1000}$$

At the wall (y=0)

$$\tau_0(x) = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$\downarrow$

$$\tau_w(x)$$

$$\tau_0(x) = \mu \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{y=0} = \mu U_\infty \sqrt{\frac{U_\infty}{\nu x}}\, F''\Big|_{\eta=0}$$

$\downarrow$

Distribution along the wall

$$\tau_0(x) = \mu \sqrt{\frac{U_\infty^3}{\nu x}}\, F''(0)$$

$\downarrow$ (for $F''(0)$)

0.332

Non dimensionalize :

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2} = \frac{2F''(0)}{\sqrt{\mathrm{Re}_x}} = \frac{0.664}{\sqrt{\mathrm{Re}_x}} \qquad \mathrm{Re}_x = \frac{U.x}{\nu}$$

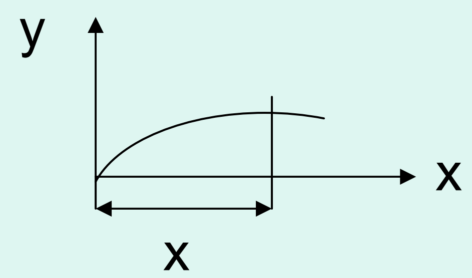
$$C_f = 0.664\sqrt{\frac{\nu}{Ux}}$$

Friction coef.

Note : $x \to 0 \quad \Rightarrow \quad \tau_0 \to \infty$

$\nu \to \infty$

B.L eqs.are not valid near the leading edge



Up to the point we are considering

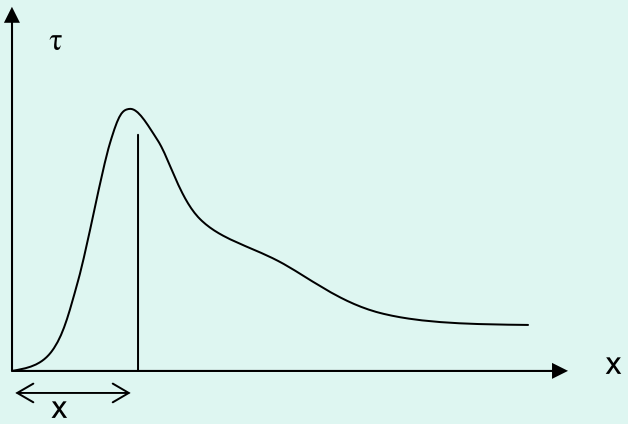
Drag force acting on the flat plate

We have to integrate shear stress

$$F_D = \int_0^x \tau_0(\zeta)\,d\zeta$$

↓

per unit width



$$2F_D = 1.328(b)\sqrt{U_\infty^3 \mu \rho x}$$

dimensionless drag coef.(C_D)

we have 2 wetted sides

$$C_D = \frac{2F_D}{\frac{1}{2}\rho U_\infty^2 A}$$

A=2bx

Width normal to the blackboard

$$C_D = \frac{1.328}{\sqrt{Re_x}}$$

valid for laminar flow i.e for $Re_x < 5.10^5$ to 10^6

for $\mathrm{Re}_x > 10^6 \quad \rightarrow$ turbulent drag becomes considerably greater

Boundary Layer Thickness : δ

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \quad \text{at } \eta = 5 \quad \Rightarrow \quad \frac{u}{U} = 0.99 \quad \rightarrow y = \delta \quad \text{(Table)}$$

$$5 \cong \delta\sqrt{\frac{U_\infty}{\nu x}} \qquad \delta \cong \frac{5x}{\sqrt{\mathrm{Re}_x}} \qquad \mathrm{Re}_x = \frac{U_\infty x}{\nu}$$

δ:defined as the distance from the wall for which u=0.99U_∞

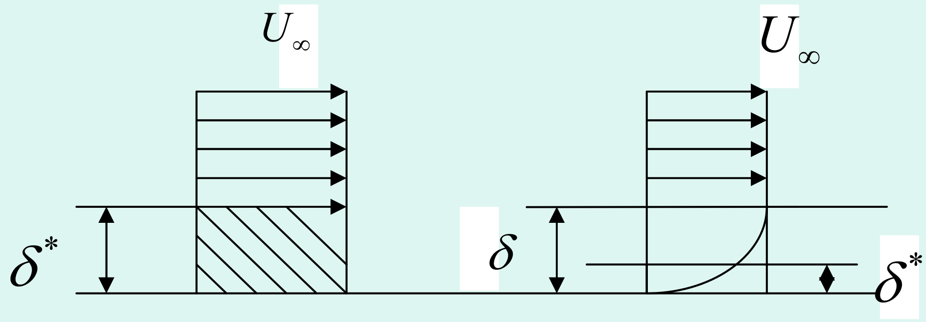
Boundary Layer Parameter (thicknesses)

Most widely used is δ but is rather arbitrary y=δ when u=0.99 U_∞

→ hard to establish

→ more physical parameters are needed

Displacement thickness: δ



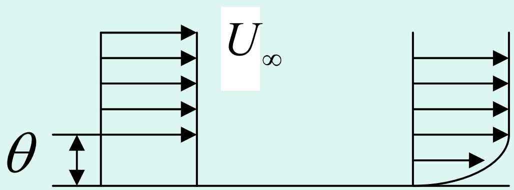
an imaginary displacement of fluid from the surface to account for “lost” mass flow in boundary layer

$$\dot{m}_{tot} = \int_0^\infty \rho u dy = \int_{y=\delta^*}^\infty \rho U_\infty dy = \int_0^\infty \rho U_\infty dy - \underbrace{\int_0^{\delta^*} \rho U_\infty dy}_{-\rho U_\infty \delta^*} \qquad \text{or}$$

$$\rho U_\infty \delta^* = \int_0^\infty (\rho U_\infty - \rho u) dy \qquad\qquad \underline{\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy}$$

if $\rho = cons.$ $\delta > \delta^*$ always by definition

Momentum thickness: θ



an imaginary displacement of fluid of velocity U_∞ to account for “lost” momentum due to the formation of a boundary layer velocity profile

$$\rho U_\infty^2 \theta = \underbrace{\int_0^\infty (\rho u dy) U_\infty}_{\text{Possible momentum}} - \underbrace{\int_0^\infty (\rho u dy) u}_{\text{actual momentum}}$$

Mass flow in B.L

"lost" momentum

$$\theta = \int_0^\infty \frac{u}{U_\infty}(1-\frac{u}{U_\infty})dy$$ will occur in B.L eqs.

notes(remaks)

* Various thinknesses defined above are,to some extend,an indication of the distance over which viscous effects extend.
* $\delta^*, \theta(x)$ only
* $\delta > \delta^* > \theta$ (always)
* Definition is same for ZPG,APG,FPG,turbulance

From flat plate analysis $\delta \cong \frac{5x}{\sqrt{\text{Re}_x}}$

and $$\delta^x = \int_0^\delta (1-\frac{u}{u_\infty})dy$$

remember $$\eta = y\sqrt{\frac{u_\infty}{\nu x}} \Rightarrow d_\eta = d_y\sqrt{\frac{u_\infty}{\nu x}}$$

$$\delta^* = \int_0^{\eta=5} (1-\frac{u}{u_\infty})\sqrt{\frac{\nu x}{u_\infty}}d_\eta = \sqrt{\frac{\nu x}{u_\infty}}\int_0^{\eta=5}(1-F')d_\eta$$

$$\sqrt{\frac{\nu x x}{u_\infty x}}[\eta - F]_0^5 = \frac{x}{\sqrt{\text{Re}_x}}[5-3.283] = \frac{1.72x}{\sqrt{\text{Re}\,x}}$$

$$F(5) = 3.283$$

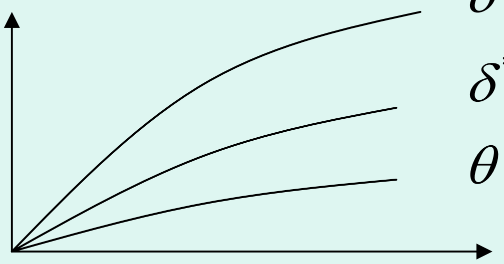
$$\delta^* = \frac{1.72x}{\sqrt{\mathrm{Re}\,x}}$$

Similarly, $$\theta = \int_0^\delta \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) d_y = \frac{0.664x}{\sqrt{\mathrm{Re}\,x}}$$

δ

δ^*

θ



δ

δ^*

θ

FALKNER-SKAN SIMILARITY SOLUTIONS

Stagnation-point flow (Hiemenz flow)

Flow over a flat plate (Blasius flow)

} Similarity methods

$$(x, y) \Rightarrow \eta$$

Falkner & Skan (1931) →general similarity solution of the B-L eqs.

Family of similarity solutions to the 2-D,steady B-L egs.

Look for general similarity solutions of the form

where (1) $$\boxed{\begin{aligned} & u(x,y) = U(x) f'(\eta) \\ & \eta = \frac{y}{\zeta(x)} \end{aligned}}$$

$\zeta(x)$ - unspecified function of x which will be determined later

(2) $$\boxed{\psi(x,y) = U(x)\zeta(x) f(\eta)}$$ check : $u = \frac{\partial \psi}{\partial y} = U(x)\cancel{\zeta(x)} f'(\eta) \frac{1}{\cancel{\zeta(x)}}$

B.L eqs. $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad (3)$$

or in terms of $\psi(x,y)$ $$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = U\frac{dU}{dx} + \nu\frac{\partial^3 \psi}{\partial y^3} \quad (3')$$

B.C.s no-slip, smooth matching

Substitute eq .(2) into (3')

$$\eta = \frac{y}{\zeta(x)}$$

$$\psi = U(x)\zeta(x)f(\eta) = \psi(x,y) \qquad f' = \frac{df}{d\eta}, \; f'' = \frac{d^2 f}{d\eta^2}$$

$$\frac{\partial \psi}{\partial y} = Uf' \quad (= u)$$

$$\frac{\partial \psi}{\partial x} = \frac{dU}{dx}\zeta f + U\frac{d\zeta}{dx}f + U\zeta\frac{df}{d\eta}\frac{d\eta}{dx}$$

$$\frac{d\eta}{dx} = -\frac{y}{\zeta^2}\frac{d\zeta}{dx} = -\eta\frac{1}{\zeta}\frac{d\zeta}{dx}$$

$$\frac{\partial \psi}{\partial x} = \frac{dU}{dx}\zeta f + U\frac{d\zeta}{dx}f - U\frac{d\zeta}{dx}\eta f'$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial x}\left[Uf'\right] = \frac{dU}{dx}f' + U\frac{\partial f'}{\partial x}$$

$$= \frac{dU}{dx}f' + U\frac{df'}{d\eta}\frac{\partial \eta}{\partial x} = \frac{dU}{dx}f' + Uf''\left[-\eta\frac{1}{\zeta}\frac{d\zeta}{dx}\right]$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{dU}{dx}f' - \frac{U}{\zeta}\frac{d\zeta}{dx}\eta f''$$

$$\frac{\partial^2\psi}{\partial y^2} = \frac{\partial}{\partial y}[Uf'] = Uf''\frac{\partial\eta}{\partial y} = \frac{U}{\zeta}f''$$

$$\frac{\partial^3\psi}{\partial y^3} = \frac{U}{\zeta^2}f'''$$

Substitute above results into (3')

$$Uf'\left[\frac{dU}{dx}f' - \frac{U}{\zeta}\frac{d\zeta}{dx}\eta f''\right] - \left[\frac{dU}{dx}\zeta f + U\frac{d\zeta}{dc}f - U\frac{d\zeta}{dx}\eta f'\right]\frac{U}{\zeta}f'' = U\frac{dU}{dx} + \nu\frac{U}{\zeta^2}f'''$$

$$U\frac{dU}{dx}(f')^2 - U\frac{dU}{dx}ff'' - U^2\frac{1}{\zeta}\frac{d\zeta}{dx}ff'' = U\frac{dU}{dx} + \nu\frac{U}{\zeta^2}f'''$$

$$u\frac{dU}{dx}(f')^2 - \frac{U}{\zeta}\frac{d}{dx}(U\zeta)ff'' = U\frac{dU}{dx} + \nu\frac{U}{\zeta^2}f'''$$

To put the eq. into standard form, multiply by $\dfrac{\zeta^2}{\nu U}$

$$f''' + \underbrace{\left[\frac{\zeta}{\nu}\frac{d}{dx}(U\zeta)\right]}_{\alpha}ff'' + \underbrace{\left[\frac{\zeta^2}{\nu}\frac{dU}{dx}\right]}_{\beta}\left[1-(f')^2\right] = 0 \qquad (4)$$

Transformed gov. Eq.

If a similarity solution exists, eq.(4) must be an ODE for the function *f* in terms of η.

So, coefficieuts α & β must be constant for a similarity solution

$$\boxed{f''' + \alpha f f'' + \beta\left[1-(f')^2\right] = 0} \quad \text{Falker-Skan eq.} \quad (5)$$

B.C same as for flat plate $f(0) = f'(0) = 0$

$f'(\eta \to \infty) \to 1$

remark : BCs don't depend on α, β

Exact solutions to the B-L. Eqs. May be obtained by pursuing the following PROCEDURE

Step 1: Select α & β . (a particular flow configuration is considered

this will not be known a priori but will be exident when step 2 is completed).

Step 2: Determine U(x) , ζ(x)

$$\alpha = \frac{\zeta}{\nu}\frac{d}{dx}(U\zeta) \quad , \quad \beta = \frac{\zeta^2}{\nu}\frac{dU}{dx} \quad \text{(6a-b)}$$

Step 3 : Determine the function $f(\eta)$ which is the solution of the following problem

$$f''' + \alpha f f'' + \beta\left[1-(f')^2\right] = 0$$

with BCs $\quad f(0) = f'(0) = 0 \;,\; f'(\eta) \to 1$ as $\eta \to \infty$

Step 4 : Calculate the stream functionin physical coord.

$$\psi(x,y) = U(x)\zeta(x) f\left(\underbrace{\frac{y}{\zeta(x)}}_{\eta}\right)$$

Remark in step #2 , instead of working with eqs. 6 a-b)

$$\frac{\zeta^2}{\nu}\frac{dU}{dx} = \beta \quad \text{(6a)'} \qquad\qquad \frac{d}{dx}\left(U\zeta^2\right) = \nu\left(2\alpha-\beta\right) \quad \text{(6b)'}$$

Example #1 Flate Plate (ZPG)

step #1 $\quad \alpha = \frac{1}{2} \;,\; \beta = 0$

step #2 $\quad \frac{d}{dx}\left(U\zeta^2\right) = \nu \quad$ (6a)'

$$\frac{\zeta}{\nu}\frac{dU}{dx} = 0 \quad \text{(6b)'}$$

$\zeta(x) \neq 0 \Rightarrow (6b)'$ leads to $\frac{dU}{dx} = 0 \Rightarrow$ U=const.

this means that flat plate at ZPG

(6a)' $\frac{d\zeta^2}{dx} = \frac{\nu}{U} \rightarrow \zeta^2 = \frac{\nu x}{U}$

$$\boxed{\zeta(x) = \sqrt{\frac{\nu x}{U}}}$$

Step #3 : $f''' + \frac{1}{2} f f'' = 0 \qquad f(0) = f'(0) = 0$

$\eta = \frac{y}{\sqrt{\frac{\nu x}{U}}} \qquad \eta \rightarrow \infty; f' \rightarrow 1$ compare with Blasius solution

Step #4 $\psi(x, y) = U\zeta f\left(\frac{y}{\zeta}\right) = U\sqrt{\frac{\nu x}{U}} f\left(\frac{y}{\sqrt{\frac{\nu x}{U}}}\right)$

$\psi(x, y) = \sqrt{U\nu x} f\left(y\sqrt{\frac{U}{\nu x}}\right) \leftarrow$ same as Blasius solution

Example #2 FLOW OVER WEDGE

Step #1 $\alpha = 1, \quad \beta =$ arbitrary constant

(6a') $\frac{d}{dx}\left(U\zeta^2\right) = \nu\left(2-\beta\right) \Rightarrow U\zeta^2 = \nu\left(2-\beta\right)x \qquad (7)$

$$(6b') \qquad \zeta^2 \frac{dU}{dx} = \nu\beta$$

Divide eq. (6b') by (7)

$$\boxed{\frac{1}{U}\frac{dU}{dx} = \frac{\beta}{2-\beta}\frac{1}{x}}$$

$$\ln U = \frac{\beta}{2-\beta}\ln x + \ln c \Rightarrow \boxed{U(x) = cx^{\frac{\beta}{2-\beta}}}$$ outer flow is that over a wedge of angle $\pi\beta$ (Fig.)

$$\zeta^2 \frac{dU}{dx} = \nu\beta \qquad \zeta^2 c \frac{\beta}{2-\beta} x^{\frac{-2(1-\beta)}{2-\beta}} = \nu\beta$$

$$\boxed{\zeta(x) = \sqrt{\frac{\nu(2-\beta)}{c} x^{\frac{1-\beta}{2-\beta}}}} \qquad (9)$$

Step #3 Solve the BVP

$$\boxed{f''' + ff'' + \beta\left[1-(f')^2\right]} = 0$$

$$f(0) = f'(0) = 0$$

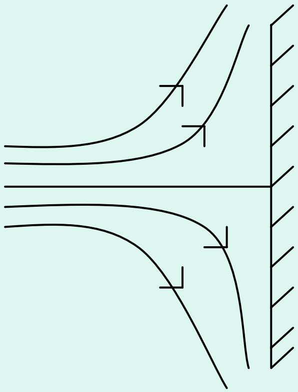
$as \quad \eta \to \infty \quad f' \to 1$ Solve numerically to get $f(\eta), f'(\eta), f''(\eta)$

Step 4: Go back to the physical coordinate

$$\psi(x,y)=U(x)\zeta(x)f\left(\frac{y}{\zeta(x)}\right)=\sqrt{c(2-\beta)\nu}\;x^{1/(2-\beta)}\,f\left(\frac{y}{\sqrt{(2-\beta)\nu/c}}\,x^{-(1-\beta)(2-\beta)}\right)$$

STAGNATION-POINT FLOW; $\beta=1\ \ \alpha=1$

Flow over a wedge $\rightarrow$ Let $\beta=1$



Eq. (8) gives,

$U(x)=cx$

$(9)\rightarrow\zeta(x)=\sqrt{\dfrac{\nu}{c}}$

$$f'''+ff''+1-(f')^2=0$$

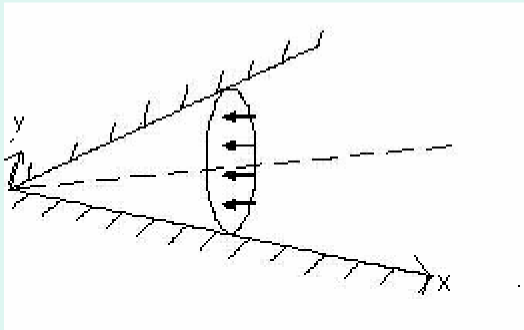
$$f(0)=f'(0)=0 \quad \text{as} \quad \eta\rightarrow\infty \quad f'(\eta)\rightarrow 1$$

$$\psi(x,y)=\sqrt{c\nu}\;\mathrm{x}\,f\left(\frac{y}{\sqrt{\nu/c}}\right)$$

Note: See Hiemenz flow

Exact solution to the full Navier-Stokes equations obtained by Hiemenz for a stagnation point.

