Vibration of single degree of freedom systems

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Course Schedule

- A short review on the dynamic behaviour of the single degree of freedom systems
- A short review on the dynamic behaviour of multi-degree of freedom structures
- Objectives for vibration monitoring
- Fourier Series Expansion, Fourier Transforms, Discrete Fourier Transform
- Digital signal processing, problems associated with analog-to-digital conversion, sampling, aliasing, leakage, windowing, filters
- Steps in instrumenting a structure, selection and installation of instruments,maintenance, vibration instrumentation, exciters, transducers, performance specification, data acquisition systems, strong-motion data processing

Course Schedule

- Random variables, stochastic processes, statistical analysis, correlation and convolution, coherence, time and frequency domain representation of random dynamic loads
- Dynamic response of single and multi degree of freedom systems to random loads
- Modal analysis
- Applications in bridges, buildings, mechanical engineering and aerospace structures
- MATLAB exercises
- Term Projects

References

Modal analysis:

- Heylen W., Lammens S. And Sas P., ‘Modal Analysis Theory and Testing’, Katholieke Universiteit Leuven, 1997.
- Ewins D.J., ‘Modal Testing, Theory, Practice, and Application’ (Mechanical Engineering Research Studies Engineering Design Series), Research Studies Pre; 2 edition (August 2001) ISBN-13: 978-0863802188
- Maia, N. M. M. and Silva, J. M. M.*Theoretical and Experimental Modal Analysis,* Research Studies Press Ltd,, Hertfordshire, 1997, 488 pp.,ISBN 0863802087

Signal processing:

- Blackburn, James A, ***Modern instrumentation for scientists and engineers*****,** New York : Springer, 2001
- Stearns S. D. and David, R. A., *Signal Processing Algorithms in Matlab*, Prentice-Hall Inc, 1996
- Mitra S.K., ‘Digital Signal Processing’, *A Computer based approach, Mc-Graw Hill, 3rd Edition, 2006.*
- Heylen W., Lammens S. And Sas P., ‘Modal Analysis Theory and Testing’, Katholieke Universiteit Leuven, 1997.
- Keith Worden ‘Signal Processing and Instrumentation’, Lecture Notes, http://www.dynamics.group.shef.ac.uk/people/keith/mec409.htm

References

Signal processing:

- Lynn, P. A. Introductory Digital Signal processing With Computer Applications. John Wiley & Sons, 1994.
- Stearns D. D. and David, R. A., Signal Processing Algorithms in Matlab, Prentice-Hall Inc, 1996
- Ifeachor E.C. and Jervis B.W. Digital Signal Processing: A Practical Approach, Addison-Wesley, 1997

General vibration theory

- Rao S.S., ‘Mechanical vibrations’, Pearson, Prentice Hall, 2004.
- Inman D.J., ‘Engineering Vibration’, Prentice Hall, 1994.
- Meirovitch L., ‘Fundamentals of vibrations’, McGrawHill, 2001.

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Random vibrations:

- Bendat J.S. and Piersol A.G., ‘Random data analysis and measurement procedures’, Wiley Series in Probability and Statistics, 3rd Edition, 2004.
- Lutes L.D. and Sarkani S., ‘Random Vibrations: Analysis of structural and mechanical systems’, Elsevier, 631 pp, 2004.
- Newland D.E., ’An introduction to random vibrations, spectral and wavelet analysis’, Longman, 1975/1984/1993.
- Soong T.T. and Grigoriu, ‘Random vibration of mechanical and structural systems’, Prentice Hall, 1993.
- Wirsching P.H. and Paez T.L. and Ortiz K. ‘Random vibrations: Theory and Practice’, John Wiley and sons, 1995.
- Bendat J.S. and Piersol A.G. ‘Engineering applications of correlation and spectral analysis’, John Wiley and Sons, 2nd Edition, 1993.

References

Vibration Instrumentation:

- Vibration, monitoring, testing and instrumentation handbook, CRC Press, Taylor and Francis, Edited by: Clarence W. De Silva.

- Aszkler C., ‘Acceleration, shock and vibration sensors’, Sensors handbook, Chapter 5, pages 137-159.

- McConnell K.G., ‘Vibration testing, theory and practice’, John Wiley and Sons, 1995.

- **Prerequisites:** Basic knowledge on structural analysis.

Vibration based health monitoring

Basic Information:

- Instructor: Assoc. Prof. Dr. Pelin Gundes Bakir (http://atlas.cc.itu.edu.tr/~gundes
- Email: gundesbakir@yahoo.com
- Office hours TBD by email appointment
- Website: http://atlas.cc.itu.edu.tr/~gundes/lectures
- Lecture time: Wednesday 14.00-17.00
- Lecture venue: NH 404

Vibration based health monitoring

'Vibration based structural health monitoring' is a multidisciplinary research topic. The course is suitable both for undergraduate and graduate students as well as the following departments:

- Civil engineering
- Earthquake engineering
- Mechanical engineering
- Aerospace engineering
- Electrical and electronic engineering

Vibration based health monitoring

Basic Information:

- 70 % attendance is required.

Grading:

- Quiz+homeworks: 35%
- Mid-term project:25%
- Final project:40%

Introduction

- Concepts from vibrations
- Degrees of freedom
- Classification of vibration

Concepts from vibrations

NEWTON'S LAWS

First law:

If there are no forces acting upon a particle, then the particle will move in a straight line with constant velocity.

Second law:

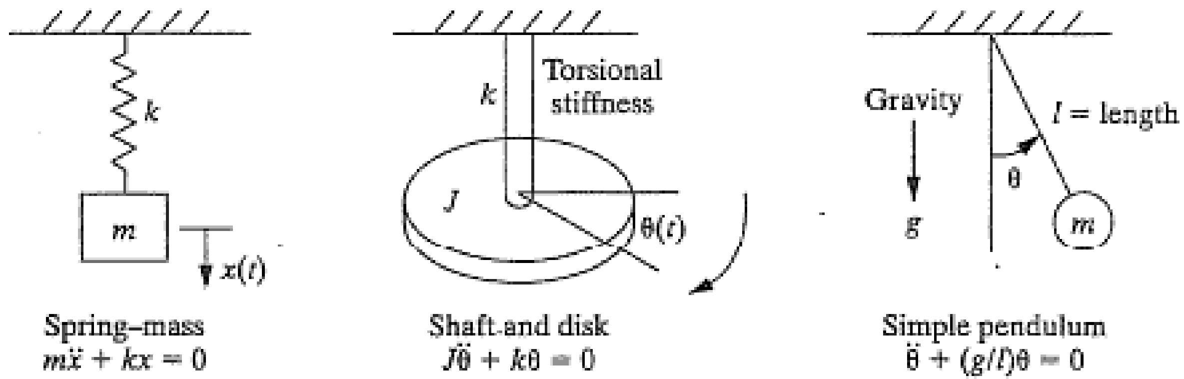
A particle acted upon by a force moves so that the force vector is equal to the time rate of change of the linear momentum vector.