FLOW IN A CONVERGENT CHANNEL $\alpha = 0 \quad , \quad \beta = 1$



Boundary layer flow on the wall of a convergent channel.

Exercise: pg. 132.

Solve the BVP (F-S. eq.)

More on similarity solutions to the B.L.

Evans (1968) "Laminar Boundary Layers"

Numerical Solutions

Finite differences

H.B. Keller (1978)

Ann. Rev. of Fluid Mech. Vol.10.pp. 417-433

Finite Element Methods, Finite Volume Methods

Spectral (Element) Methods

APPROXIMATE SOLUTIONS:

Solve exact eq. approximately

Von Karman Momentum Integral Eqn

(General Momentum Integral Equation for Boundary Layer)

Idea: Develop an eqn. which can accept "approximate" vel. profiles as input & yield accurate (close, but approximate) shear stress δ, δ^*, θ as output.

Approach: Integrate the differential B-L. eqs. across the B-L. $0 \le y \le \delta$

Start with B-L. eqs.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$B.C \qquad y = 0 \qquad u, v = 0$

$\qquad\qquad y = \delta \qquad u = U$

First note $\quad v\frac{\partial u}{\partial y} = \frac{\partial (uv)}{\partial y} - u\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(uv) + u\frac{\partial u}{\partial x} \qquad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}(continuity)$

Substitute into B.L eq. & integrate from y=0 to y= δ.

$$\underset{(1)}{\int_0^\delta 2u\frac{\partial u}{\partial x}dy} + \underset{(2)}{\int_0^\delta \frac{\partial (uv)}{\partial y}dy} = \underset{(3)}{\int_0^\delta U\frac{dU}{dx}dy} + \underset{(4)}{\nu\int_0^\delta \frac{\partial^2 u}{\partial y^2}dy}$$

Consider term (2) $\qquad \int_0^\delta \frac{\partial (uv)}{\partial y}dy = uv\Big|_0^\delta = U\underbrace{v(x,\delta)}_{?} - 0$

Integrate cont. eq. $\int_0^\delta \quad dy$

$$\int_0^\delta \frac{\partial u}{\partial x}dy + \int_0^\delta \frac{\partial v}{\partial y}dy = 0 \quad \Rightarrow \quad \int_0^\delta \frac{\partial u}{\partial x}dy + v(x,\delta) - 0 = 0 \qquad \underline{Uv(x,\delta) = -U\int_0^\delta \frac{\partial u}{\partial x}dy}$$

Integrate term (4) $$\int_0^\delta \frac{\partial^2 u}{\partial y^2}dy = \int_0^\delta \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)dy = \left.\frac{\partial u}{\partial y}\right|_0^\delta = \left.\frac{\partial u}{\partial y}\right|_{y=\delta} - \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

=0

$$\tau_0 = \mu \left.\frac{du}{dy}\right|_{y=0} \Rightarrow \quad (4) \quad \Rightarrow -\frac{\tau_0}{\mu}\nu = -\frac{\tau_0}{\rho}$$

Term(1) $$\Rightarrow \int_0^\delta 2u\frac{\partial u}{\partial x}dy = \int_0^\delta \frac{\partial(u^2)}{\partial x}dy$$

B-L. eq. becomes $$\int_0^\delta \frac{\partial(u^2)}{\partial x}dy - U\int_0^\delta \frac{\partial u}{\partial x}dy = \int_0^\delta U\frac{dU}{dx}dy - \frac{\tau_0}{\rho}$$

$$U\int_0^\delta \frac{\partial u}{\partial x}dy = \int_0^\delta U\frac{\partial u}{\partial x}dy = \int_0^\delta \left(\frac{\partial(uU)}{\partial x} - u\frac{dU}{dx}\right)dy$$

Thus, get $$\underbrace{\int_0^\delta \frac{\partial(u^2)}{\partial x}dy - \int_0^\delta \frac{\partial(uU)}{\partial x}dy} + \underbrace{\int_0^\delta u\frac{dU}{dx}dy - \int_0^\delta U\frac{dU}{dx}dy} = -\frac{\tau_0}{\rho}$$

$$\underbrace{\int_0^\delta \frac{\partial}{\partial x}\left(u^2 - uU\right)dy} + \int_0^\delta (u-U)\frac{dU}{dx}dy = -\frac{\tau_0}{\rho}$$

$$\frac{\partial}{\partial x}\int_0^\delta \left(u^2 - uU\right)dy$$

Using Leibnitz's rule permits the order of integ. & dif. to be interchanged

$$\frac{\partial}{\partial x}\int_0^\delta U^2\left(\frac{u^2}{U^2} - \frac{u}{U}\right)dy + \int_0^\delta \left(\frac{u}{U} - 1\right)U\frac{dU}{dx}dy = -\frac{\tau_0}{\rho}$$

Multiply by -1 & factor U terms out of integrals,

$$\frac{\partial}{\partial x}\left[U^2\underbrace{\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy}_{\theta(x)}\right] + U\frac{dU}{dx}\underbrace{\int_0^\delta \left(1 - \frac{u}{U}\right)dy}_{\delta^*(x)} = \frac{\tau_0}{\rho}$$

$$\frac{\partial}{\partial x}(U^2\theta) + \delta^* U \frac{dU}{dx} = \frac{\tau_0}{\rho}$$

$$\downarrow$$

$$U^2 \frac{\partial \theta}{\partial x} + \theta 2U \frac{dU}{dx} \quad , \qquad \frac{\partial \theta}{\partial x} \to \frac{d\theta}{dx} \qquad \theta(x) \;\; \text{only}$$

Divide eq. by U^2 & get

$$\boxed{\frac{d\theta}{dx} + (\delta^* + 2\theta)\frac{1}{U}\frac{dU}{dx} = \frac{\tau_0}{\rho U^2}}$$

or $$\frac{d\theta}{dx} + (H+2)\frac{\theta}{U}\frac{dU}{dx} = \frac{C_f}{2} \qquad H = \frac{\delta^*}{\theta} \quad C_f = \frac{\tau_0}{\frac{1}{2}\rho U^2}$$ H= shape factor

Ordinary Differential eq. for θ(x) & is called von Karman Momentum Integral eqn. or Generalized momentum integral equation

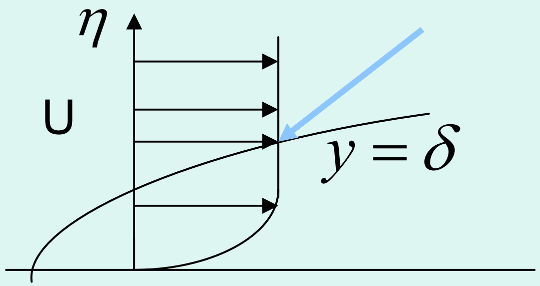
To solve the integral eq. we first "*assume*" an approximate velocity profile, i.e. one that "fits" & has proper "shape" and satisfies the proper B.C we do thin by using similarity concept again & writing potential similarity velocity profiles in terms of the variable , $\eta = \frac{y}{\delta(x)}$ & apply B.C & get particular form.

⇨ evaluate θ(x), δ^*(x) and τ_0 from their definitions.

⇨ integral equation can be solved for the B.L. thickness, δ(x)

An approximate velocity profile, for example

$$\frac{u}{U} = a + b\eta + c\eta^2$$



$$\tau_0 \sim \left(\frac{\partial u}{dy}\right)^n$$

$$\frac{d\theta}{dx} + (\delta^* + 2\theta)\frac{1}{\theta}\frac{dU}{dx} = \frac{\tau_0}{\rho U^2}$$

Steady ,incompressible 2-D flow with no body forces. Valid for laminar and turbulent flow

O.D.E for $\theta(x)$

To solve eq. we first ”assume” an approximate velocity profile inside the B.L

Relate the wall shear stress to the velocity field

Typically the velocity profile is taken to be a polynomial in y, and the degree of this polynomial determines the number of boundary conditions which may be satisfied

EXAMPLE: $\frac{u}{U} = a + b\eta + c\eta^2 = f(\eta)$ LAMINAR FLOW OVER A FLAT PLATE:

laminar profile

later as an example

or $u = a + by + cy^2$

B.C 1-)u=0 at y=0 (η=0) $\Longrightarrow$ a=0

2-)u=U at y= δ (η =1) $\Longrightarrow$ 1=b+c

3-) $\frac{\partial u}{\partial y} = 0$ at y= δ (η =1) $\Longrightarrow$ 0=b+2c

$\Big\}$ b=2, c=-1

$$\frac{u}{U} = 2\eta - \eta^2 = 2(\frac{y}{\delta}) - (\frac{y}{\delta})^2$$

Now use the approximate velocity profile to obtain terms in the momentum integral eq.

NOTE: Using the approximate velocity profile across the B.L will reduce the momentum integral to an O.D.E for the B.L thickness, δ (x).

$$\delta^* = \int_0^\delta (1 - \frac{u}{U}) dy \qquad \eta = \frac{y}{\delta(x)} \qquad d\eta = \frac{dy}{\delta}$$

$$\delta^* = \int_0^{\eta=1} (1 - \frac{u}{U}) \delta d\eta = \delta \int_0^1 (1 - 2\eta + \eta^2) d\eta \qquad \delta^* = \delta (\eta - \eta^2 + \eta^3 \frac{1}{3}) \Big|_0^1 = \frac{\delta}{3}$$

$$\theta = \int_0^{\delta} \frac{u}{U}(1-\frac{u}{U})dy = \delta \int_0^{\eta=1} \frac{u}{U}(1-\frac{u}{U})d\eta$$

$$\theta = \delta \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)d\eta \quad \Longrightarrow \quad \theta = \frac{2}{15}\delta$$

$$\tau_0 = \mu \frac{du}{dy}\bigg|_{y=0} = \mu \frac{du}{d\eta}\frac{d\eta}{dy}\bigg|_{\eta=0} = \mu \frac{1}{\delta}\frac{du}{d\eta}\bigg|_{\eta=0} = 2\mu \frac{U}{\delta}$$

$$U(2 - \cancel{2\eta})\Big|_{\eta=0}$$

or

$$\tau_0 = 2\eta U \frac{\partial}{\partial y}\left[2(\frac{y}{\delta}) - (\frac{y}{\delta})^2 \right]\bigg|_{y=0} = 2\mu \frac{U}{\delta}$$

Momentum Integral eq. becomes

$$\frac{d\theta}{dx} + (\delta^* + 2\theta)\frac{1}{U}\frac{dU}{dx} = \frac{\tau_0}{\rho U^2}$$

$$\frac{d}{dx}\left(\frac{2\delta}{15}\right)+\left(\frac{\delta}{3}+\frac{4\delta}{15}\right)\frac{1}{U}\frac{dU}{dx}=\frac{2\mu U}{\delta\rho U^2}=\frac{2\nu}{\delta U}$$

For a flow over a flat plate ⇨ U=const. ⇨ $\frac{dU}{dx}=0$

$$\boxed{\frac{2}{15}\frac{d\delta}{dx}=\frac{2\nu}{\delta U}}$$

ODE for δ (x).solve δ (x)first then δ^*,θ,τ_0

Solving for δ,

$$\int_0^\delta \delta d\delta=\frac{15\nu}{U}\int_0^x dx \Rightarrow \frac{\delta^2}{2}=\frac{15\nu x}{U}$$

$$\delta=\sqrt{30\frac{\nu x}{U}}=5.477\sqrt{\frac{\nu x}{U}}=\frac{5.477x}{\sqrt{\mathrm{Re}_x}},\qquad \mathrm{Re}_x=\frac{Ux}{\nu},\qquad \delta^*=\frac{\delta}{3}=1.826\sqrt{\frac{\nu x}{U}}$$

$$\theta=\frac{2\delta}{15}=0.73\sqrt{\frac{\nu x}{U}},\qquad C_f=\frac{\tau_0}{\frac{1}{2}\rho U^2}=\frac{0.73}{\sqrt{\mathrm{Re}_x}}\qquad \tau_0=2\mu\frac{U}{\nu}$$

Comparing to (exact) blasius solution

$$\frac{\delta}{\delta_{blasius}} = \frac{5,477}{5} = 1.095$$

$$\frac{\delta^*}{\delta^*_{blasius}} = \frac{1,826}{1,72} = 1.061 \qquad \sim 10\%$$

$$\frac{\theta}{\theta_B} = \frac{0.73}{0.664} = 1.099$$

note:2nd order profile but it should be zero $\frac{\partial^2 u}{\partial y^2}(x,0) = -\frac{2U}{\delta^2} \neq 0$

$$\frac{u}{U} = A + B\eta + C\eta^2 + D\eta^3 + E\eta^4$$

Additional BCs need to be imposed

$$\not{u}\left.\frac{\partial u}{\partial x}\right|_{y=0} + \not{v}\left.\frac{\partial u}{\partial y}\right|_{y=0} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0}$$

$$\nu\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = \frac{1}{\rho}\frac{\partial P}{\partial x} = -U\frac{dU}{dx}$$ (=0 for flat plate)

BC#5 at y= δ $\frac{\partial^2 u}{\partial y^2} = 0$ all higher derivates should also be zero at y= δ for a smooth transition from the B-L. to the outer flow

Note: $\frac{u}{U} = 2(\frac{y}{\delta}) - (\frac{y}{\delta})^2$ 2nd order profile

BC#4 $\frac{\partial^2 u}{\partial y^2}(x,0) = -\frac{2U}{\delta^2} \neq 0$

by employing 3rd order profile , i.e $\frac{u}{U} = a + b\eta + c\eta^2 + d\eta^3$ the above condt. may be imposed

More accurate results are obtained

Flat Plate at zero incidence

Vel. Dist.

$$\frac{u}{U} = f(\frac{y}{\delta}) = f(\eta)$$

$$f(\eta) = \eta$$

$f(\eta) = 2\eta - \eta^2$,

$f(\eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3$ → $\delta = \frac{4.64}{\sqrt{\mathrm{Re}_x}}$ $C_f = \frac{0.647x}{\sqrt{\mathrm{Re}_x}}$

$f(\eta) = 2\eta - 2\eta^3 + \eta^4$ → $\delta = \frac{5.84}{\sqrt{\mathrm{Re}_x}}$ $C_f = \frac{0.685x}{\sqrt{\mathrm{Re}_x}}$

$f(\eta) = \sin(\frac{\pi}{2}\eta)$ → $\delta = 4.80$ $C_f = 0.65$

Note 1: Once the variation of τ_0 is known, viscous drag on the surface can be evaluated by integration over the area of the flat plate.

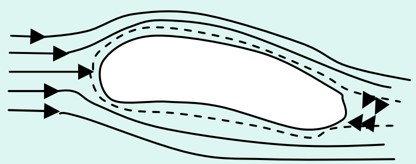
Note 2: B-L thickness at transition $\mathrm{Re}_x = 5.10^5$

$U = 30\, m/s$ $x = 0.24m$ air (ν) $\frac{\delta}{x} = \frac{5.48}{\sqrt{\mathrm{Re}_x}} = 0.00775$

$\delta = 0.00775x = 1.86mm \leftarrow$ less than 1% of development length,x.

viscous effects are confined to a very thin layer near surface of body

Boundary layer seperation



Separation ⇨ wake formation

increase in **drag**

total force exerted on body in direction of fluid motion

Boundary layers have a tendency to separate and form wake

Wake leads to large streamwise pressure differentials across the body

⇨ results in substantial pressure drag (form drag)

For large Re (10^4 or higher) bluff bodies (e.g circular cylinder) pressure drag constitutes almost all the total drag

Total drag = pressure drag + viscous drag

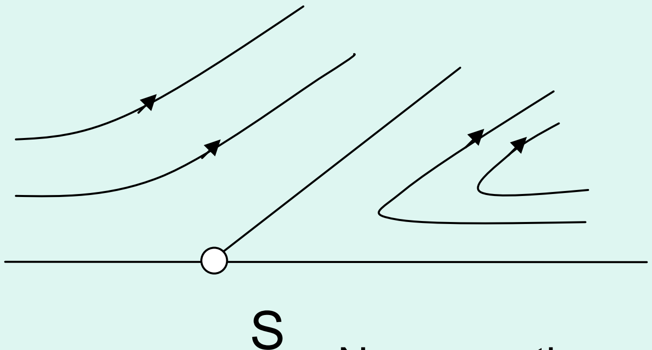
viscous drag: due to shear stress along the surface

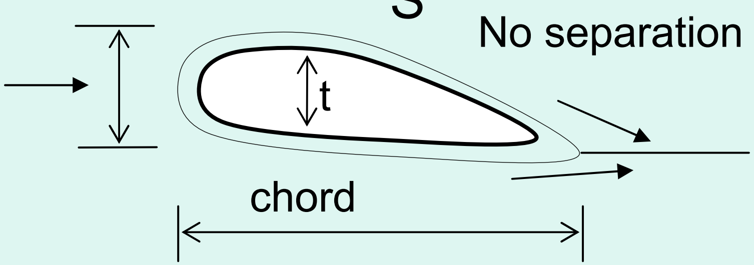
pressure drag: due to pressure differences caused by separation of flow

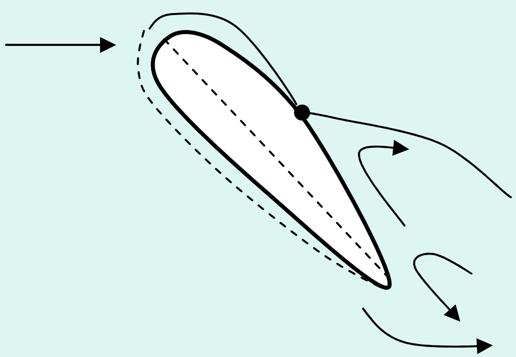
2) <u>AIRFOILS</u>-LIFT drops sharply → "STALL" due to separation

↳ force normal to flow direction

Shape of streamlines near point of separation

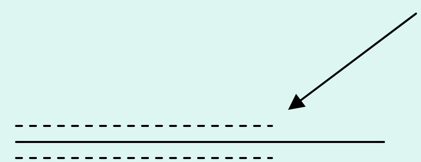






3) <u>FLAT PLATE</u>

~ No separation



REMARK: After separation point ,external (decelerating) stream ceases to flow nearly parallel to the boundary surface