Third law:

When two particles exert forces upon one another, the forces lie along the line joining the particles and the corresponding force vectors are the negative of each other.

Definition

- The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time defines the degree of freedom of the system. **A single degree of freedom system** requires only one coordinate to describe its position at any instant of time.

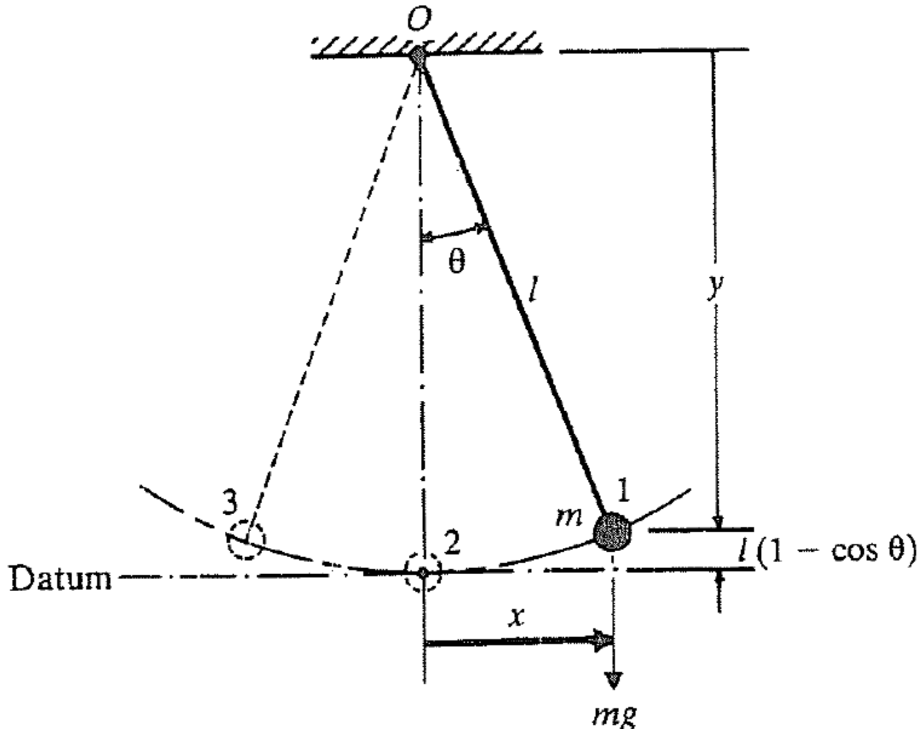


Single degree of freedom system

- For the simple pendulum in the figure, the motion can be stated either in terms of θ or x and y. If the coordinates x and y are used to describe the motion, it must be recognized that these coordinates are not independent. They are related to each other through the relation

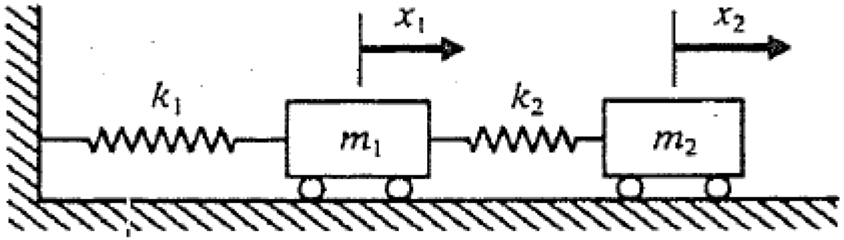
$$x^2 + y^2 = l^2$$

where l is the constant length of the pendulum. Thus any one coordinate can describe the motion of the pendulum. In this example, we find that the choice of θ as the independent coordinate will be more convenient than the choice of x and y.

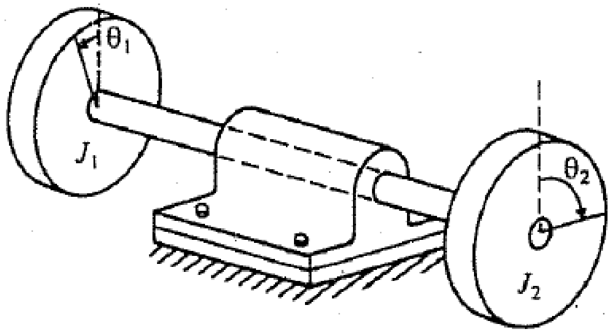


Two degree of freedom system

- Some examples of two degree of freedom systems are shown in the figure. The first figure shows a two mass – two spring system that is described by two linear coordinates x_1 and x_2. The second figure denotes a two rotor system whose motion can be specified in terms of $\theta 1$ and $\theta 2$. The motion of the system in the third figure can be described completely either by X and θ or by x,y and X.



(a)



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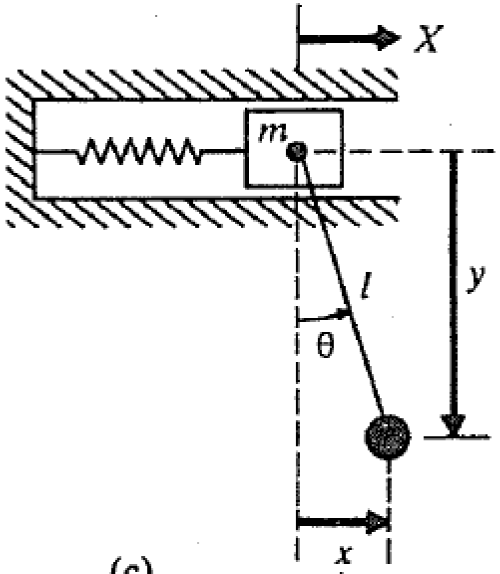
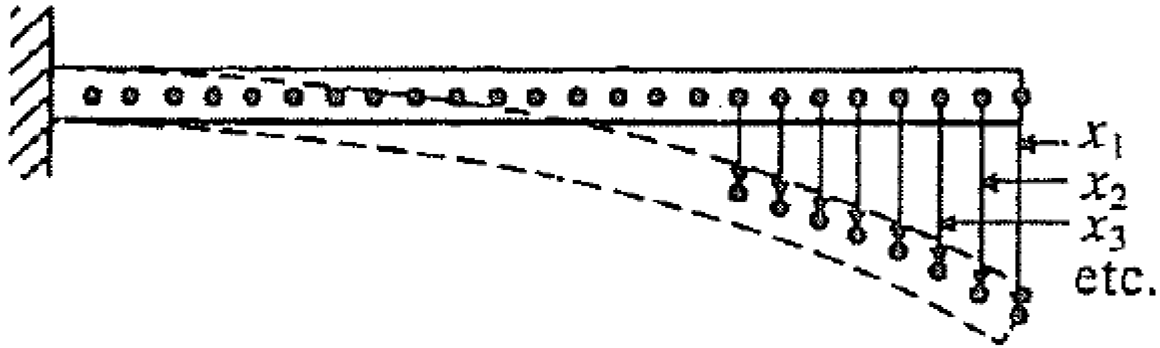