Condition for separation

Pressure gradient , $\frac{dP}{dx}$

$\frac{dP}{dx} > 0$ adverse pressure gradient (decelerating external stream) increasing pressure in the flow direction

$\frac{dP}{dx} < 0$ favourable P.G and $\frac{dP}{dx} = 0$ (zero pressure gradient)

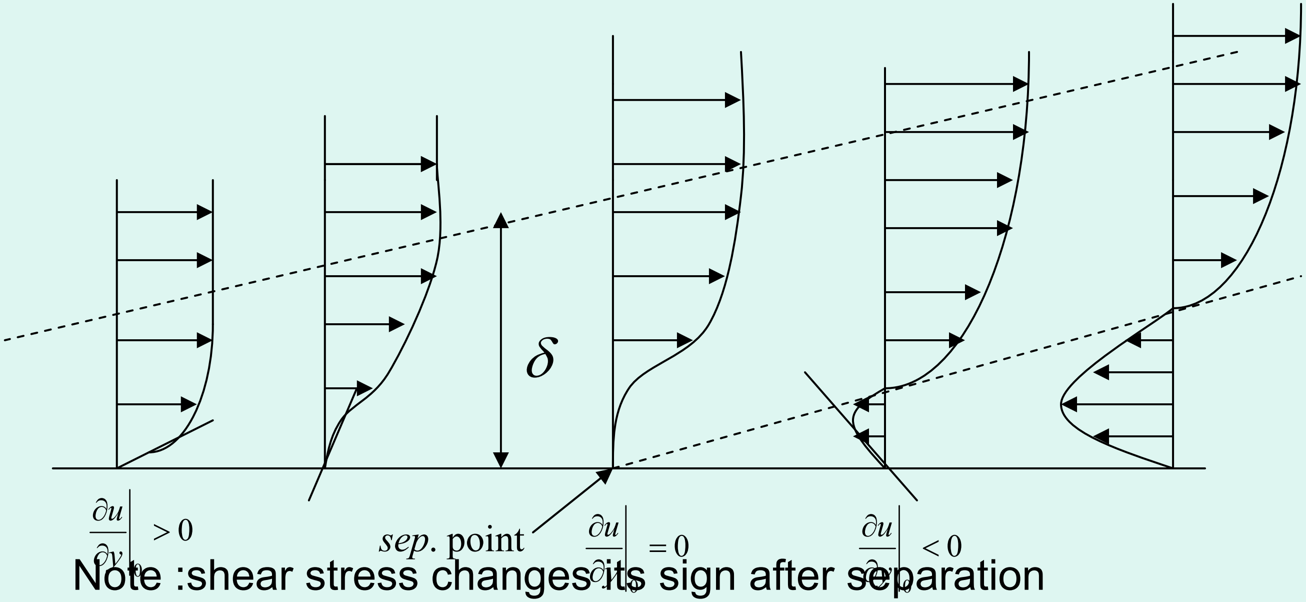
NOTE: pressure gradient along a B.L is determined by the outer flow

$$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dP}{dx} \quad \text{(Bern. Eq.)}$$

Separation occurs only for APG condition

- Momentum contained in the fluid layers adjacent to surface will be insufficient to overcome the force exerted by the pressure gradient , so that a region of reverse flow occurs.

i.e at some point downstream, the APG will cause the fluid layers adjacent to the surface to flow in a direction opposite to that of the outer flow ⇒ B.L separation



velocity profiles in a B.L near separation

Note :shear stress changes its sign after separation

Definition of separation point = point at which the shear (or velocity gradient) vanishes

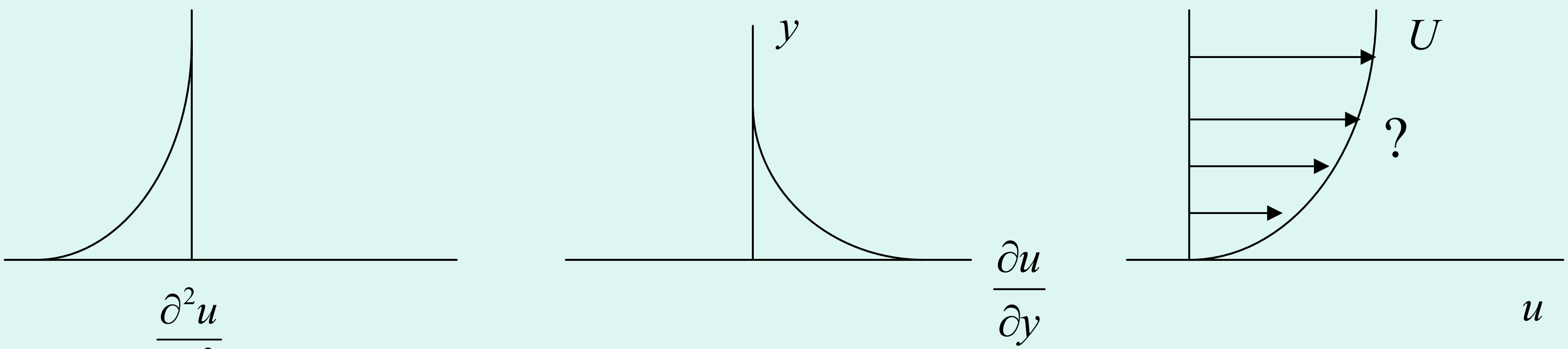
$$\frac{\partial u}{\partial y}(x,0) = 0, \quad \text{for separation}$$

- **Question** show that separation can occur only in region of adverse pressure gradient !

Steady state B.L eqs.

$$\cancel{u}\left.\frac{\partial u}{\partial x}\right|_{y=0} + \cancel{v}\left.\frac{\partial u}{\partial y}\right|_{y=0} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} \qquad \mu\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = \frac{dP}{dx} \qquad \left.\frac{\partial^3 u}{\partial y^3}\right|_{y=0} = 0$$

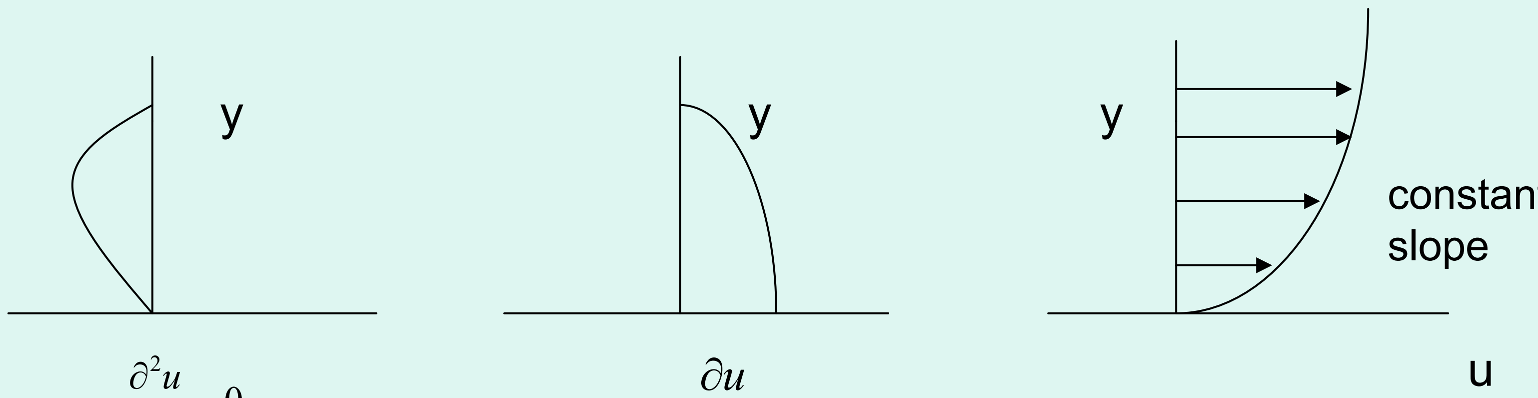
If $\dfrac{dP}{dx} < 0$



$y=0$	$\frac{\partial^2 u}{\partial y^2} = \frac{\partial P}{\partial x}$	$\frac{\partial u}{\partial y} > 0$	$u=0$
$y=\delta$	$\frac{\partial^2 u}{\partial y^2} = 0$	$\frac{\partial u}{\partial y} = 0$	$u=U$

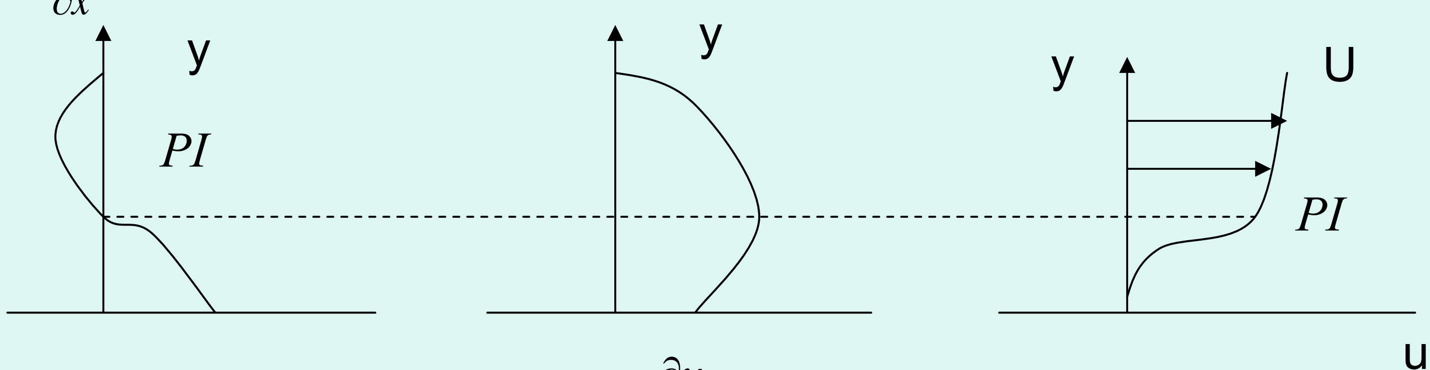
the same

case $\frac{\partial P}{\partial x} = 0$



$\frac{\partial^2 u}{\partial y^2} = 0$ $\frac{\partial u}{\partial y}$

Case $\frac{\partial P}{\partial x} > 0$ APG



$\frac{\partial^2 u}{\partial y^2} = 0$ $\frac{\partial u}{\partial y}$

PI= point of inflection where $\frac{\partial^2 u}{\partial y^2} = 0$ $\mu \frac{\partial^2 u}{\partial y^2}\Big|_{wall} = \frac{dP}{dx} > 0$

Control of separation by suction

Control of separation by variable geometry and by blowing

How to calculate the separation point ?

Goldstein

Stewartson

The Karman – Pohlhausen Approximate Method

Fourth order polynomial for u (y). **Pohlhausen** (1921)

Step #1

:coefs. a,b,c,d,e, in general, will be functions of x, so that solutions which are **not similar** may be obtained.

$$\frac{u}{U} = a + b\eta + c\eta^2 + d\eta^3 + e\eta^4$$

$$\eta = \frac{y}{\delta}$$

y=0	y= δ
u=0	u=U
$\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{U(x)}{\nu}\dfrac{dU}{dx}$ $= \dfrac{1}{\mu}\dfrac{dp}{dx}$	$\left.\dfrac{\partial u}{\partial y}\right\rvert_{y=\delta} = 0$ $\left.\dfrac{\partial^2 u}{\partial y^2}\right\rvert_{y=\delta} = 0$

$$\left.\frac{\partial^2 u}{\partial y^2}\right\rvert_{y=0} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{1}{\delta}\frac{\partial}{\partial \eta}\left(\frac{1}{\delta}\frac{\partial u}{\partial \eta}\right)$$

$$= \left.\frac{1}{\delta^2}\frac{\partial^2 u}{\partial \eta^2}\right\rvert_{\eta=0} = -\frac{U}{\nu}\frac{dU}{dx}$$

impose B.C.s

η=0   0=a

Λ:dimensionless variable; a measure of pressure gradient in outer flow

η=0 $$\frac{\partial^2 (u/U)}{\partial \eta^2} = -\Lambda = -\frac{\delta^2}{\nu}\frac{dU}{dx} = 2c$$

$\eta = 1$ 1=a+b+c+d+e

η=1   0=b+2c+3d+4e

η=1   0=2c+6d+12e

solution $\rightarrow$ a=0   b=2+$\dfrac{\Lambda}{6}$ c=-$\dfrac{\Lambda}{2}$ d=-2+$\dfrac{\Lambda}{2}$ e=1-$\dfrac{\Lambda}{6}$

$$\frac{u}{U} = F(\eta) + \Lambda G(\eta) \qquad (1)$$

where $F(\eta) = 1-(1+\eta)(1-\eta)^3$

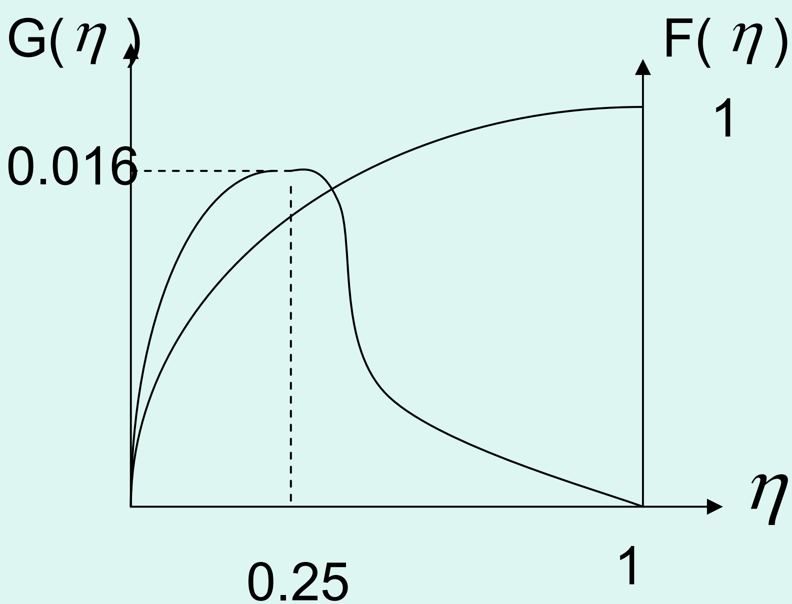
$G(\eta) = \eta(1-\eta^3)/6$

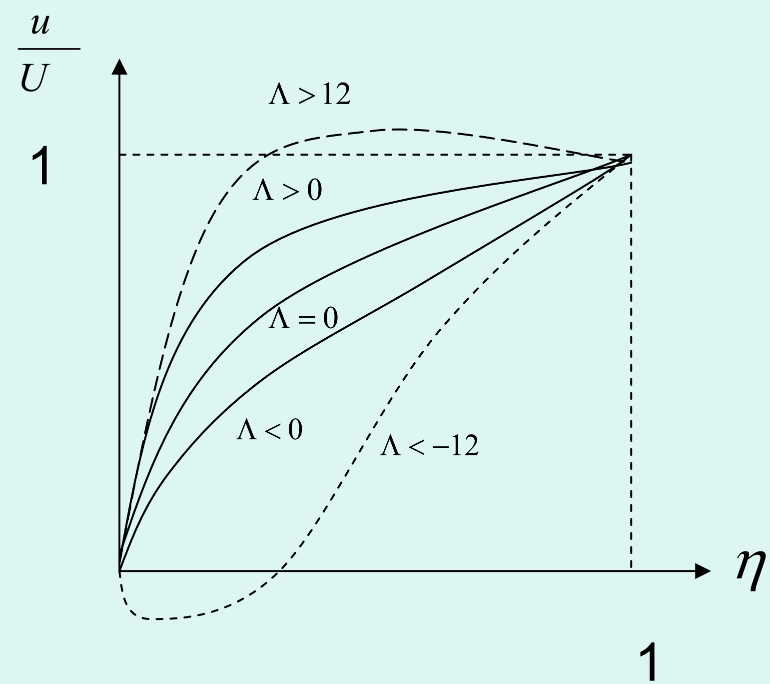
$$\Lambda(x) = \frac{\delta^2}{\nu}\frac{dU}{dx} \qquad -12 \le \Lambda \le 12$$

Pohlhausen parameter

Note : for $\Lambda = 0$ velocity profile corresponds to a flat plate

Plot function F(η) & G(η)





$\Lambda = 0:\quad \frac{\mathrm{u}}{\mathrm{U}} = F(\eta)$ Flat surface in which the represantation is a 4^{th} order polynominal

$\Lambda > 12 \quad \frac{\mathrm{u}}{\mathrm{U}} > 1$ vel. in B.L. is not expected to exceed that of the outer flow locally.