(c)

Discrete and continuous systems

- A large number of practical systems can be described using a finite number of degrees of freedom, such as the simple system shown in the previous slides.

- Some systems, especially those involving continuous elastic members, have an infinite number of degrees of freedom as shown in the figure. Since the beam in the figure has an infinite number of mass points, we need an infinite number of coordinates to specify its deflected configuration. The infinite number of coordinates defines its elastic deflection curve. Thus, the cantilever beam has infinite number of degrees of freedom.



Discrete and continuous systems

- Systems with a finite number of degrees of freedom are called **discrete** or **lumped parameter systems**, and those with an infinite number of degrees of freedom are called **continuous** or **distributed systems**.

- Most of the time, continuous systems are approximated as discrete systems, and solutions are obtained in a simple manner. Although treatment of a system as continuous gives exact results, the analytical methods available for dealing with continuous systems are limited to a narrow selection of problems, such as uniform beams, slender rods and thin plates.

- Hence, most of the practical systems are studied by treating them as finite lumped masses, springs and dampers. In general, more accurate results are obtained by increasing the number of masses, springs and dampers- that is by increasing the number of degrees of freedom.

Classification of vibration

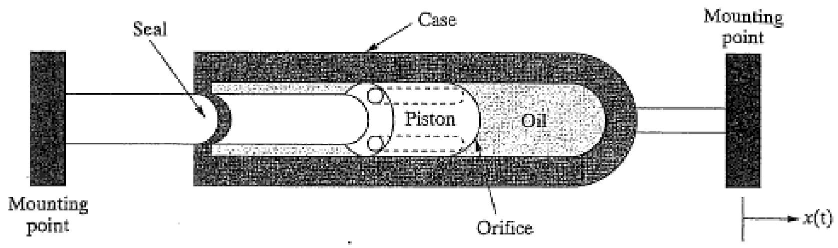
- **Free vibration:** If a system, after an initial disturbance is left to vibrate on its own, the ensuing vibration is known as free vibration. No external force acts on the system. The oscillation of a simple pendulum is an example of free vibration.

- **Forced vibration:** If a system is subjected to an external force (often a repeating type of force), the resulting vibration is known as forced vibration.

  – If the frequency of the external force coincides with one of the natural frequencies of the system, a condition known as resonance occurs, and the system undergoes dangerously large oscillations. Failures of such structures as buildings, bridges, turbines, and airplane wings have been assoicated with then occurrence of resonance.

Classification of vibration

- **Undamped vibration:** If no energy is lost or dissipated in friction or other resistance during oscillation, the vibration is known as undamped vibration.
- If any energy is lost in this way however, it is called **damped vibration**.

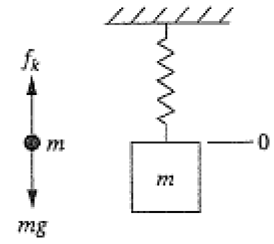


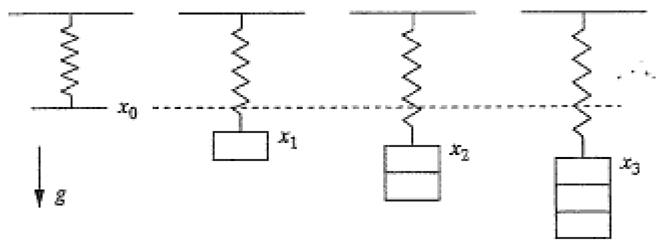
- While the spring forms a physical model for storing kinetic energy and hence causing vibration, the dashpot, or damper, forms the physical model for dissipating energy and damping the response of a mechanical system. A dashpot consists of a piston fit into a cylinder filled with oil. This piston is perforated with holes so that motion of the piston in the oil is possible. The laminar flow of the oil through the perforations as the piston moves causes a damping force on the piston.

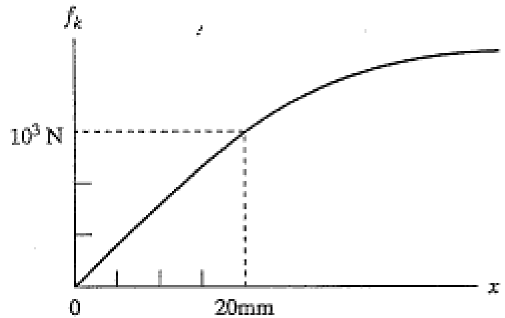
Classification of vibration

- **Linear vibration:** If all the basic components of a vibratory system-the spring, the mass, and the damper, behave linearly, the resulting vibration is known as linear vibration. The differential equations that govern the behaviour of vibratory linear systems are linear. Therefore, the principle of superposition holds.

- **Nonlinear vibration:** If however, any of the basic components behave nonlinearly, the vibration is called '**nonlinear vibration**'. The differential equations that govern the behaviour of vibratory non-linear systems are non-linear. Therefore, the principle of superposition does not hold.



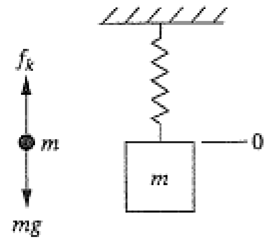


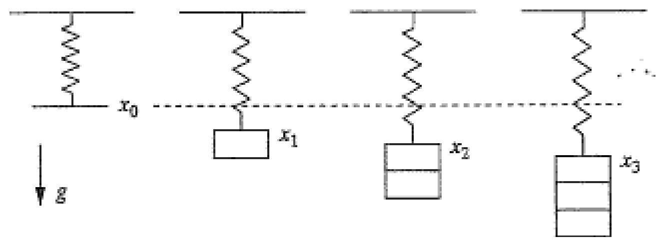


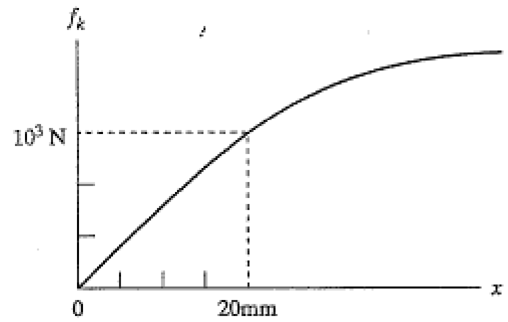
Classification of vibration

Linear and nonlinear vibrations contd:

- The nature of the spring force can be deduced by performing a simple static experiment. With no mass attached, the spring stretches to a position labeled as $x_0=0$ in the figure.
- As successively more mass is attached to the spring, the force of gravity causes the spring to stretch further. If the value of the mass is recorded, along with the value of the displacement of the end of the spring each time more mass is added, the plot of the force (mass denoted by m, times the acceleration due to gravity, denoted by g), versus this displacement denoted by x, yields a curve similar to that shown in the figure.





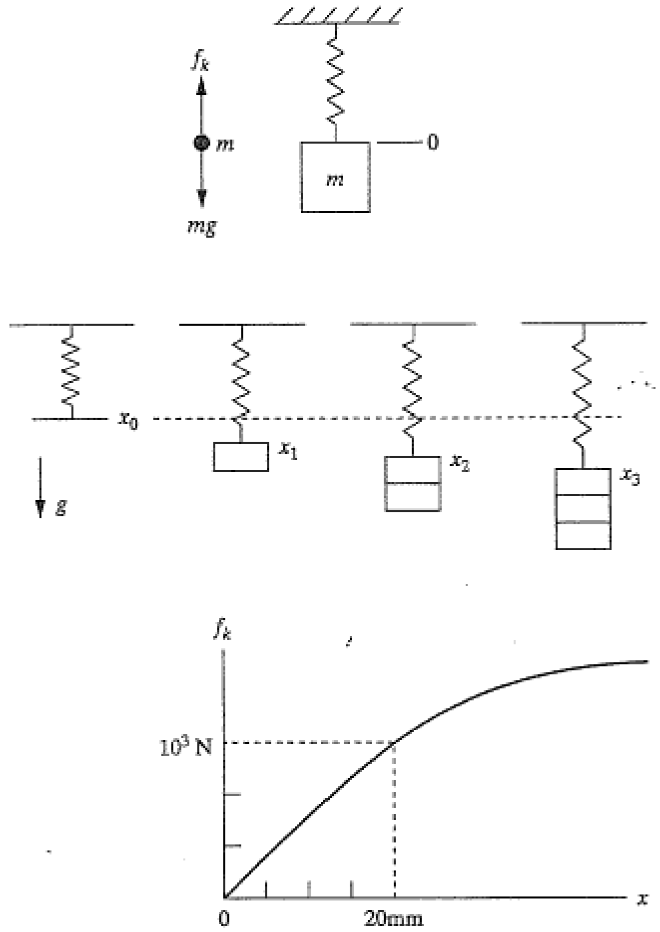


Classification of vibration

Linear and nonlinear vibrations contd:

- Note that in the region of values for x between 0 and about 20 mm, the curve is a straight line. This indicates that for deflections less than 20 mm and forces less than 1000 N, the force that is applied by the spring to the mass is proportional to the stretch of the spring.

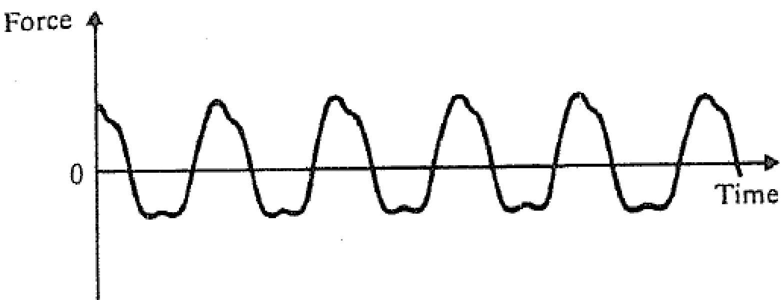
- The constant of proportionality is the slope of the straight line.



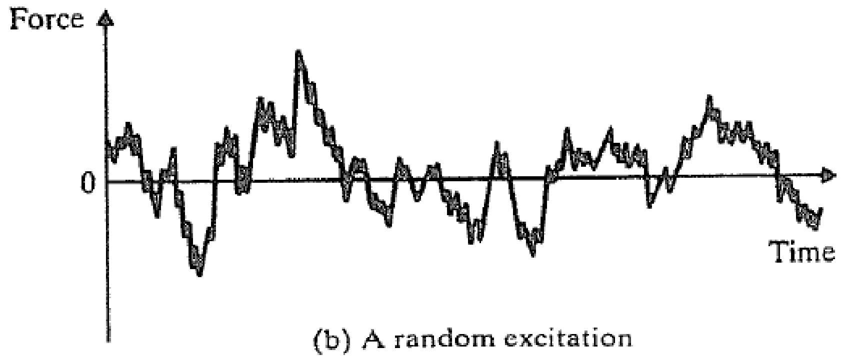
Classification of vibration

- **Deterministic vibration:** If the value or magnitude of the excitation (force or motion) acting on a vibratory system is known at any given time, the excitation is called '**deterministic**'. The resulting vibration is known as '**deterministic vibration**'.

- **Nondeterministic vibration:** In some cases, the excitation is **non-deterministic** or **random**; the value of excitation at a given time cannot be predicted. In these cases, a large collection of records of the excitation may exhibit some statistical regularity. It is possible to estimate averages such as the mean and mean square values of the excitation.