So Λ must be less than 12

$\Lambda < -12 \quad \Rightarrow$ negative velocity $\therefore$ reverse flow.B.L. theory is not applicable after separation

Step#2 Displacement thickness δ^*

$$\delta^*(x) = \int_0^\delta (1 - \frac{u}{U})dy = \delta\int_0^1 (1 - \frac{u}{U})d\eta$$

$$= \delta\int_0^1 \left[(1+\eta)(1-\eta)^3 - \frac{\Lambda}{6}\eta(1-\eta)^3 \right] d\eta = \delta(\frac{3}{10} - \frac{\Lambda}{120}) \qquad (2)$$

momentum thickness

$$\theta(\mathrm{x}) = \delta\int_0^1 \frac{u}{U}(1 - \frac{u}{U})d\eta = \delta(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072}) \qquad (3)$$

wall shear stress : τ_0

$$\tau_0 = \mu \frac{U}{\delta} \frac{\partial(u/U)}{\partial\eta}\bigg|_{\eta=0}$$

$$\boxed{\tau_0 = \mu \frac{U}{\delta} \underbrace{(2 + \frac{\Lambda}{6})}_{b}}$$

Step #3 Plug into the general momentum eq. Multiply the mom. Eq. by $\frac{U\theta}{\nu}$

$$\frac{U\theta}{\nu}\frac{d\theta}{dx}+(2\theta+\delta^*)\frac{\theta}{\nu}\frac{dU}{dx}=\frac{\tau_0\theta}{\mu U} \qquad \text{or}$$

$$\frac{1}{2}U\frac{d}{dx}\left(\frac{\theta^2}{\nu}\right)+\left(2+\frac{\delta^*}{\theta}\right)\frac{\theta^2}{\nu}\frac{dU}{dx}=\frac{\tau_0\theta}{\mu U} \qquad (5)$$

$\Lambda(x)=\frac{\delta^2}{\nu}\frac{dU}{dx}$ evaluate each term in terms of $\Lambda(x)$

$$\underline{\frac{\theta^2}{\nu}\frac{dU}{dx}=\frac{\theta^2}{\delta^2}}\Lambda=\left(\frac{37}{315}-\frac{\Lambda}{945}-\frac{\Lambda^2}{9072}\right)^2\Lambda=K(x) \qquad \frac{\theta^2}{\nu}\frac{dU}{dx}=K(x)$$

$$\frac{\delta^*}{\theta}=\frac{\left(\frac{3}{10}-\frac{\Lambda}{120}\right)}{\left(\frac{37}{315}-\frac{\Lambda}{945}-\frac{\Lambda^2}{9072}\right)}=f(K) \qquad (6)$$

f(Λ) → f(x) but K=K(x) ⇒ f(K)

$$\boxed{\frac{\tau_0\theta}{\mu U}=g(K)}\ ,\quad g(K)=\left(2+\frac{\Lambda}{6}\right)\left(\frac{37}{315}-\frac{\Lambda}{945}-\frac{\Lambda^2}{9072}\right)$$

$$\tau_0 = \mu \frac{U}{\delta}(2 + \frac{\Lambda}{6})$$

$$\frac{1}{2} U \frac{d}{dx}(\frac{\theta^2}{\nu}) + \left[2 + f(K)\right] K = g(K) \qquad (7)$$

where $K = \frac{\theta^2}{\nu}\frac{dU}{dx} = K(x)$

Now, let us take $Z = \frac{\theta^2}{\nu}$ as the new dependent variable so that $K = Z\frac{dU}{dx}$ and the mom,int. becomes

$$U\frac{dZ}{dx} = 2\left\{g(K) - \left[2 + f(K)\right]K\right\} = H(K) \quad \text{or} \quad \boxed{U\frac{dZ}{dx} = H(K)} \qquad (8)$$

H(K) is known (1st order nonlinear, ODE for Z, solve numericallay, start x=0 $\rightarrow$ stop Λ=-12 [separation])

but complex H(Λ)

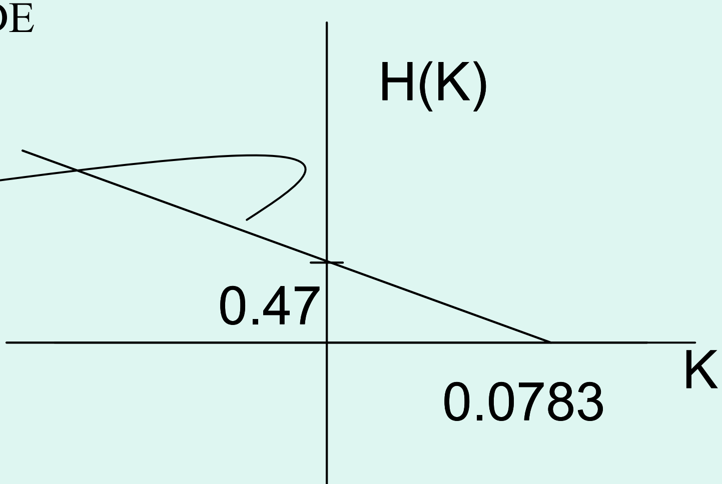
ODE for Z(x) - mom. int. reduces to above form IVP for ODE

for any Λ(x) $\rightarrow$ K & H(K) may be evaluated

H(K)=0.47-6K (9)

approximation

Linear in K over the range of interest



Mom. Int. eq. becomes

$$U\frac{dZ}{dx} = 0.47 - 6K = H(K) = 0.47 - 6Z\frac{dU}{dx} \qquad \text{or}$$

$$\frac{1}{U^5}\frac{d}{dx}(ZU^6) = 0.47$$

$$Z(x) = \frac{0.47}{U^6(x)}\int_0^x U^5(\varsigma)d\varsigma$$

Mom. int. may be expressed interms of this quadrature

then , since $Z = \theta^2/\nu$, the value of θ will be

$$\boxed{\theta^2(x) = \frac{0.47\nu}{U^6(x)}\int_0^x U^5(\varsigma)d\varsigma} \qquad (10)$$

Procedure: Potential flow problem should be solved to yield the outer velocity U(x) (for a given boundary shape)

Use eq. (10) to evaluate the momentum thickness $\theta(x)$

Pr*essure* parameter Λ(x) may be evaluated from the relation

$$\frac{\theta^2}{\nu}\frac{dU}{dx} = K(x) = \left(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072}\right)^2 \qquad (11) \qquad \text{difficult to find } \Lambda(x)$$

having found Λ(x), δ(x) is evaluated from eq. (3)

$$\theta(x) = \delta\left(\frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072}\right)$$

and δ^* eq. (2)

$$\delta^* = \delta\left(\frac{3}{10} - \frac{\Lambda}{120}\right)$$

$$\frac{u}{U} = F(\eta) + \Lambda G(\eta)$$

← vel. distribution eq (1)

shear stress at the surface is given by eq. (4)

$$\tau_0 = \mu \frac{U}{u}\left(2 + \frac{\Lambda}{6}\right)$$

In practice it is difficult to evaluate the quality $\Lambda(x)$ from eq (11) unless Λ is a constant

<u>Instead :</u> choose specific functions $\Lambda(x)$ and use foregoing eqs. to determine the outer-flow vel. & hence the nature of the boundary shape

EXAMPLE Karman-Pohlhausen approx. applied to the case of flow over a flat plate

$$U = constant \qquad \text{eq. (10)} \rightarrow \theta^2 = 0.47\frac{\nu x}{U} \rightarrow \theta = 0.686\sqrt{\frac{\nu \text{x}}{\text{U}}} \qquad \theta = 0.686\frac{\text{x}}{\sqrt{\text{Re}_\text{x}}}$$

$$\frac{dU}{dx} = 0 \Rightarrow \text{eq. (11)}$$

$$\Downarrow$$

$$\Lambda = 0 \ \left(\frac{\delta^2}{\nu}\frac{dU}{dx}\right)$$

From eq. (3) $\theta(x) = \frac{37}{315}\delta \rightarrow \boxed{\delta = 5.84\sqrt{\frac{\nu x}{U}} = \frac{5.84x}{\sqrt{Re_x}}}$

$eq.(2) \Rightarrow \delta^* = \delta\frac{3}{10} \rightarrow \boxed{\delta^* = \frac{1.75x}{\sqrt{Re_x}}}$

$eq.(4) \Rightarrow \tau_0 = \mu\frac{U}{\delta}2 \Rightarrow \boxed{\frac{\tau_0}{\frac{1}{2}\rho U^2} = \frac{0.686}{\sqrt{Re_x}}}$ Θ

3.5% error

Exact → 0.664

4th order vel. pr → 0.686

2nd order vel. pr → 0.73

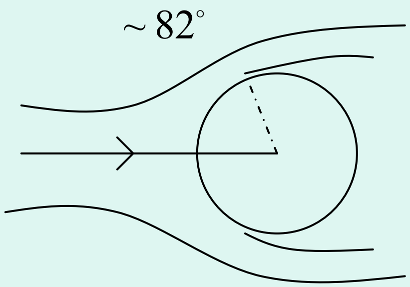
STABILITY OF STEADY FLOWS

Boundary – Layers

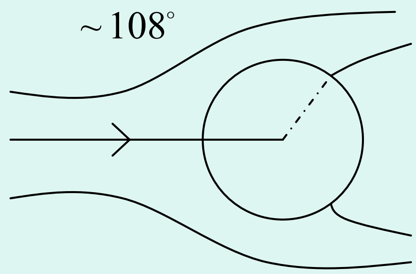
Instabilities

Usually laminar flow becomes turbulent flow

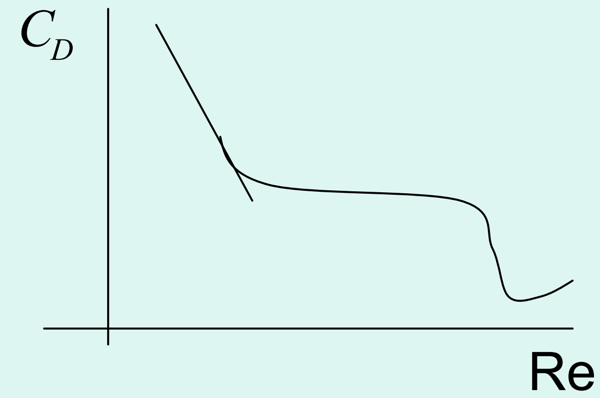
EXAMPLE: Flow over a circular cylinder



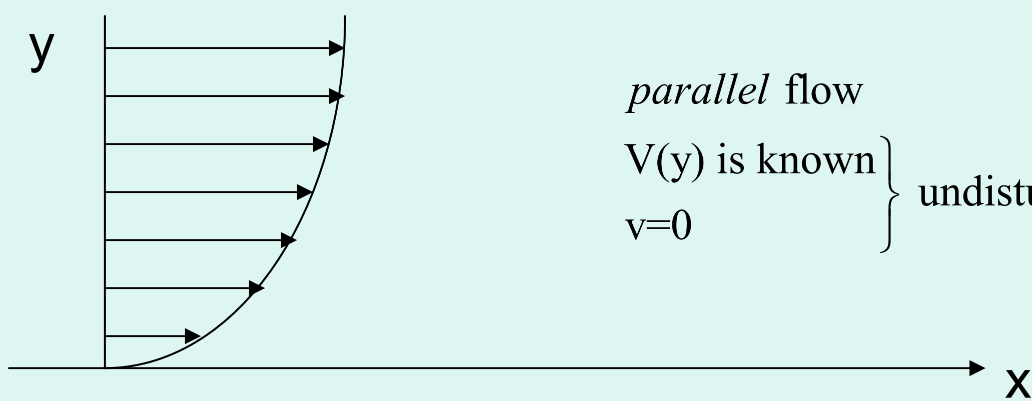
Laminar B.L



Turbulent B.L



- Significant drop in the drag coefficent C_D
- Due to vel. profile difference between lam. & turb. flow



parallel flow

V(y) is known
v=0

} undisturbed flow "base flow"

Linear Stability Analysis: **The Method of Small Perturbations**

Introduce arbitrary **small** (infinitesimal) disturbance into the flow eqs. & determine whether this disturbance *grows or decays* with time

if the disturbance grows with time, the flow (the B.L) will be classified as **unstable**

if the disturbance decays with time, the flow (the B.L) will be classified as **stable**

marginal stability (neutral): the disturbance neither grows nor decays

Non linear stability analysis: no restriction on disturbance size

A1

Introduce **small** disturbance to the velocity profile

u(x,y,t) = V(y) + u'(x,y,t)

$$u(x,y,t) = V(y) + u'(x,y,t)$$
$$v(x,y,t) = 0 + v'(x,y,t)$$
$$p(x,y,t) = p_0(x) + p'(x,y,t)$$

$$\textit{where} \quad \left|\frac{\mathrm{u'}}{\mathrm{V}}\right| << 1 \quad ; \quad \left|\frac{v'}{\mathrm{V}}\right| << 1 \quad ; \quad \left|\frac{\mathrm{p'}}{\mathrm{p_0}}\right| << 1$$

A2

Substitute A1 into the N-S eqs. & continuity

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0$$

$$x\ ;\ \frac{\partial u'}{\partial t}+(V+u')\frac{\partial u'}{\partial x}+v'(\frac{dV}{dy}+\frac{\partial u'}{\partial y})=-\frac{1}{\rho}(\frac{dp_0}{dx}+\frac{dp'}{dx})+\nu(\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 u'}{\partial y^2})$$

$$y\ ;\ \frac{\partial v'}{\partial t}+(V+u')\frac{\partial v'}{\partial x}+v'\frac{\partial v'}{\partial x}=-\frac{1}{\rho}\frac{dp'}{dy}+\nu(\frac{\partial^2 v'}{\partial x^2}+\frac{\partial^2 v'}{\partial y^2})$$

A3

When the perturbation is zero, the above eqs. reduce to

$$\boxed{0=-\frac{1}{\rho}\frac{dp_0}{dx}+\nu\frac{d^2V}{dy^2}}$$

Undisturbed flow

(parallel)

A4

Drop term A3 in x-mom. Eq.

Since the perturbation is assumed to be small, ***products of all primed quantities may be neglected as being small***

Thus , **Linearized** eqs. governing the motion of the disturbances are

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0$$

$$X \;\; ; \;\; \frac{\partial u'}{\partial t}+V\frac{\partial u'}{\partial x}+v'\frac{dV}{dy}=-\frac{1}{\rho}\frac{dp'}{dx}+\nu\left(\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial y^2}\right)$$

$$Y \;\; ; \;\; \frac{\partial v'}{\partial t}+V\frac{\partial v'}{\partial x}=-\frac{1}{\rho}\frac{dp'}{dy}+\nu\left(\frac{\partial^2 v'}{\partial x^2}+\frac{\partial^2 v'}{\partial y^2}\right)$$

A5 Introduce a perturbation stream - function ψ (to reduce number or eqs. by one)

$$u'=\frac{\partial\psi}{\partial y} \quad , \quad v'=-\frac{\partial\psi}{\partial x}$$

In terms of this stream function the governing eqs. become

$$\frac{\partial^2\psi}{\partial y\partial t}+V\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{dV}{dy}=-\frac{1}{\rho}\frac{dp'}{dx}+\nu\left(\frac{\partial^3\psi}{\partial x^2\partial y}+\frac{\partial^3\psi}{\partial y^3}\right)$$

$$-\frac{\partial^2\psi}{\partial x\partial t}-V\frac{\partial^2\psi}{\partial x^2}=-\frac{1}{\rho}\frac{dp'}{dy}-\nu\left(\frac{\partial^3\psi}{\partial x^3}+\frac{\partial^3\psi}{\partial x\partial y^2}\right)$$

A6

Eliminate the pressure term by forming $\frac{\partial^2 p'}{\partial x \partial y}$ mixed derivative, above two eqs.

above two eqs. may be reduced to one ,

$$(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x})(\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2}) - \frac{d^2 V}{dy^2}\frac{\partial \psi}{\partial x} = \nu(\frac{\partial^4 \psi}{\partial y^4} + 2\frac{\partial^4 \psi}{\partial y^2 \partial x^2} + \frac{\partial^4 \psi}{\partial^4 x})$$

Stream function for the disturbance must satisfy this linear , 4^{th} order , PDE

A7

Since the disturbance under consideration is arbitrary in form, Perturbation stream function may be represented by the following Fourier – Integral:

$$\psi(x, y, t) = \int_0^\infty \phi(y) e^{i\alpha(x-ct)} d\alpha$$

c:time coefficient

$\alpha : real$ & positive (inverse wavelength)

$$\lambda = \frac{2\pi}{\alpha} \quad [\mathrm{m}]$$

$\mapsto$ wave length of the disturbances *note*: time variation $e^{-i\alpha ct}$

$\underline{\underline{c = c_r + c_i i}}$ $\rightarrow$ if $c_i > 0$ $\rightarrow$ $e^{-i\alpha ct}$ $\rightarrow$ ∞ as $t \rightarrow \infty$

disturbance will grow $\rightarrow$ unstable

<u>in general complex number:</u> if $c_i < 0$ $\rightarrow$ $e^{-i\alpha ct}$ $\rightarrow$ 0 as $t \rightarrow \infty$

disturbance will decay $\rightarrow$ stable

$c_i = 0$ $\rightarrow$ neutrally stable

(c=0)

Plug in <u>A6</u> yields the integro – differential equation:

$$\int_0^\infty \left[\left(-i\alpha c + i\alpha V\right)(\phi'' - \alpha^2 \phi) - i\alpha\phi V'' \right] e^{i\alpha(x-ct)} d\alpha$$

$$= \int_0^\infty \nu \left[(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi) \right] e^{i\alpha(x-ct)} d\alpha \quad , \quad i^2 = -1 \quad i^4 = 1$$

$$\phi'' = \frac{d^2\phi}{dy^2} \quad , \quad \phi'''' = \frac{d^4\phi}{dy^4}, \ldots$$

Above equation should be valid for arbitrary α. *Thus*, the integrand should vanish (because eq. should be valid for arbitrary disturbance)

$$\boxed{(V-c)(\phi''-\alpha^2\phi)-V\phi=\frac{\nu}{i\alpha}(\phi''''-2\alpha^2\phi''+\alpha^4\phi)} \qquad (A)$$

Orr-Sommerfield equation

B.C disturbance should vanish at the surface y=0 and at the edge of the Boundary Layer

$$u'(x,y=0,t)=0 \quad , \quad v'(x,y=0,t)=0$$
$$u'(x,y,t)=v'(x,y,t)=0 \qquad \text{as} \quad y\rightarrow\infty$$

in terms of the stream function ψ(y)

$$u'=\left.\frac{\partial\psi}{\partial y}\right|_{y=0}=0 \quad \rightarrow \quad \phi'(0)=0$$

$$v'=-\left.\frac{\partial\psi}{\partial x}\right|_{y=0}=0 \quad \rightarrow \quad \phi(0)=0$$

$$\phi'(y)=\phi(y)\rightarrow 0 \quad \text{as} \quad y\rightarrow\infty \qquad (B)$$

Solution of the Orr – Sommerfeld Equation

Undisturbed vel. profile V(y) and disturbance wavelength α is specified

$$V(y)\ \ \&\ \ \alpha\ \ \text{known}$$

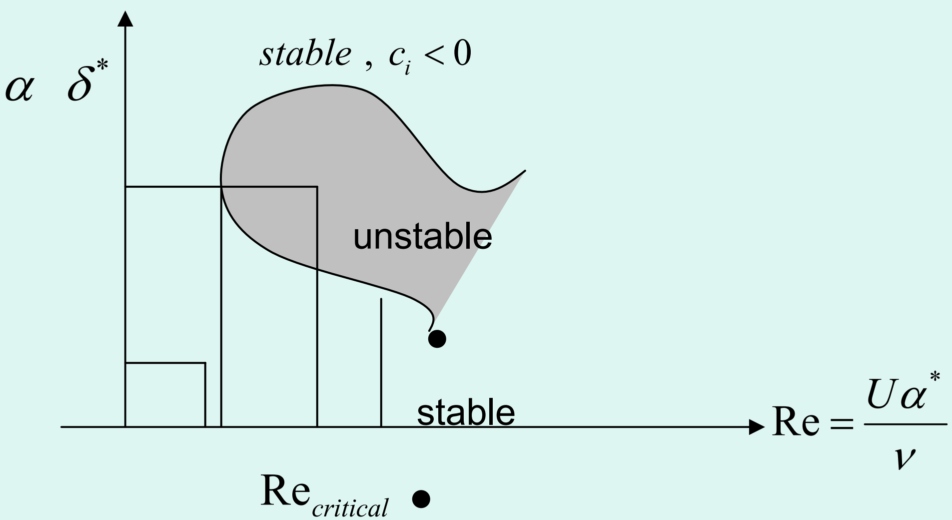
Eq. (A) with BC.(B)represent an eigenvalue problem for the time coefficient , c

$$c = c_r + i\, c_i \quad , \quad c_i < 0 \quad \Rightarrow \quad \text{flow stable}$$