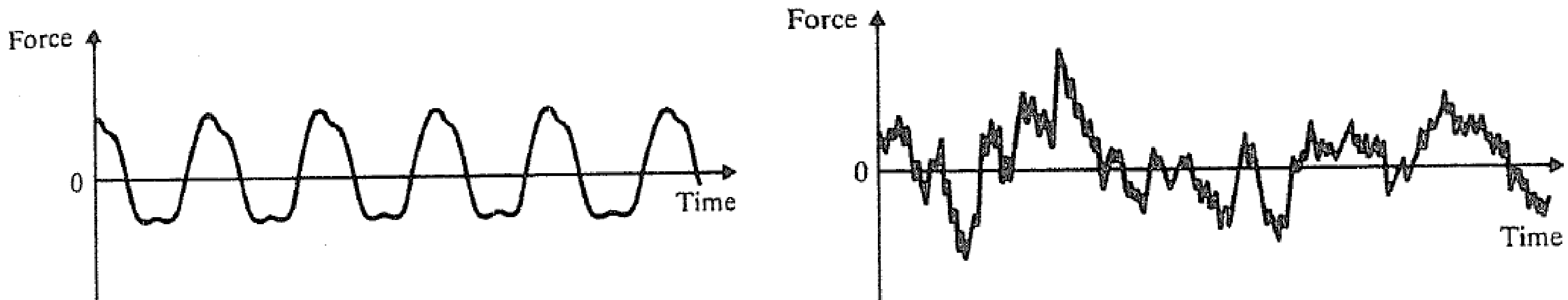
(a) A deterministic (periodic) excitation



(b) A random excitation

Classification of vibration

- Examples of random excitations are wind velocity, road roughness, and ground motion during earthquakes.
- If the excitation is random, the resulting vibration is called **random vibration**. In the case of random vibration, the **vibratory response** of the system is also **random**: it can be described only in terms of statistical quantities.



(a) A deterministic (periodic) excitation

(b) A random excitation

Mathematical background

- Homogeneous linear ODEs with constant coefficients
- Nonhomogeneous ODEs

Introduction

- The dynamic behaviour of mechanical systems is described by what we call second order Ordinary Differential Equations.
- The input to the mechanical structure appears on the right hand side of the equation and is the Force and the solution of the equation gives the output which is usually the displacement.
- In order to be able solve these equations, it is imperative to have a solid background on the solution of homogeneous and nonhomogeneous Ordinary Differential Equations.
- Homogeneous Ordinary Differential Equations represent the ‘Free Vibrations’ and the non-homogeneous Ordinary Differential Equations represent ‘Forced Vibrations’.

Homogeneous linear ODEs with constant coefficients

- We shall now consider second-order homogeneous linear ODEs whose coefficients a and b are constant.

$$y'' + ay' + by = 0$$

- The solution of a first order linear ODE:

$$y' + ky = 0$$

- By separating variables and integrating, we obtain:

$$\frac{dy}{y} = -kdx \qquad \ln|y| = -\int kdx + c^*$$

- Taking exponents on both sides:

$$y(x) = ce^{-\int kdx} = ce^{-kx}$$

- Let's try the above solution in the first equation. Using a constant coefficient k:

$$y = e^{\lambda x}$$

$$\text{Substituting its derivatives: } y' = \lambda e^{\lambda x} \quad \text{and} \quad y'' = \lambda^2 e^{\lambda x}$$

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0$$

Homogeneous linear ODEs with constant coefficients

- Hence if λ is a solution of the important characteristic equation (or auxiliary equation)

$$\lambda^2 + a\lambda + b = 0$$

- Then the exponential solution $y = e^{\lambda x}$ is a solution of the

$$y'' + ay' + by = 0$$

- Now from elementary algebra we recall that the roots of this quadratic equation are:

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b})$$

$$\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$$

- The functions below are solutions to $y'' + ay' + by = 0$

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}$$

Homogeneous linear ODEs with constant coefficients

- From algebra we know that the quadratic equation below may have three kinds of roots:

$$\lambda^2 + a\lambda + b = 0$$

- **Case I:** Two real roots if $a^2 - 4b > 0$
- **Case II:** A real double root if $a^2 - 4b = 0$
- **Case III:** Complex conjugate roots if $a^2 - 4b < 0$

- **CASE I:** In this case, a basis of solutions of $y'' + ay' + by = 0$

  in any interval is: $y_1 = e^{\lambda_1 x} \quad and \quad y_2 = e^{\lambda_2 x}$

  because y_1 and y_2 are defined and real for all x and their quotient is not constant. The corresponding general solution is:

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

Homogeneous linear ODEs with constant coefficients

CASE II: Real double root λ=-a/2

- If the discriminant $a^2 - 4b$ is zero, we see from

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b})$$
$$\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$$

that we get only one root: $\lambda = \lambda_1 = \lambda_2 = -a/2,$
hence only one solution:
$y_1 = e^{-(a/2)x}$

To obtain a second independent solution y_2 needed for a basis, we use the method of order of reduction. Setting

$y_2 = uy_1,$ Substituting this and its derivatives $y_2' = u'y_1 + uy_1'$ *and* y_2'' into

$$y'' + ay' + by = 0$$

Homogeneous linear ODEs with constant coefficients

CASE II: Real double root λ=-a/2

- We have : $(u''y_1 + 2u'y_1' + uy_1'') + a(u'y_1 + uy_1') + buy_1 = 0$
- Collecting terms $u''y_1 + u'(2y_1' + ay_1) + u(y_1'' + ay_1' + by_1) = 0$
- This expression in the last paranthesis is zero, since y_1 is a solution of

$$y'' + ay' + by = 0$$

- The expression in the second paranthesis is zero too since $2y_1' = -ae^{-ax/2} = -ay_1$
- We are thus left with

$$u''y_1 = 0$$

Hence

$$u'' = 0$$

By two integrations

$$\mathrm{u} = \mathrm{c}_1 x + c_2$$

Homogeneous linear ODEs with constant coefficients

CASE II: Real double root λ=-a/2

To get a second independent solution $y_2=uy_1$, we can simply choose $c_1=1$ and $c_2=0$ and take u=x. Then $y_2=xy_1$. Since these solutions are not proportional, they form a basis. Hence in the case of a double root of

$$\lambda^2 + a\lambda + b = 0$$

a basis of solutions of $y'' + ay' + by = 0$

on any interval is: $e^{-ax/2}, \quad xe^{-ax/2}$

- The corresponding general solution is: $y = (c_1 + c_2 x)e^{-ax/2}$

Homogeneous linear ODEs with constant coefficients

CASE III: Complex roots –a/2+iω and –a/2-iω

- This case occurs if the discriminant of the characteristic equation $\lambda^2 + a\lambda + b = 0$

  is negative. In this case, the roots of the above equation and thus the solutions of the ODE

  come at first out complex. However, we show that from them we can obtain a basis of real solutions:

$$y'' + ay' + by = 0$$

where $$y_1 = e^{-ax/2} \cos \omega x, \qquad y_2 = e^{-ax/2} \sin \omega x$$

$$\omega^2 = b - \frac{1}{4}a^2$$

- This is proved in the next slides. It can be verified by substitution that these are solutions in the present case. They form a basis on any interval since their quotient cotωx is not constant. Hence, a real general solution in Case III is:

$$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x) \qquad (A, B \text{ arbitrary })$$