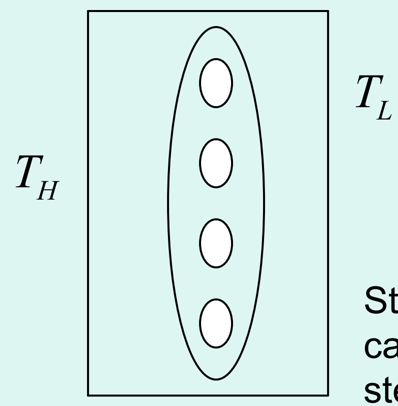
$$c_i > 0 \quad \Rightarrow \quad \text{flow unstable}$$

$$i\alpha(x - ct) \quad , \quad c_i = 0 \quad \Rightarrow \quad \text{neutral stablity}$$

$$\psi = \phi(y)\, e^{i\alpha(x-ct)}$$



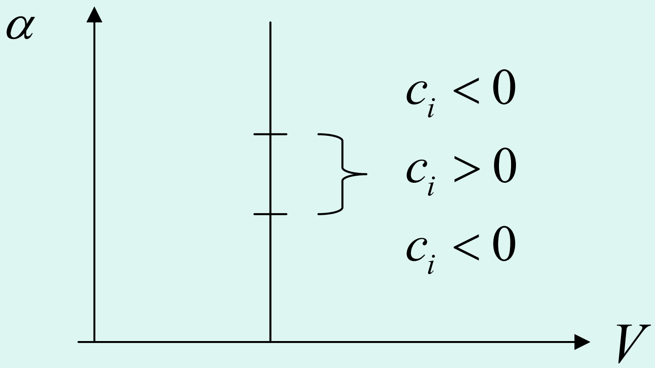
Stability Diagram:



$Ra = Gr\ .\ \mathrm{Pr}$

Steady laminar flow
can become another
steady lam. flow

Orszag (1971):



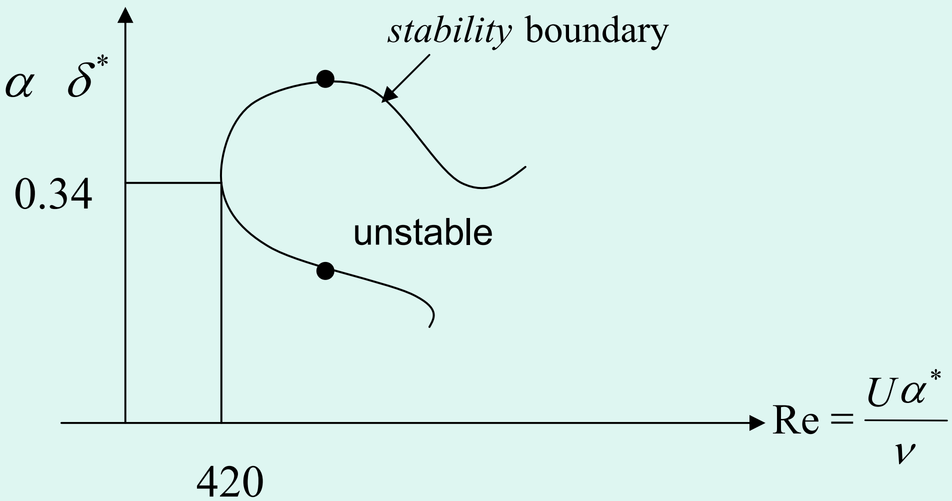
typical stability – calculation result for fixed V , α is varied. Then, by considering all possible values of the undisturbed B.L vel. (which less than the outer –flow vel.) a stability diagram is constructed

All possible values of $V(y)$ in the range

$$0 \le V(y) \le U(x)$$

Flow over a flat surface

$$\mathrm{Re}_{cr} = \frac{U\alpha^*_{cr}}{\nu} = 420$$



$$Schlichting\ 575 = \mathrm{Re}_{cr}$$

Re >420 → arbitrary disturbance will be unstable.

⇨ manifest themselves in the form of turbulence

FREE – SHEAR FLOWS (LAYERS)

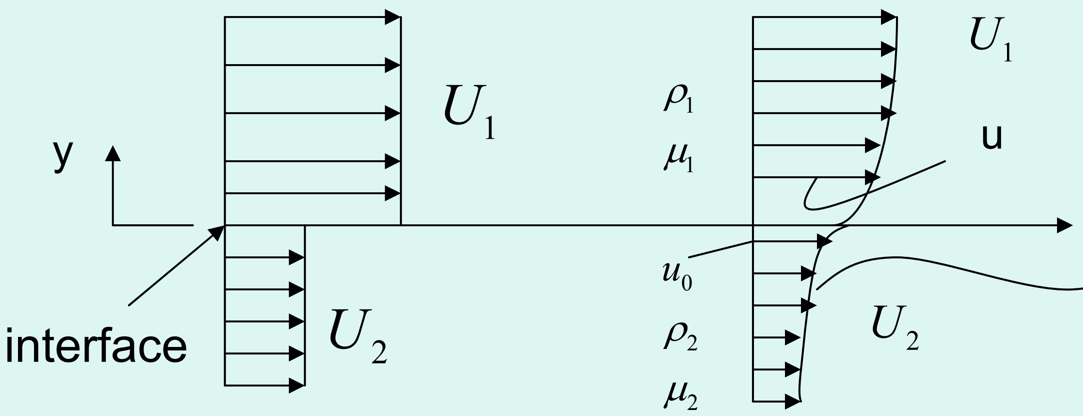
Unaffected by walls

Develop and spread in an open ambient fluid

Possess vel. gradient created upstream mechanism

viscous diffusion $\Leftrightarrow$ *convective deceleration*

EXAMPLE: 1) The free-shear layer between parallel moving streams:



S - shaped free-shear layer is due to viscous diffusion

At x=0 , upper free stream U_1 } meets as x=0
lower free stream U_2

U_1 & U_2 uniform

For each stream , can define a Blasius – type similarity variable

Lock(1951) – two different fluids with physical parameters

$$(\rho_1, \mu_1) \ \& \ (\rho_2, \mu_2)$$

$$\eta_j = y\sqrt{\frac{U_1}{2x\nu_j}} \ , \ f_j' = \frac{u_j}{U_1} \ , \ \text{j=1,2}$$

$$\psi_{\text{j}} = \sqrt{2\nu_j U_1 x}\, f_j(\eta_j)$$

Following the same procedure as in derivation of Blasius equation, one can obtain Blasius-type eq. for <u>each</u> layer

$$f_j''' + f_j f_j'' = 0 \qquad \text{j=1,2}$$

B.C.s 1) $f_1'(+\infty)=1$ asymptotic approach to the two stream velocities

$$y \to (-\infty) \;\to\; \eta \to -\infty \quad \Rightarrow \quad u_2 \to U_2 \to f_2' = \frac{U_2}{U_1}$$

$$u_1 \to U_1 \text{ as } \eta \to +\infty$$

B.C.s 2) Kinematics equality , $u_1 = u_2$ and $v_1 = v_2$ at the interface

$$\eta_j = 0 \quad \to \quad f_1'(0) = f_2'(0) \neq 0 = u_0 \qquad u_1 = u_2$$

$$f_1(0) = f_2(0) = 0 \qquad v_1 = v_2 \;\Rightarrow\; \frac{\partial \psi_1}{\partial x} = \frac{\partial \psi_2}{\partial x}$$

B.C.s 3) Equality of shear stress at the interface

$$\mu_1 \frac{\partial u_1}{\partial y}(0) = \mu_2 \frac{\partial u_2}{\partial y}(0) \quad \text{or} \quad \eta_j = \frac{y\sqrt{U_1}}{\sqrt{2x\nu_i}}$$

$$\mu_1 \left.\frac{\partial u_1}{\partial y}\right|_{y=0} = \mu_1 U_1 \left.\frac{\partial f_1'}{\partial \eta_1}\right|_0 \frac{\partial \eta_1}{\partial y} = \mu_1 U_1 f_1'' \frac{\sqrt{U_1}}{\sqrt{2x\nu_1}} \qquad (1)$$

$$\mu_2 \frac{\partial u_2}{\partial y}\bigg|_{y=0} = \mu_2 U_1 f_2\,'' \frac{\sqrt{U_1}}{\sqrt{2x\nu_2}} \qquad (2)$$

$$(1)=(2) \quad \Rightarrow f_1\,''(0)\mu_1 \frac{1}{\sqrt{\nu_1}} = f_2\,''(0)\mu_2 \frac{1}{\sqrt{\nu_2}} \quad \rightarrow \quad \underline{\underline{f_1\,''(0) = \sqrt{\frac{\rho_2\mu_2}{\rho_1\mu_1}} f_2\,''(0)}}$$

$$f_1\,''(0) = \sqrt{k}\, f_2\,''(0) \qquad k = \frac{\rho_2\mu_2}{\rho_1\mu_1}$$

Most practical cases

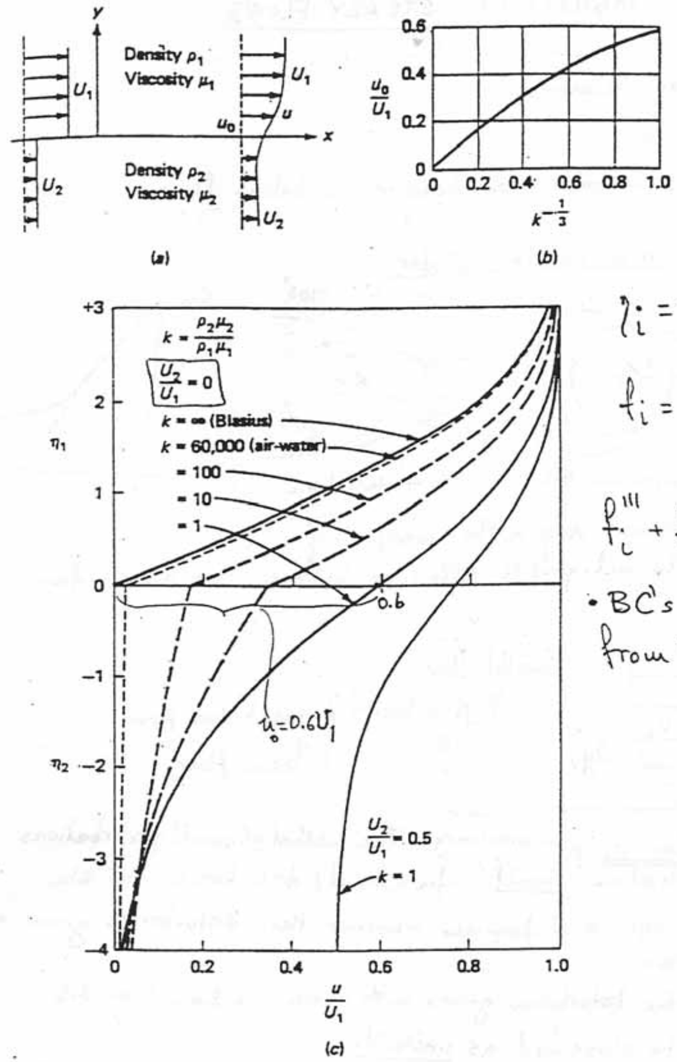
Case 1 : k=1 (identical fluids) $\rho_1 = \rho_2$; $\mu_1 = \mu_2$

Case 2 : a gas flowing over a liquid k>>1

ex. air-water interface k ≈ 60000 $\Rightarrow$ $\underline{\underline{\sqrt{k} \approx 245}}$

Free-shear layer between two different streams:

(F3)



$U_2 = 0$

$$\eta_i = y\sqrt{\frac{U_1}{2\nu_i x}}$$

$$f_i = \frac{u_i}{U_1} \qquad i = 1, 2$$

$$f_i''' + f_i f_i'' = 0$$

- BC's different from the flat plate

FIGURE 4-17
Velocity distribution between two parallel streams of different properties: (*a*) geometry; (*b*) velocities at the interface ($U_2 = 0$). [*After Lock (1951).*] (*By permission of The Clarendon Press, Oxford.*)

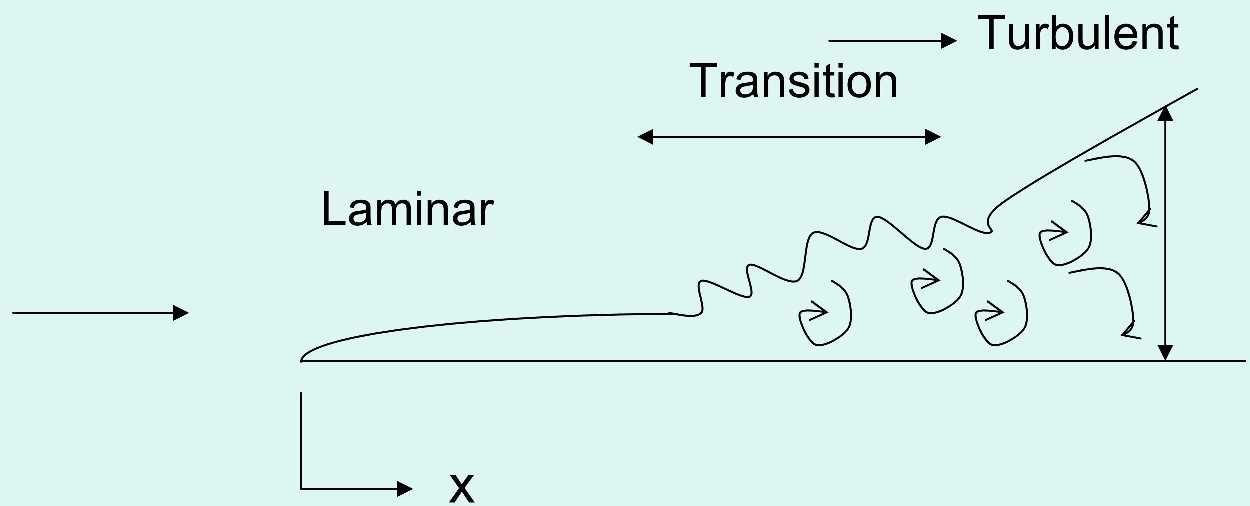
- as k increases, the lower layer moves slower.
- $k = 1$ & $U_2 = 0$.
  - Vel. dist. is not antisymmetric
  - Interface velocity is greater than $0.5U_1$ ($u_0 \approx 0.6U_1$) because two layers have different convective deceleration.

TURBULENCE

INTRODUCTION

LAMINAR FLOW : Smooth , orderly flow ← limited to finite values of critical parameters: Re, Gr, Ta, Ri

Beyond the critical parameter, Laminar flow is **unstable** a new flow regime ⇒ turbulent flow



Characteristics

1) **Disorder** : not merely white noise but has spatial structure (Random variations)

2) **Eddies** : (or fluid packets of many sizes) Large & small varies continuously from shear – layer thickness δ down to the Kolmogorov length scale , $L = (\frac{\nu^3 \delta}{U^3})^{1/4}$

3) **Enhanced mixing** in laminar flow ⇨ molecular action

mixing in turbulent flow ⇨ turbulent eddies actively about in 3-D and cause rapid diffusion of mass, momentum & energy

Heat transfer & friction are greatly enhanced compared to Lam. Flow

4) **Fluctuations :** (in pressure, vel. & temp.)

Velocity fluctuates in all three directions

5) **Self-sustaining motion**: Once trigged turbulent flow can maintain. Itself by producing new eddies to replace those lost by viscous dissipation

Experimental measurement :

Hot-wire anemometer

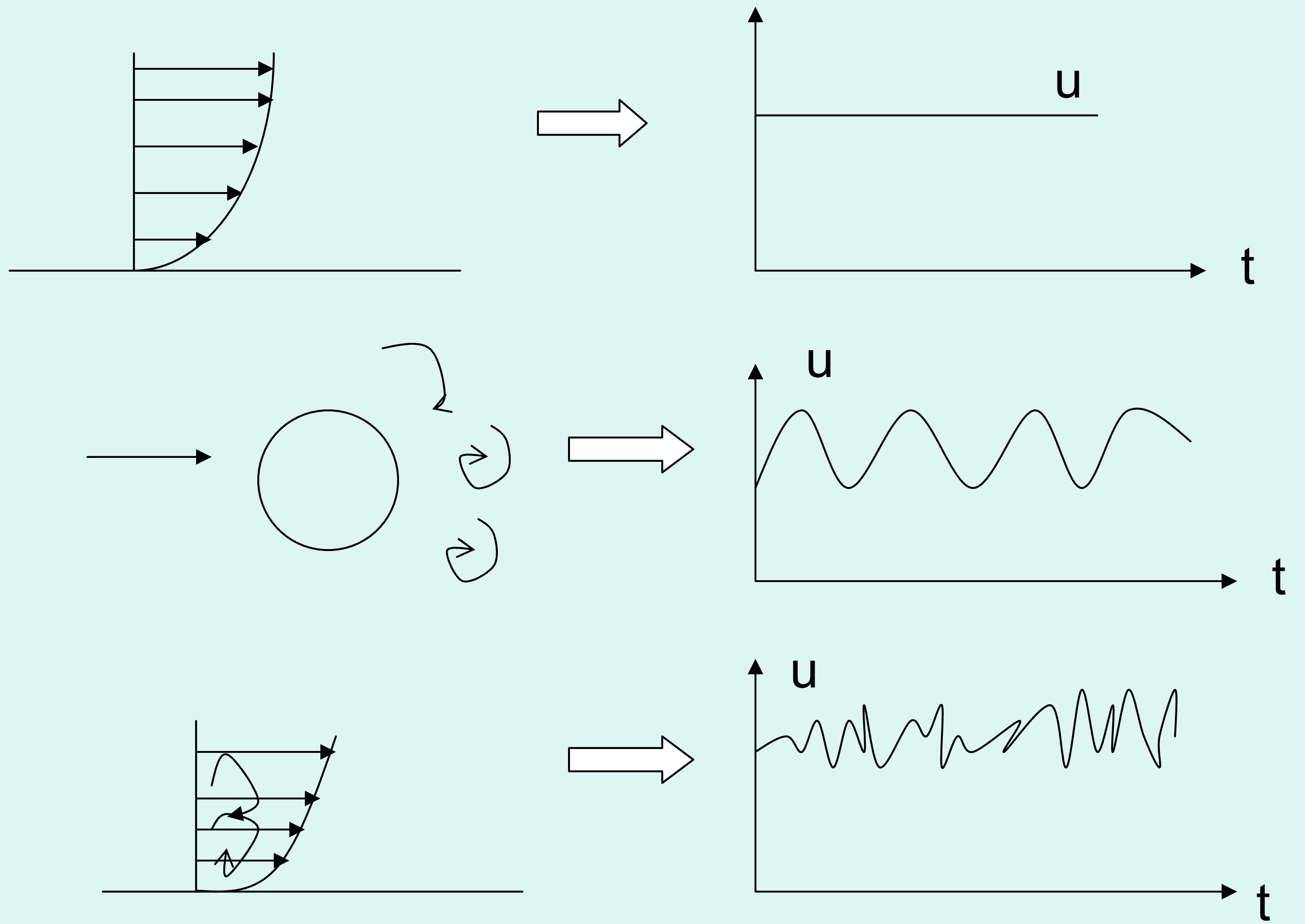
measure fluctuations in velocity via heat transfer

Examine change in resistance assoc. with temp. (use wire ~ 0.0001” dia.)