Homogeneous linear ODEs with constant coefficients: Proof

- Complex number representation of harmonic motion: Since

$$\vec{X} = O\vec{P}$$

this vector can be represented as a complex number:

$$\vec{X} = a + ib$$

where $i = \sqrt{-1}$ and a and b denote x and y components of $\vec{X}$. Components a and b are also called the real and the imaginary parts of the vector X. If A denotes the modulus or the absolute value of the vector X, and θ denotes the argument of the angle between the vector and the x-axis, then $\vec{X}$ can also be expressed as:

$$\vec{X} = A \cos\theta + iA \sin\theta$$

$$A = (a^2 + b^2)^{1/2}$$

$$\theta = \tan^{-1}\frac{b}{a}$$

Homogeneous linear ODEs with constant coefficients: Proof

Complex number representation of harmonic motion

Noting that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, . . . , $\cos\theta$ and $i\sin\theta$ can be expanded in a series as

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots = 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \cdots \quad (1.39)$$

$$i\sin\theta = i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right] = i\theta + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \cdots \quad (1.40)$$

Equations (1.39) and (1.40) yield

$$(\cos\theta + i\sin\theta) = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = e^{i\theta} \quad (1.41)$$

and

$$(\cos\theta - i\sin\theta) = 1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \cdots = e^{-i\theta} \quad (1.42)$$

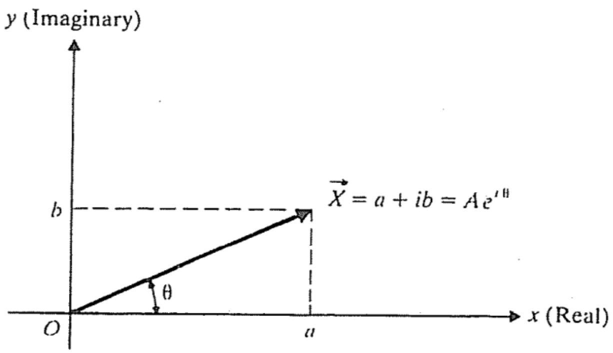


FIGURE 1.40

$$\vec{X} = A\cos\theta + iA\sin\theta \quad (1.36)$$

Thus Eq. (1.36) can be expressed as

$$\vec{X} = A(\cos\theta + i\sin\theta) = Ae^{i\theta}$$

Homogeneous linear ODEs with constant coefficients: Proof

- As apparent we have two complex roots. These are:

$$\lambda_1 = \frac{1}{2}a + i\omega \quad \text{and} \quad \lambda_1 = \frac{1}{2}a - i\omega$$

- We know from basic mathematics that a complex exponential function can be expressed as:

$$e^{r+it} = e^r e^{it} = e^r(\cos t + i \sin t)$$

- Thus the roots of the second order Ordinary Differential Equation can be expressed as:

$$e^{\lambda_1 x} = e^{-(a/2)x + i\omega x} = e^{-(a/2)x}(\cos \omega x + i \sin \omega x)$$
$$e^{\lambda_2 x} = e^{-(a/2)x - i\omega x} = e^{-(a/2)x}(\cos \omega x - i \sin \omega x)$$

- We now add these two lines and multiply the result by ½. This gives:

$$y_1 = e^{-ax/2} \cos \omega x$$

- Then we subtract the second line from the first and multiply the result by 1/2i. This gives:

$$y_2 = e^{-ax/2} \sin \omega x$$

Homogeneous linear ODEs with constant coefficients

Case	Roots	Basis	General solution
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -a/2$	$e^{-ax/2}, \quad xe^{-ax/2}$	$y = (c_1 + c_2 x) e^{-ax/2}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$	$y_1 = e^{-ax/2} \cos \omega x$	$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$
	$\lambda_2 = -\frac{1}{2}a - i\omega$	$y_2 = e^{-ax/2} \sin \omega x$	

Nonhomogeneous ODEs

- In this section, we proceed from homogeneous to nonhomogeneous ODEs.

$$y'' + p(x)y' + q(x)y = r(x)$$

- The general solution consists of two parts:

$$y(x) = y_h(x) + y_p(x)$$

where $y_h = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous ODE.

Term in r(x)	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0,1,2,\ldots)$	$K_n x^n + K_{n-1} x^{n-1} + \ldots + K_1 x + K_0$
$k \cos \omega x$ $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

Nonhomogeneous ODEs

Choice rules for the method of undetermined coefficients

a) Basic rule: If r(x) is one of the functions in the first column in the Table, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into

$$y'' + p(x)y' + q(x)y = r(x)$$

b) Modification rule: If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to the above equation, multiply your choice of y_p by x (or x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE)

c) Sum rule: If r(x) is a sum of functions in the first column of the table, choose for y_p the sum of the functions in the corresponding lines of the second column.

Free Vibration of Single Degree of Freedom Systems

- Harmonic Motion
- Free vibration of undamped SDOF systems
- Free vibration of damped SDOF systems

Harmonic motion

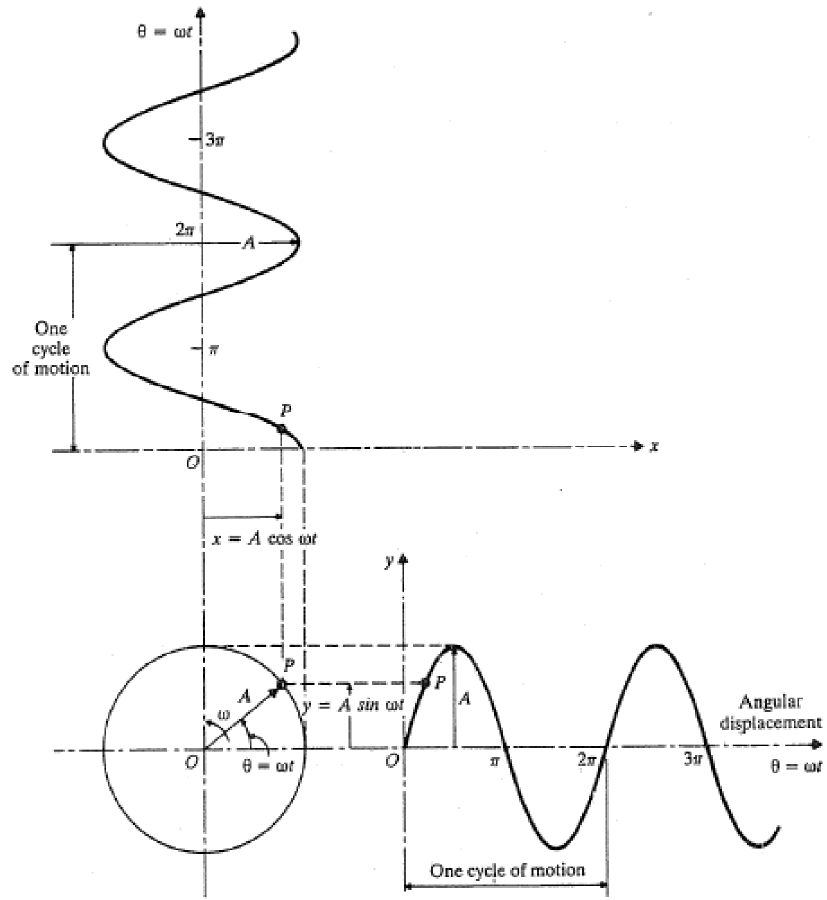
- Oscillatory motion may repeat itself regularly, as in the case of a simple pendulum, or it may display considerable irregularity, as in the case of ground motion during an earthquake.

- If the motion is repeated after equal intervals of time, it is called **periodic motion**.

- The simplest type of **periodic motion** is **harmonic motion**.

Harmonic motion

- Shown in the figure is a vector OP that rotates counterclockwise with constant angular velocity ω.
- At any time t, the angle that OP makes with the horizontal is $\theta=\omega t$.
- Let y be the projection of OP on the vertical axis. Then y=A sin ωt. Here y, a function of time is plotted versus ωt.



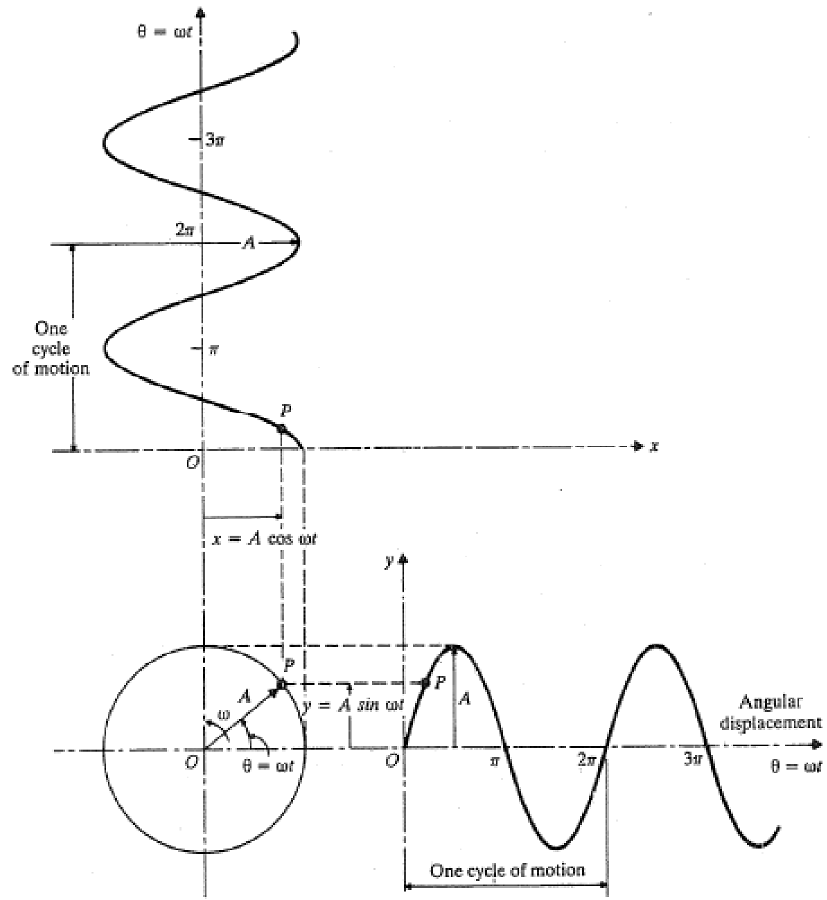
Harmonic motion

- A particle that experiences this motion is said to have **harmonic motion**.

- The maximum displacement of a vibrating body from its equilibrium position is called the **amplitude of vibration**. Amplitude A is shown in the figure.

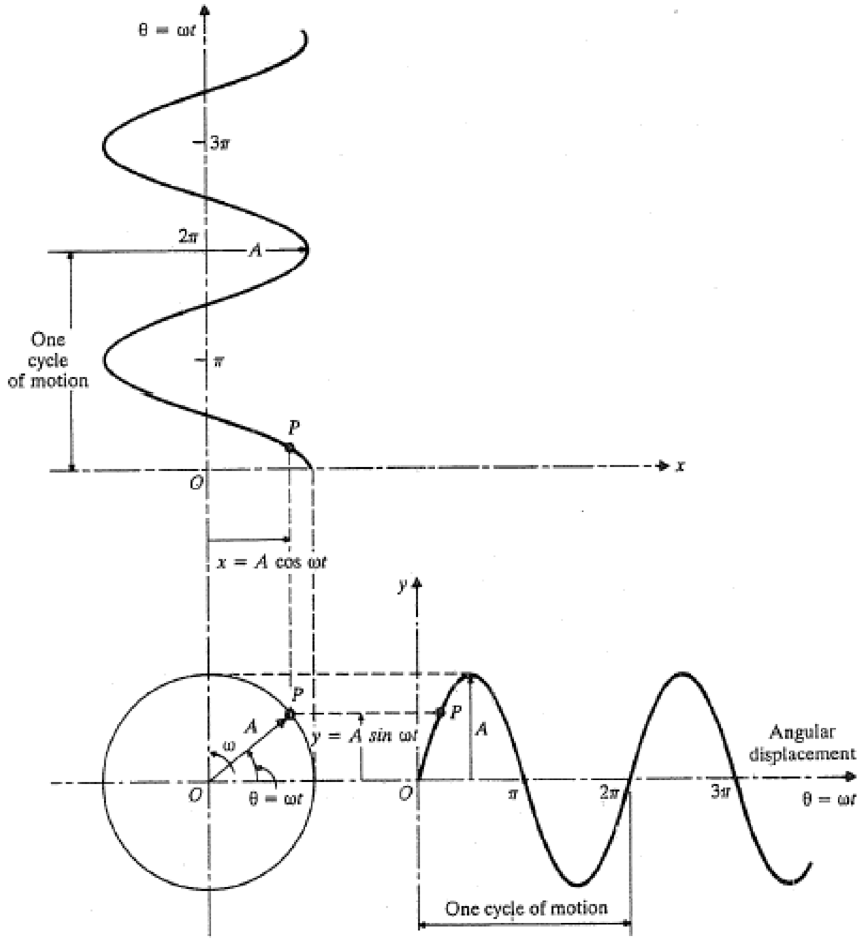
- Range 2A is the peak to peak displacement.