Laminar B.L

Shedding cylinder

Turbulent B.L



Mathematical Description

N-S eqs. do apply to turbulent flow

Direct Numerical Simulation :Solve the N-S eqs. directly using computers

Problem: wide range of flow scales involved ⇨ solutions requires supercomputers and even then are limited to very low Reynolds numbers

Mesh points : beyond the capacity of present computers (trillions)

Eq. Turbulent flow in a pipe

At $Re_d = 10^7 \rightarrow$ requires 10^{22} numerical operatious $\Rightarrow$ computation would take thousand years to complete (for the fine details of the turbulent flow)

Direct numerical simulation DNS

Because of complexity of the fluctuations, a purely numerical computation of turbulent flow has only been possible in a few special cases.

Therefore, consider time average of turbulent motion

Difficulties in setting up eqs. of motion for mean motion

Turbulent fluctuations $\Leftrightarrow$ coupled with mean motion

Time averaging N-S $\Rightarrow$ additional terms (determined by turbulent fluctuations)

Additional unknowns in computation of mean motion

We have more unknowns than eqs.

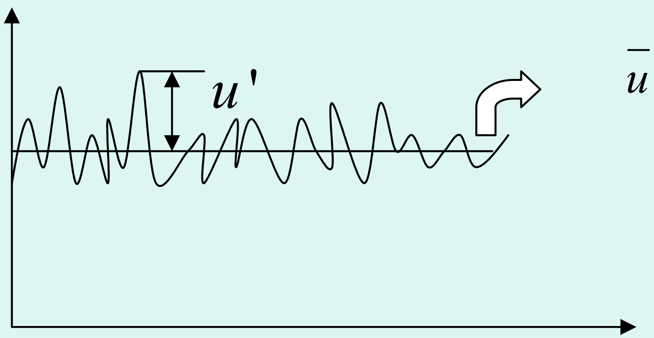
To close system of eqs. of motion $\Rightarrow$ need additional eqs

These eqs. can no longer be set up purely from the balances of mass momentum & energy

But, they are model eqs. which model relation between the fluctuations & mean motion

called turbulence modelling $\Rightarrow$ central problem in computing the mean motion of turbulent flows

Mean Motion & Fluctuations



$\bar{u}$, time average value

Decompose the motion into a mean motion & a fluctuating motion

$$u = \bar{u} + u'$$
$$v = \bar{v} + v'$$
$$w = \bar{w} + w'$$
$$p = \bar{p} + p'$$

In compressible turbulent flows

$$\rho = \bar{\rho} + \rho' \quad ; \quad T = \bar{T} + T'$$

Average is formed as the time average at a fixed point in space

$$\bar{u} = \frac{1}{T} \int_{t_0}^{t_0+T} u \, dt$$

$\leftarrow$ integral is to be taken over a sufficently large time interval T so that $\bar{u} \neq f(t)$

Characterization of fluctuation $\Rightarrow$ RMS

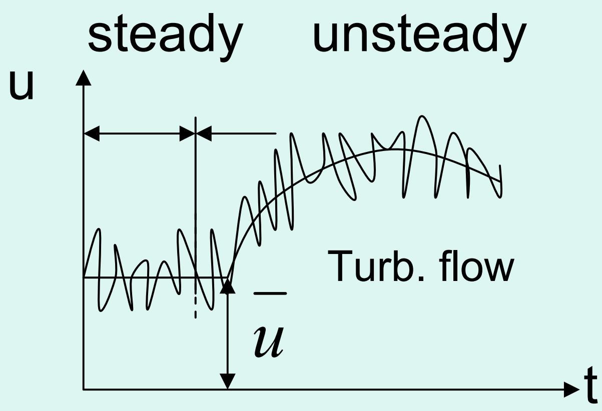
$$\bar{u} = \left\{ \frac{1}{T} \int_0^T (u - \bar{u})^2 \, dt \right\}^{1/2}$$

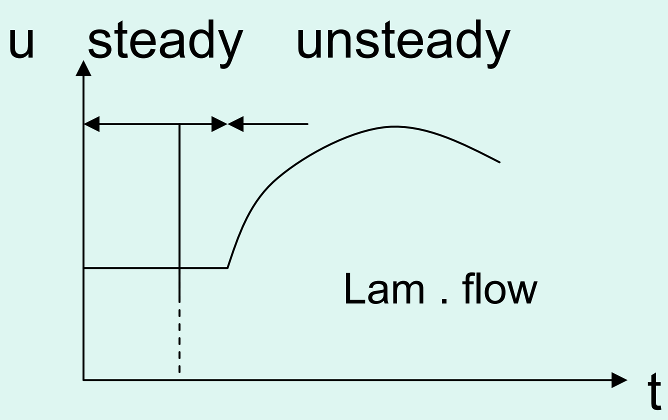
$$u' = g(t)$$
$$u = \bar{u} + u' = f(t)$$

By definition time average of fluctuating quautities are zero i.e.

$$\overline{u'} = 0 \quad , \quad \overline{v'} = 0 \quad , \quad \overline{w'} = 0 \quad , \quad \overline{p'} = 0$$

First assume that mean motion indep. of time $\Rightarrow$ steady turbulent flow



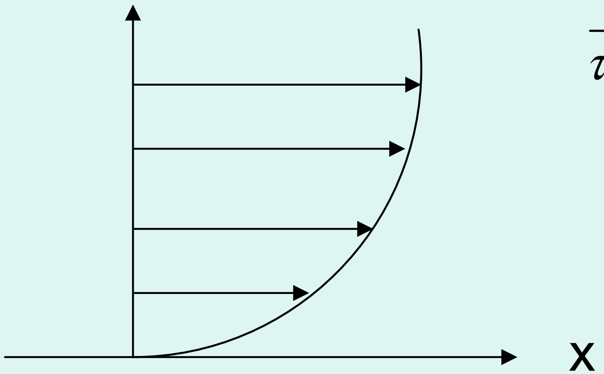


Fluctuations u', v', w' influence the progrees of mean motion $\overline{u}$, $\overline{v}$, $\overline{w}$, so that mean motion exhibit an apparent increase in resistance aganist deformation.Increased apparent viscosity *is cenral of* all theoretical considerations on turbulent flow

Rules of computation

$$\overline{\overline{u}} = \overline{u} \quad , \quad \overline{u+v} = \overline{u} + \overline{v} \quad , \quad \overline{\overline{u}\,.v} = \overline{u}\,.\,\overline{v}$$

$$\overline{\frac{\partial u}{\partial x}} = \frac{\partial \overline{u}}{\partial x} \quad , \quad \overline{\int u dx} = \int \overline{u} dx \quad ; \quad \overline{uv} = \overline{u}\;\overline{v} + \overline{u'v'} \quad ; \quad \overline{u'\overline{v}} = 0$$



$$\overline{\tau_{xy}} = \tau_{xy}\big|_{lam} + \tau_{xy}\big|_{tur} = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u'v'}$$

Additional shear stress

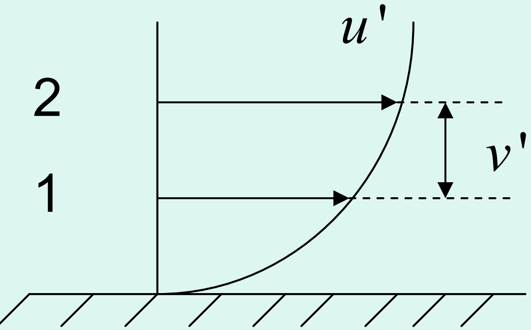
(Reynolds stress)

$Ex:$

$$uv = (\underbrace{\bar{u} + u'}_{u})\,(\underbrace{\bar{v} + v'}_{v}) = \bar{u}\bar{v} + \bar{u}v' + \bar{v}u' + u'v'$$

$$uv = \bar{u}\bar{v} + \overline{u'v'} \qquad\qquad \overline{u'v'} \neq 0$$

Physical Interpretation of $\rho u'v'$ as a stress



a)Consider fluid particle moving up from 1 to 2

$v' > 0 \quad u' < 0$ (since particle has velocity deficit i.e $u_1 < u_2$)

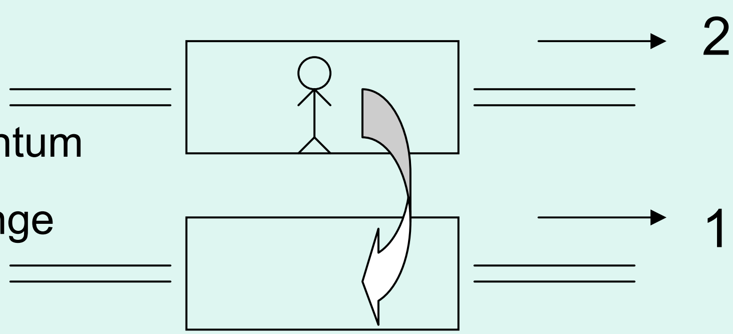
$u'v' < 0 \;\Rightarrow\; \tau_{\text{turb}} > 0 \;\Rightarrow$ *de*cel. of flow at 2

b)if particle moves down from 2 to 1

$v' < 0 \quad u' > 0$ (particle has excess vel.)

$\therefore\; u'v' < 0 \;\Rightarrow\; \tau_{\text{turb}} > 0 \;\Rightarrow$ accel. of flow at 1

Momentum exchange



Turbulent shear stress is higher

Basic Eqs. for Mean Motion of Turbulent Flows

Consider flows with constant properties

Continuity equation

(1) $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$$ $$u=\bar{u}+u'$$

Time – averaging of (1) $$\overline{\frac{\partial u}{\partial x}}=\frac{\partial \bar{u}}{\partial x}+\cancel{\frac{\partial \overline{u'}}{\partial x}}$$

(2) $$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0$$

(3) Also , using (1) $$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

Both time average values &fluctuations satisfy laminar flow continuity eq

Momentum Eqs.(Reynolds eqs.)

Incomp. N-S eqs. $$\rho\left(\frac{\partial V}{\partial t}+(\vec{V}.\nabla)\vec{V}\right)=-\nabla p+\mu\nabla^2\vec{V} \qquad (4)$$

1) Substitute $u = \bar{u} + u'$ $v = \bar{v} + v'$ $w = \bar{w} + w'$ $p = \bar{p} + p'$ into N-S egs

2) Time average the equations

3) Drop-out terms which `average` to zero . Use "Rules of Computation"

$$\overline{\frac{\partial u'}{\partial t}} = 0 \qquad \overline{\frac{\partial^2 u'}{\partial x^2}} = 0 \quad \leftarrow \text{terms which are linear in fluctuating quantities} \;\Rightarrow\; 0$$

$$\overline{u'^2} \neq 0 \qquad \overline{u'v'} \neq 0 \quad \leftarrow \text{terms which are quadratic in fluctuating quantities} \;\Rightarrow\; 0$$

Resultant eqs. (called Reynolds eqs.)

$$\rho(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z}) = -\frac{\partial \bar{p}}{\partial x} + \mu\nabla^2\bar{u} - \rho(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z})$$

$$\rho(\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + \bar{w}\frac{\partial \bar{v}}{\partial z}) = -\frac{\partial \bar{p}}{\partial y} + \mu\nabla^2\bar{v} - \rho(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'^2}}{\partial y} + \frac{\partial \overline{v'w'}}{\partial z})$$

$$\rho\left(\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial z}+\mu\nabla^2\bar{w}-\underbrace{\rho\left(\frac{\partial \overline{u'w'}}{\partial x}+\frac{\partial \overline{v'w'}}{\partial y}+\frac{\partial \overline{w'^2}}{\partial z}\right)}$$

$\therefore$ treat unsteady "fluctuations" as added stresses $\Rightarrow$ called Reynolds stresses(turbulent stresses)

additional terms due to turbulent fluctuating motion $\Rightarrow$ momentum exchange due to fluctuations $\Rightarrow$"stresses"

Complete stresses consist of

$$\sigma_{xx}=-p+2\mu\frac{\partial \bar{u}}{\partial x}-\rho\overline{u'^2} \quad \rightarrow \text{ fluctuatios}$$

$$\tau_{xy}=\underbrace{\mu\left(\frac{\partial \bar{u}}{\partial y}+\frac{\partial \bar{v}}{\partial x}\right)}_{\text{viscous stresses laminar}}-\underbrace{\rho\overline{u'v'}}_{\text{Reynolds stress apparent turbulent stresses}} \quad ,.......$$

In general , Reynolds stresses dominate over viscous stresses, except for regions directly at the wall

Closure problem

too few eqs : 4

too many unknowns : 10

Figure some way to approximate Reynolds stresses

Objective : Establish relationship between Reynolds stresses & mean motions, i.e $\bar{u}$, $\bar{v}$, $\bar{w}$

$\Rightarrow$ model eqs. must be developed

$\therefore$ turbulence models or turbulence modeling.

model equations contain empirical elements

A. Eddy viscosity

– Attempt to approximate a "turbulent" viscosity

idea : Since $\tau_{\text{lam}} = \mu \dfrac{\partial \bar{u}}{\partial y} = \rho \nu \dfrac{\partial \bar{u}}{\partial y}$

Let $\tau_{\text{turb}} = \rho \in \dfrac{\partial \bar{u}}{\partial y} = -\rho \overline{u'v'}$

$\mapsto$ *Eddy* viscosity $\Rightarrow$ $\in >> \nu$

Problem : how to model $\in$?

For some situations $\Rightarrow \quad \in \;\approx$ const.

In general $\in \neq const. \;\Rightarrow\; \in \; = f(\bar{u}, y, \frac{\partial \bar{u}}{\partial y}, etc.)$

In general, many wild guesses are made, not many work

Energy Equation

Consider the energy equation for incompressible flow with constant properties

$$\rho c_p \frac{DT}{Dt} = k\nabla^2 T + \Phi$$

Taking the time-average of the energy eq. , we obtain following eq. for the average temp.

field $\overline{\mathrm{T}} = (x, y, z)$

$$\left.\rho c_p (\vec{u}\frac{\partial T}{\partial x} + \vec{v}\frac{\partial T}{\partial y} + \vec{w}\frac{\partial T}{\partial z})\right\} \text{ convection}$$

$$=k\left(\frac{\partial^2 \overline{T}}{\partial x^2}+\frac{\partial^2 \overline{T}}{\partial y^2}+\frac{\partial^2 \overline{T}}{\partial z^2}\right)\Big\}\ \text{molecular heat transport}$$

$$-\rho c_p\left(\frac{\partial \overline{u'T'}}{\partial x}+\frac{\partial \overline{v'T'}}{\partial y}+\frac{\partial \overline{w'T'}}{\partial z}\right)\Big\}\ \text{turbulent heat transport("apparent" heat conduction)}$$

$$+\mu\left[2\left(\frac{\partial \overline{u}}{\partial x}\right)^2+2\left(\frac{\partial \overline{v}}{\partial y}\right)^2+2\left(\frac{\partial \overline{w}}{\partial z}\right)^2+\left(\frac{\partial \overline{u}}{\partial y}+\frac{\partial \overline{v}}{\partial x}\right)^2+\left(\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial x}\right)^2+\left(\frac{\partial \overline{v}}{\partial z}+\frac{\partial \overline{w}}{\partial y}\right)^2\right]\Big\}\ \text{direct dissipation}$$

The same eq.holds for the average temp. fields as for laminar temp. fields, apart from

two additional terms

"apparent" heat conduction $\Rightarrow$ $\text{div}(\overline{\vec{V}'T'})$

"turbulent" dissipation , $\rho\,\tilde{\in}$

$$\rho\tilde{\in}=\mu\left[2\overline{\left(\frac{\partial \overline{u}}{\partial x}\right)^2}+2\overline{\left(\frac{\partial \overline{v}}{\partial y}\right)^2}+2\overline{\left(\frac{\partial \overline{w}}{\partial z}\right)^2}+\overline{\left(\frac{\partial \overline{u}}{\partial y}+\frac{\partial \overline{v}}{\partial x}\right)^2}+\overline{\left(\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial x}\right)^2}+\overline{\left(\frac{\partial \overline{v}}{\partial z}+\frac{\partial \overline{w}}{\partial y}\right)^2}\right]$$

In turbulent flows mechanical energy is transformed into internal energy in two different ways:

a) Direct dissipation : transfer is due to the viscosity (as in laminar flow)

b) Turbulent dissipation : transfer is due to the turbulent fluctuations

The Turbulence Kinetic Energy Equation (K-equation)

Many attemps have been made to add “turbulence conservation” relations to the time-averaged continuity, momentum and energy equations derived.

A relation for the *turbulence kinetic energy* K of fluctuations.

$$K \equiv \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}\right) = \frac{1}{2}\overline{u_i'u_i'}$$

Einstein summation notation,

$$u_i = (u_1, u_2, u_3) = (u, v, w)$$

A conservation relation for K can be derived by forming the mechanical energy equation i.e., dot product of u_i ve i^{th} momentum equation subtract instantaneous mechanical energy equation from its time averaged value.

Result: *Turbulence kinetic energy* relation for an incompressible fluid.

$$\underbrace{\frac{DK}{Dt}}_{\text{I}} = \underbrace{-\frac{\partial}{\partial x_i}\left[\overline{u_i'\left(\frac{1}{2}u_j'u_j' + \frac{p'}{\rho}\right)}\right]}_{\text{II}} \underbrace{-\overline{u_i'u_j'}\frac{\partial u_j'}{\partial x_i}}_{\text{III}} +$$

$$\underbrace{\frac{\partial}{\partial x_i}\left[\overline{\nu u_j'\left(\frac{\partial u_i'}{\partial x_j'} + \frac{\partial u_j'}{\partial x_i'}\right)}\right]}_{\text{IV}} \underbrace{-\overline{\nu\frac{\partial u_j'}{\partial x_i'}\left(\frac{\partial u_i'}{\partial x_j'} + \frac{\partial u_j'}{\partial x_i'}\right)}}_{\text{V}}$$

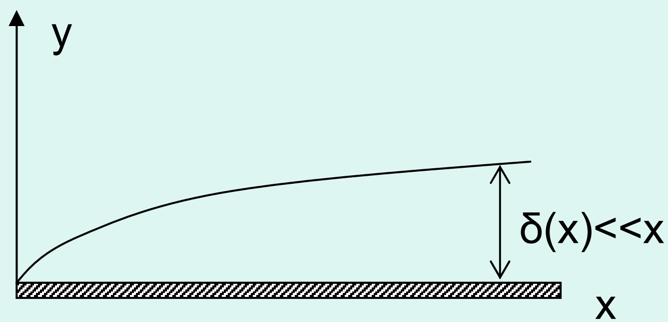
I. Rate of change of turbulent (kinetic) energy
II. Convective diffusion of turbulence energy
III. Production of turbulent energy
IV. Viscous diffusion (work done by turbulence viscous stresses)
V. Turbulent viscous dissipation

Reynolds stress equation: conservation equations for Reynolds stresses see F. White pg. 406

2-D Turbulent Boundary Layer Equations

Just as laminar flows, turbulent flows at high Re also have *boundary layer* character, i.e. large lateral changes and small longitudinal changes in flow properties.

Ex.: Pipe flow, channel flow, wakes and jets.



Same approximations as in laminar boundary layer analysis,

$$\overline{v} << \overline{u} \qquad \frac{\partial}{\partial x} << \frac{\partial}{\partial y}$$

Assume that mean flow structure is 2D

$$\overline{w} = 0 \qquad \frac{\partial}{\partial z} = 0 \quad \text{but} \quad \overline{w'^2} \neq 0$$

Basic turbulent equations (Reynolds equations) reduce to

Continuity:
$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \qquad (1)$$

x-momentum:
$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} \approx U_e \frac{dU_e}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y} \qquad (2)$$

U_e: free stream velocity

Thermal energy:
$$\rho c_p \left(\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} \right) \approx \frac{\partial q}{\partial y} + \tau \frac{\partial \bar{u}}{\partial y} \qquad (3)$$

where $\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'}$

$$q = \underbrace{k\frac{\partial \bar{T}}{\partial y}}_{\text{molecular flux}} - \underbrace{\rho c_p \overline{v'T'}}_{\text{turbulent flux}} \qquad (4)$$

Above equations closely resemble the laminar flow equations except that τ and q contain turbulent shear stress and turbulent heat flux (Reynolds Stress) must be modelled.

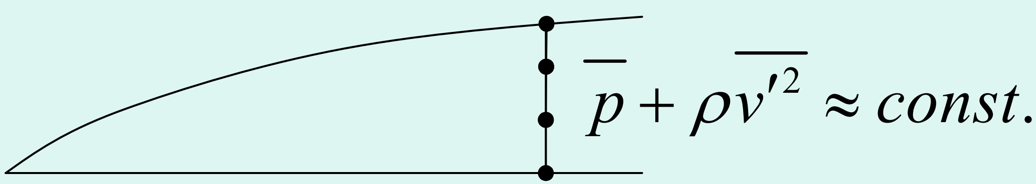
y-momentum equation reduces to

$$\frac{\partial \overline{p}}{\partial y} \approx -\rho \frac{\partial \overline{v'^2}}{\partial y} \qquad (5)$$

Integrating over the boundary layer yields:

$$\overline{p} \approx p_e(x) - \rho \overline{v'^2}$$

Unlike laminar flow, $\overline{p}$ varies slightly across the boundary layer due to velocity fluctuations normal to the the wall



Note: $\overline{p}_w$: wall pressure

no-slip $\Rightarrow$ $v' \equiv 0$ $\Rightarrow$ $\overline{p}_w = p_e(x)$

Bernoulli equation in the (inviscid) free stream $\quad dp_e \approx -\rho U_e dU_e$

Boundary Conditions:

Free stream conditions Ue(x) and Te(x) are known.

No-slip, no jump: $\bar{u}(x,0)=\bar{v}(x,0)=0 \quad , \quad \bar{T}(x,0)=T_w(x)$

Free stream matching: $\bar{u}(x,\delta)=U_e \quad , \quad \bar{T}(x,\delta_T)=T_e(x)$

The velocity and thermal boundary layer thicknesses (δ, δ_T) are not necessarily equal but depend upon the Pr, as in laminar flow. Eqs. 1 and 2 can be solved for $\bar{u}$ $\bar{v}$ if a suitable correlation for total shear τ is known.

Turbulent Boundary Layer Integral Relations:

The integral momentum equation has the *identical* form as laminar flow

$$\frac{d\theta}{dx}+(2+H)\frac{\theta}{U_e}\frac{dU_e}{dx}=\frac{\tau_w}{\rho U_e^2}=\frac{c_f}{2}$$

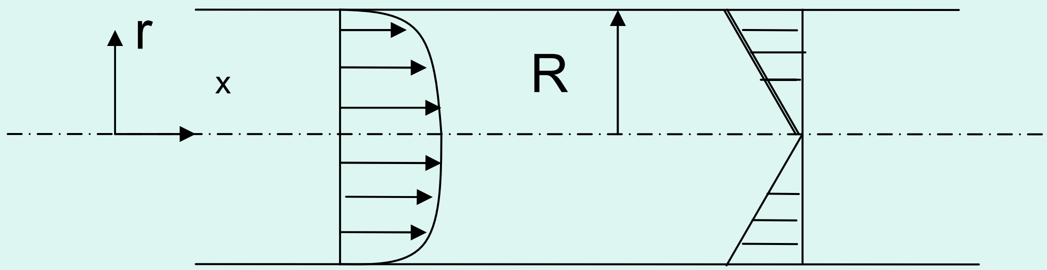
$$\theta=\int_0^\infty \frac{\bar{u}}{U_e}\left(1-\frac{\bar{u}}{U_e}\right)dy \quad , \quad \text{H}=\frac{\delta^*}{\theta} \text{ (momentum shape factor)}$$

$$\delta^*=\left(1-\frac{\bar{u}}{U_e}\right)dy$$

Turbulent velocity profile is *more complicated* in shape and many different correlations have been proposed.

Example: *Turbulent pipe flow*

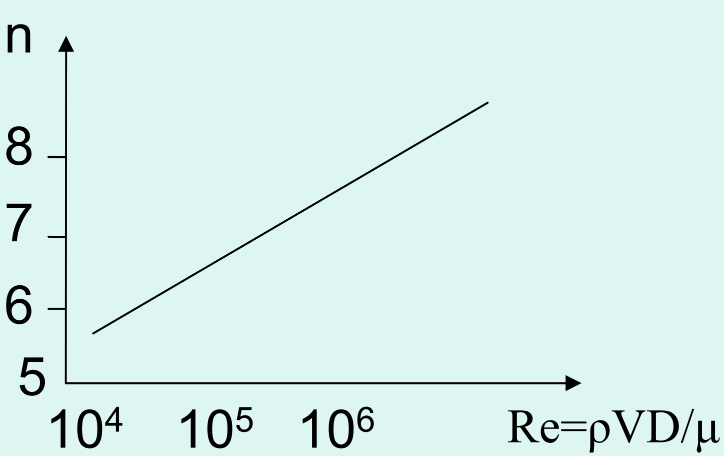
Often used correlation is the empirical power-law velocity profile

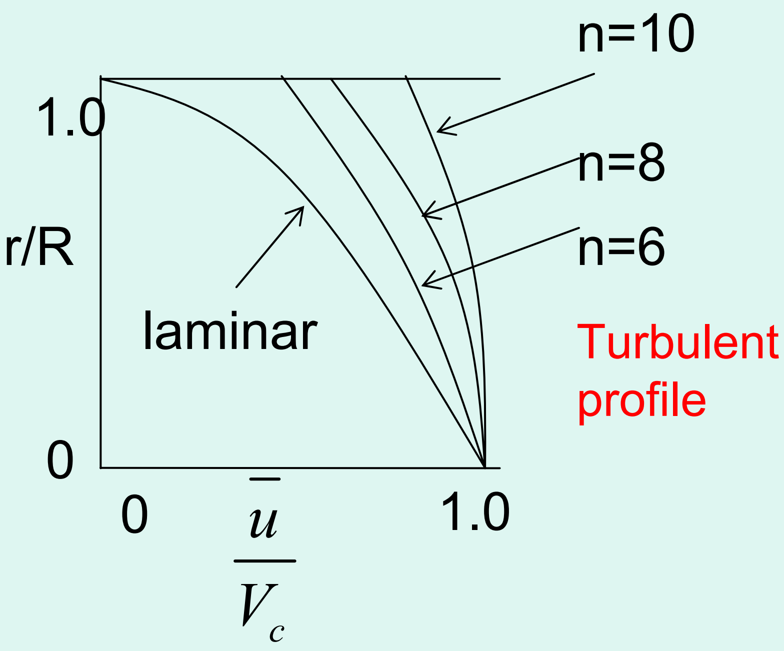


$$\frac{\bar{u}}{V_c} = \left(1 - \frac{r}{R}\right)^{1/n}$$

n=f(Re)

for many practical flows n = 7



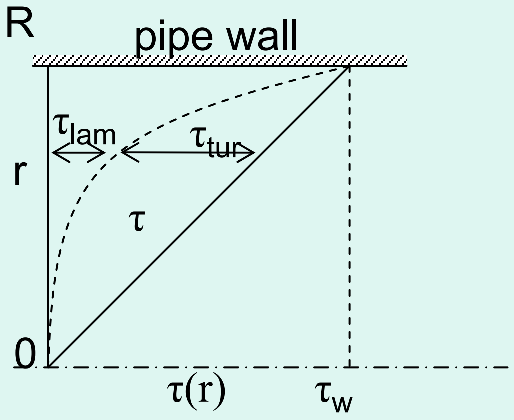


- Turbulent profiles are much “flatter” than laminar profile
- Flatness increases with Reynolds number (i.e., with n)

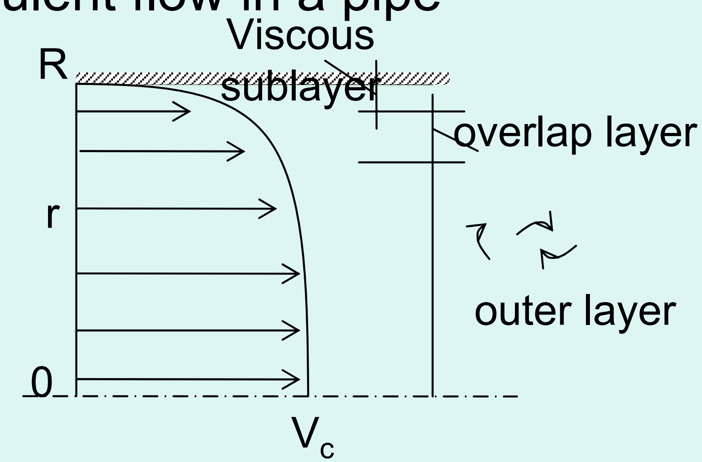
Turbulent velocity profile(s): The *inner, outer*, and *overlap* layers.
Key profile shape consist of 3 layers

Inner layer: very narrow region near the wall (viscous sublayer)
viscous (molecular) shear dominates
laminar shear stress is dominant, random eddying nature of flow is absent
Outer layer: turbulent (eddy) shear (stress) dominates
Overlap layer: both types of shear important; profile smoothly connects inner and outer regions.

Example: Structure of turbulent flow in a pipe



Shear stress



Average velocity

Inner law:

$$\overline{u} = f(\tau_w, \rho, \mu, y) \qquad (1)$$

Velocity profile would not depend on free stream parameters.

Outer law:

$$U_e - \overline{u} = g(\tau_w, \rho, y, \delta, \frac{dp_e}{dx}) \qquad (2)$$

Wall acts as a source of retardation, independent of μ.

Overlap law:

$$\overline{u}_{inner} = \overline{u}_{outer} \qquad (3)$$

We specify inner and outer functions merge together smoothly.

Dimensionless Profiles:

The functional forms in Eqs.(1)-(3) are determined from experiment after use of dimensional analysis.

Primary Dimensions: (mass, length, time) : 3

Eq.(1) : 5 variables

Π groups : 5-3 = 2 (dimensionless parameters)

Proper dimensionless inner law:

$$\frac{\bar{u}}{v^*} = f\left(\frac{yv^*}{\nu}\right) \quad ; \quad v^* = \left(\frac{\tau_w}{\rho}\right)^{1/2}$$

Variable v* [m/s] called wall friction velocity.
v* is used a lot in turbulent flow analyses.

Outer law using Π - theorem:

$$\frac{U_e - \bar{u}}{v^*} = g\left(\frac{y}{\delta}, \xi\right) \quad ; \quad \xi = \frac{\delta}{\tau_w}\frac{dp_e}{dx}$$

Often called velocity defect law, with $U_e - \bar{u}$ being "defect" or retardation of flow due to wall effects. At any given position x, defect g(y/δ) will depend on local pressure gradient ξ.
Let ξ have some particular value. Then overlap function requires

Overlap law:

$$\frac{\bar{u}}{v^*} = f\left(\frac{\delta v^*}{\nu}\frac{y}{\delta}\right) = \frac{U_e}{v^*} - g\left(\frac{y}{\delta}\right)$$

From functional analysis: both f and g must be logarithmic functions.

Thus, in overlap layer:

Inner variables: $\frac{\bar{u}}{v^*} = \frac{1}{k}\ln\frac{yv^*}{\nu} + B$

Outer variables: $\frac{U_e - \bar{u}}{v^*} = -\frac{1}{k}\ln\frac{y}{\delta} + A$

Where K and B are near-universal constants for turbulent flow past smooth, impermeable walls.
K≈0.41 , B≈5.0 pipe flow measurements, data correlations
A varies with pressure gradient ξ (perhaps with other parameters also).

Let $u^+ = \frac{\bar{u}}{v^*}$, and $y^+ = \frac{yv^*}{\nu}$

Inner layer details, Law of the wall.

At very small y, velocity profile is linear.

$y^+ \le 5: \quad \tau_w = \mu\frac{\bar{u}}{y} \quad or \quad u^+ = y^+$

Example: Thickness of viscous sublayer

$$\delta_{sub} = \frac{5\nu}{v^*} \qquad \frac{\nu}{v^*}\text{: viscous length scale of a turbulent boundary layer}$$

Flat plate airfoil data: v*=1.24 m/s , v_{air}≈1.51x10-5 m2/s

Between 5 ≤y^+≤30 buffer layer.

Velocity profile is neither linear nor logarithmic but is a smooth merge between two.

Spalding (1961) single composite formula.

$$y^+ = u^+ + e^{-KB}\left[e^{Ku^+} - 1 - Ku^+ - \frac{\left(Ku^+\right)^2}{2} - \frac{\left(Ku^+\right)^3}{6}\right]$$

Notes:

$$\frac{\bar{u}}{V_c} = \left(1 - \frac{r}{R}\right)^{\frac{1}{n}}$$

$$\frac{d\bar{u}}{dr} = \frac{V_c}{n}\left(1 - \frac{r}{R}\right)^{\frac{1}{n}-1}\left(-\frac{1}{R}\right)$$

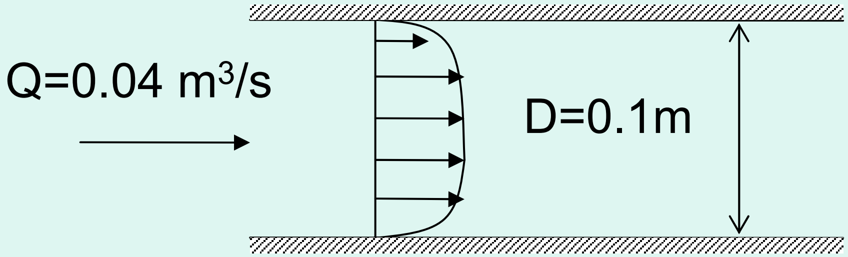
$$r = R \quad \frac{d\bar{u}}{dr} = \infty$$

$$r = 0 \quad \frac{d\bar{u}}{dr} \neq 0$$

Power law profile cannot be valid near the wall.
Power law profile cannot be precisely valid near the centreline.
However, it does provide a reasonable approximation to measured velocity profiles across most of the pipe.

Example:

Water at 20 °C (ρ=998 kg/m^3), ν=1.004x10^{-6} m^2/s



$$\frac{dp}{dx} = 2.59 kPa/m$$

δs = ? thickness of viscous sublayer?

centreline velocity, V_c = ?

ratio of turbulent to laminar shear stress, τ_{turb}/τ_{lam} = ? at a point midway between the centreline and pipe wall i.e., at r = 0.025 m.

Law of the wall valid $y^{\pm} \leq 5$ viscous sublayer

$$y^{\pm} = \frac{y v^*}{\nu} \leq 5$$

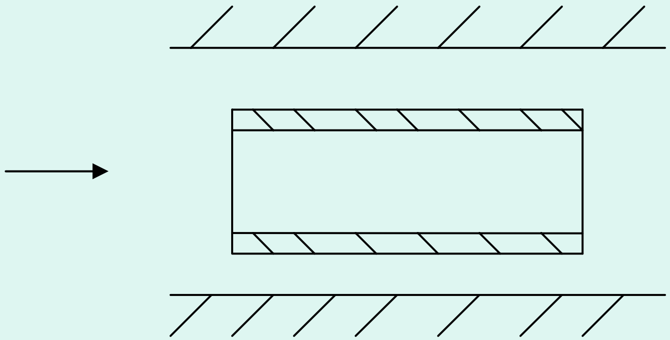
$$y = \delta_s \quad y^{\pm} = 5 \quad \Rightarrow \quad \frac{\delta_s v^*}{\nu} = 5 \quad \delta_s = \frac{5\nu}{v^*}$$

$$v^* = \sqrt{\frac{\tau_w}{\rho}}$$

Pressure drop and wall shear stres in a fully developed pipe flow is related by

$$\Delta p = \frac{4l\tau_w}{D}$$ (Valid for both laminar & turbulent flow)

(Exercise: Obtain the above equation considering the force balance of a fluid element)



$$\tau_w = \frac{D\Delta p}{4l} = \frac{(0,1)(2,59.10^3)}{4(1m)} Pa = 64,8 N/m^2$$

$$\text{So, } v^* = \sqrt{\frac{64,8 N/m^2}{998 kg/m^3}} = 0,255 m/s$$

$$\delta_s = \frac{5.1,004.10^{-6}}{0,255} = 1,97.10^{-5} m \cong 0,02 mm$$

Imperfections on pipe wall will protrude into this sublayer and affect some of the characteristics of flow(i.e.,wall shear stres & pressure drop)

$$V = \frac{Q}{A} = \frac{0,04 m^3/s}{\pi(0,1)^2/4 m^2} = 5,09 m/s$$

$$\mathrm{Re} = \frac{VD}{\nu} = \frac{5,09.(0,1)}{1,004.10^{-6}} = 5,07.10^5$$

$$\mathrm{Re} = 5,07.10^5 \quad \Rightarrow \quad n = 8,4$$

Power-law profile

$$\frac{\overline{u}}{V_c} \cong (1 - \frac{r}{R})^{1/8,4}$$

$$Q = A.V = \int \overline{u} dA = V_c \int_0^R (1 - \frac{r}{R})^{1/n} (2\pi r) dr$$

$$Q = 2\pi R^2 V_c \frac{n^2}{(n+1)(2n+1)}$$

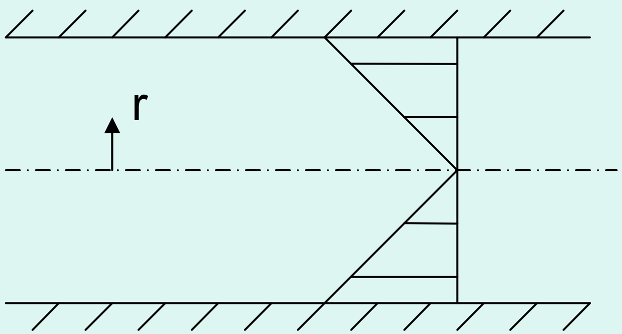
Recall that V_c=2V for laminar pipe flow:

$$Q = \pi R^2 V \quad \therefore \quad \frac{\mathrm{V}}{\mathrm{V_c}} = \frac{2n^2}{(n+1)(2n+1)}$$

$\underline{n = 8,4:}$ $\quad \mathrm{V}_c = 1,186 V = 1,186(5,09) = 6,04 m/s$

$$\left.\frac{\tau_{turb}}{\tau_{lam}}\right|_{r=0,025m} = ?$$ Shear stres distribution throughout the pipe

$$\tau = \frac{2\tau_w r}{D}$$ (Valid for laminar or turbulent flow)



R=D/2

$$\tau(r=0,025) = \frac{2(64,8).0,025}{0,1} = 32,4 N/m^2$$

$$\tau = \tau_{lam} + \tau_{turb} = 32,4$$

$$\tau_{lam} = -\mu\frac{d\bar{u}}{dr}; \quad \bar{u} = V_c(1-\frac{r}{R})^{1/n} \Rightarrow \frac{d\bar{u}}{dr} = -\frac{V_c}{nR}(1-\frac{r}{R})^{(1-n)/n}$$

$$\left.\frac{d\bar{u}}{dr}\right|_{r=0,025} = -\frac{6,04}{8,4(0,05)}(1-\frac{0,025}{0,05})^{(1-8,4)/8,4} = -26,5$$

$$\tau_{lam} = -\mu\frac{d\bar{u}}{dr} = -(\nu\rho)\frac{d\bar{u}}{dr}$$

Thus $= -(1,004.10^{-6}).(998).(-26,5) = 0,0266 N/m^2$

$$\frac{\tau_{turb}}{\tau_{lam}} = \frac{32,4-0,0266}{0,0266} = 1220$$

As expected

$$\tau_{turb} >> \tau_{lam}$$

Turbulent Boundary Layer on a Flat Plate

Problem of flow past a sharp flat plate at high Re has been studied extensively, numerous formulas have been proposed for friction factor.

- -curve fits of data
- -use of Momentum Integral Equation and/or law of the wall
- -numerical computation using models of turbulent shear

Momentum Integral Analysis

$$\frac{dp}{dx} = 0 \ (U = const.) \qquad \frac{d\theta}{dx} = \frac{C_f}{2} = \frac{\tau_w}{\rho U^2}$$

Momentum Interal Equation valid for either laminar or turbulent flow.

For turbulent flow
a reasonable approximation to the velocity profile $\dfrac{\bar{u}}{U} = f(y/\delta)$

Functional relationship describing the wall shear stress

Need to use some *empirical relationship*

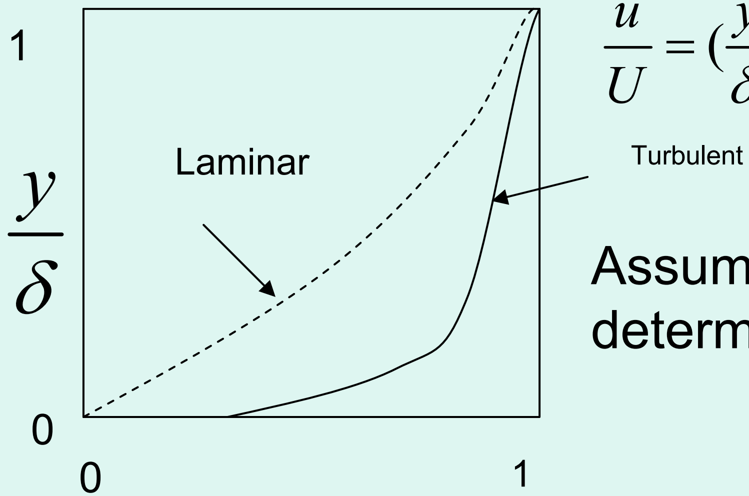
For laminar flow $\tau_w = \mu \left.\dfrac{\partial u}{\partial y}\right|_{y=0}$

Example: Turbulent flow of an incompressible fluid past a flat plate
Boundary layer velocity profile is assumed to be

$$\frac{\bar{u}}{U} = \left(\frac{y}{\delta}\right)^{1/7} \leftarrow$$ power law profile suggested by Prandtl (taken From pipe data!)

Reasonable approximation of experimentally observed profiles, except very near the plate,

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \infty!$$



$$\frac{\bar{u}}{U} = \left(\frac{y}{\delta}\right)^{1/7}$$

$$\eta = \frac{y}{\delta}$$

Assume shear stress aggrees with experimentally determined formula

$$C_f = 0,045 \, \mathrm{Re}_\delta^{-1/4} \left\{ \text{or } \tau_w = 0,0225 \rho U^2 \left(\frac{\nu}{U\delta}\right)^{1/4} \right\}$$

Determine; δ, δ^*, θ and τ_w as a function of x. $\mathrm{Re} = \dfrac{U\delta}{\nu}$

What is the friction drag coefficient $C_{D,f}$=?

Momentum Integral Equation (with U=constant)

$$\frac{d\theta}{dx} = \frac{C_f}{2} = \frac{\tau_w}{\rho U^2} \qquad \eta = \frac{y}{\delta}; \frac{\overline{u}}{U} = (\frac{y}{\delta})^{1/7} = \eta^{1/7}$$

$$\theta = \int_0^\infty \frac{\overline{u}}{U}(1-\frac{\overline{u}}{U})dy = \delta\int_o^1 \frac{\overline{u}}{U}(1-\frac{\overline{u}}{U})d\eta = \delta\int_0^1 \eta^{1/7}(1-\eta^{1/7})d\eta = \frac{7\delta}{72}$$

$$\frac{7}{72}\frac{d\delta}{dx} = 0,0225\,\mathrm{Re}_\delta^{-1/4} = 0,0225(\frac{\nu}{U\delta})^{1/4}$$

$$\int_0^\delta \delta^{1/4} d\delta = 0,231(\frac{\nu}{U})^{1/4}\int_0^x dx$$

$\delta = 0,370(\frac{\nu}{U})^{1/5} x^{4/5}$ or in dimensionless form $\boxed{\frac{\delta}{x} = \frac{0,370}{\mathrm{Re}_x^{1/5}}}$

Boundary layer at leading edge of plate is laminar but in practice,laminar boundary layer often exists over a relatively short portion of plate.

$\therefore$ error associated with starting turbulent boundary layer with δ=0 at x=0 can be negligible.

$$\delta^* = \int_0^\infty (1-\frac{\overline{u}}{U})dy = \delta\int_0^1 (1-\frac{\overline{u}}{U})d\eta = \delta\int_0^1 (1-\eta^{1/7})d\eta = \frac{\delta}{8}$$

$$\frac{\delta^*}{x} = \frac{0,0463}{\mathrm{Re}_x^{1/5}}$$

$$\theta = \frac{7}{72}\delta = 0,0360(\frac{\nu}{U})^{1/5} x^{4/5}$$

$$\frac{\theta}{x} = \frac{0,036}{\mathrm{Re}_x^{1/5}} \qquad \theta < \delta^* < \delta$$

$$\tau_w = 0,0225\rho U^2 \left[\frac{\nu}{U(0,37)(\nu/U)^{1/5} x^{4/5}}\right]^{1/4} = \frac{0,0288\rho U^2}{\mathrm{Re}_x^{1/5}}$$

$$C_f = \frac{0,058}{\mathrm{Re}_x^{1/5}}$$

Friction drag on one side of plate, D_f

$$D_f = \int_o^l b\tau_w dx = b(0,0288\rho U^2)\int_0^l (\frac{\nu}{Ux})^{1/5} dx$$

$$D_f = 0,0360\rho U^2 \frac{A}{\mathrm{Re}_l^{1/5}} \qquad \text{where } A=b.l \text{ area of plate}$$

$$C_{Df} = \frac{D_f}{\frac{1}{2}\rho U^2 A} = \frac{0,0720}{\mathrm{Re}_l^{1/5}}$$

Turbulent flow: $\delta(x) \sim x^{4/5}; \tau_w(x) \sim x^{-1/5}$

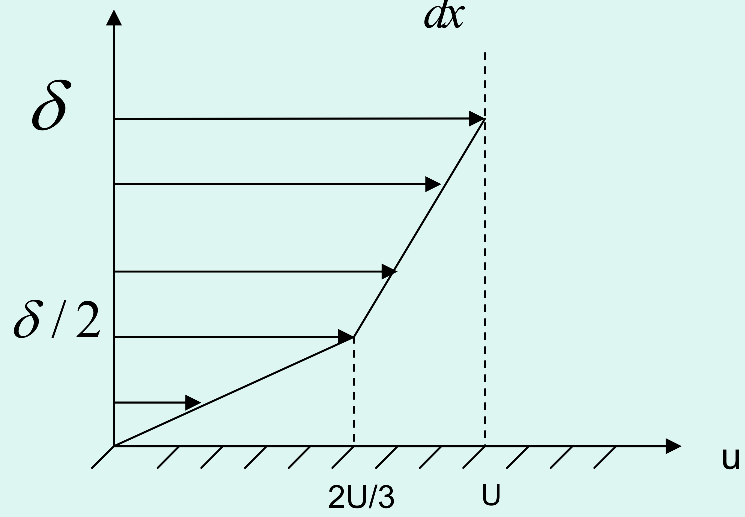
Laminar flow: $\delta(x) \sim x^{1/2}; \tau_w(x) \sim x^{-1/2}$

Note: Results presented in this example are valid only in the range of validity of original data, assumed velocity profile & shear stres. The range covers smooth flat plates with $5x10^5 < Re_l < 10^7$
See Fig 6-20 (White, page 432)

Example 1 : Momentum Integral Equation-Approximate vel. profile

$$\frac{d\theta}{dx} = \frac{\tau_w}{\rho U^2}$$

$$U = const. \Rightarrow (\frac{dU}{dx} = 0)$$



$$\frac{u}{U} = f(\eta) \qquad \eta = \frac{y}{\delta}$$

For $0 \le \eta \le 1/2 \quad f = a_1 + b_1 \eta$

$$f = \frac{2}{3} \text{ at } \eta = \frac{1}{2} \quad \& \quad f = 0 \text{ at } \eta = 0$$

$$\therefore \; a_1 = 0, b_1 = 4/3$$

$$\frac{u}{U} = \frac{4}{3}\eta \quad : \quad 0 \le \eta \le 1/2$$

Similarly, $\frac{u}{U} = \frac{1}{3} + \frac{2}{3}\eta$ for $\frac{1}{2} \le \eta < 1$

$$\theta = \int_0^1 \frac{u}{U}(1 - \frac{u}{U})\delta d\eta = \delta \int_0^{1/2} \frac{4}{3}\eta(1 - \frac{4}{3}\eta) d\eta + \delta \int_{1/2}^1 (\frac{1}{3} + \frac{2}{3}\eta)(1 - \frac{1}{3} - \frac{2}{3}) d\eta$$

$$= 0{,}1574\delta$$

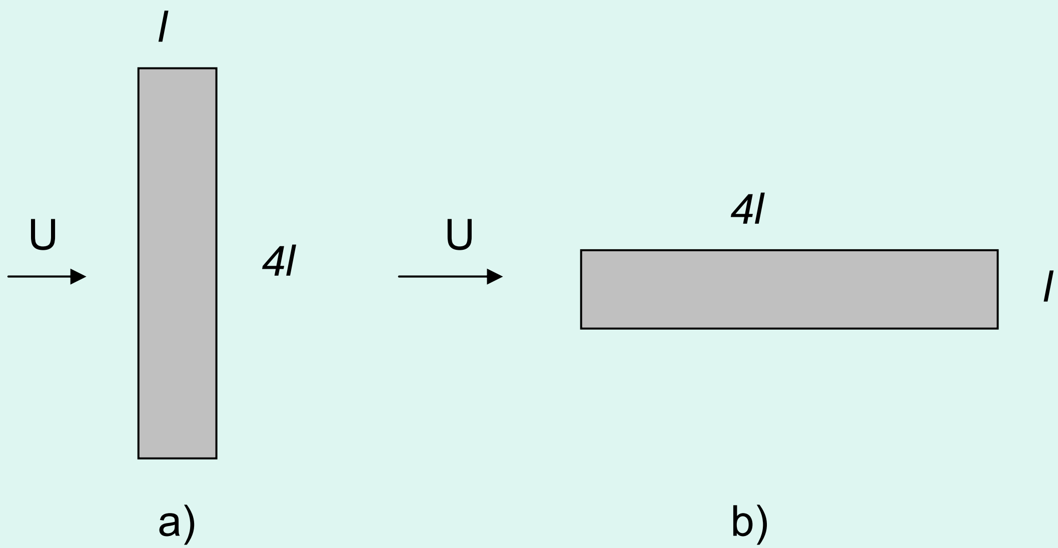
$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu \frac{\partial u}{\partial \eta}\bigg|_{\eta=0} = \frac{4}{3}\mu \frac{U}{\delta}$$

$$0{,}1574 \frac{d\delta}{dx} = \frac{4}{3}\frac{\nu}{\delta U}$$

$$0{,}1574 \int_0^{\delta} \delta d\delta = \frac{4}{3}\frac{\nu}{U} dx \quad \Rightarrow \quad \delta(x) = 4{,}12\sqrt{\frac{\nu x}{U}}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0{,}648}{\sqrt{\text{Re}_x}}$$

Example 2 : Viscous drag in thin plate



a) $F_{D,a} = \frac{1}{2}\rho U^2 A C_{D,a}$

$$C_{D,a} = \frac{1,328}{\sqrt{\mathrm{Re}_l}} = \frac{1,328}{\sqrt{\frac{Ul}{\nu}}} \qquad \& \qquad A = 4l^2$$

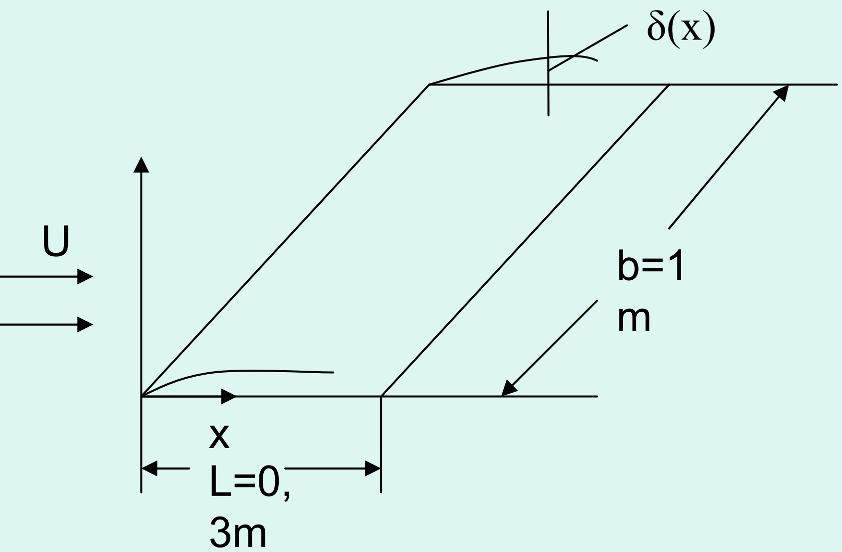
ρ, U, A is the same

$$\mathrm{C_{D,b}} = \frac{1,328}{\sqrt{\mathrm{Re}_{4l}}} = \frac{1,328}{\sqrt{\frac{U4l}{\nu}}}$$

$$\frac{F_{D,a}}{F_{D,b}} = \frac{C_{D,a}}{C_{D,b}} = 2$$

The shear stres decreases with distance from the leading edge of the plate. Thus, even though the plate area is the same for case (a) or (b), the average shear stress (and the drag) is greater for case (a).

Example 3: Thin flat plate in water tunnel



Parabolic velocity profile:

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 = 2\eta - \eta^2$$

$$\mathrm{Re}_l = \frac{Ul}{\nu} = \frac{1,6.(0,3)}{10^{-6}} = 4,8.10^5 < 5.10^5$$

$\therefore$ Flow is laminar

$$\text{Viscos drag} = F_D = 2\int_0^L \tau_w b\,dx \quad \text{(2 sides of plate)}$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu \left.\frac{\partial u}{\partial \eta}.\frac{\partial \eta}{\partial y}\right|_{\eta=0} = \frac{\mu}{\delta} U\,(2-2\eta)\Big|_{\eta=0}$$

$$= \frac{2\mu U}{\delta}$$

$$\delta = \frac{5,48x}{\sqrt{\mathrm{Re}_x}}$$

$$F_D = 2\int_0^L \frac{2\mu U}{\delta} b\,dx = \frac{4}{5,48} b\mu U \sqrt{\frac{U}{\nu}} \int_0^L \frac{dx}{\sqrt{x}} = \frac{8b\mu U}{5,48}\sqrt{\frac{UL}{\nu}}$$

$$F_D = 1,62N$$

Continuity eq. for incompressible flow,

$$Q_{inlet} = d_0^{\,2} U = (0.3*0.3)*0.7 = 0.063\ m^3/s$$

$$Q_{inlet} = Q(x) = UA = U(d - 2\delta^*)^2$$

A: effective area of the duct (allowing for the decreased flowrate in the b.l.)

Thus,

$$d_0^{\,2} = (d - 2\delta^*)^2 = 0.09 \Rightarrow d = d_0 + 2\delta^* = 0.3 + 2\delta^* \ [m]$$

$$\delta^* = 1.72\sqrt{\frac{\nu x}{U}} = 1.72\sqrt{\frac{1.5*10^{-5}x}{0.7}} = 0.00796\sqrt{x}\ [\mathrm{m}]$$

$$d = 0.3 + 0.0159\sqrt{x}\ [m]$$

$$d(x = 3m) \cong 0.328\ [m]$$