- Now consider the units of θ. Let C be the circumference of the circle shown in the figure.



Harmonic motion

- Thus C=2πA. Or we can write C=Aθ, where θ=2π for one revolution. Thus defined, θ is said to be in radians and is equivalent to 360°. Therefore, one radian is approximately equal to 58.3°.
- In general, for any arc length, s=A θ , where θ is in radians. It follows that ω in the figure would be in radians per second.
- As seen in the figure, the vectorial method of representing harmonic motion requires the description of both the horizontal and vertical components.
- The time taken to complete one cycle of motion is known as the period of oscillation or time period and is denoted by τ. The period is the time for the motion to repeat (the value of τ in the figure).

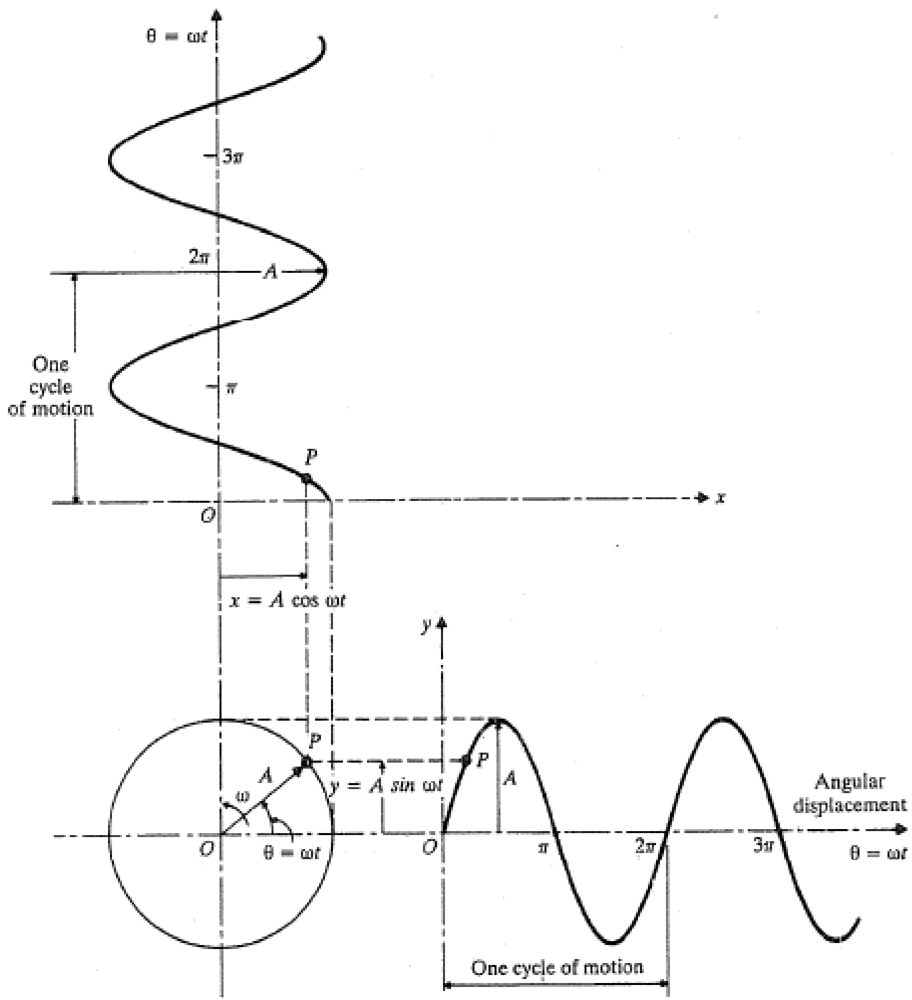


Harmonic motion

- Note that ω τ=2 π where ω denotes the angular velocity of the cyclic motion. The angular velocity ω is also called the **circular frequency**.

- The movement of a vibrating body from its undisturbed or equilibrium position to its extreme position in one direction, then to the equilibrium position, then to its extreme position in the other direction, and back to equilibrium position is called **a cycle of vibration**.

- One revolution (i.e., angular displacement of 2π radians) of the pin P in the figure or one revolution of the vector OP in the figure constitutes a **cycle**. Cycle is the motion in one period, as shown in the figure.



Harmonic motion

- Frequency is the number of cycles per unit time.
- The most common unit of time used in vibration analysis is seconds. Cycles per second is called Hertz.
- The time the cycle takes to repat itself is the period T. In terms of the period, the frequency is:

$$f = \frac{1}{\tau}$$

- The frequency f is related to ω: $f = \frac{\omega}{2\pi}$

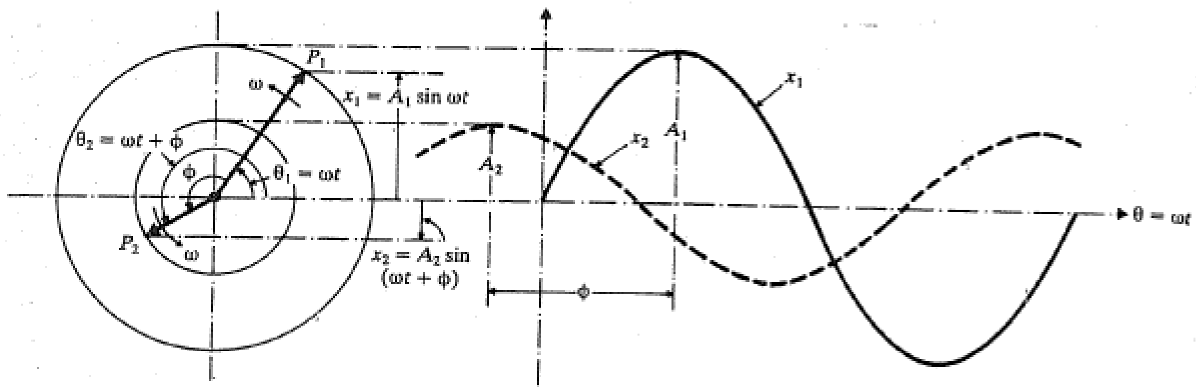
$$\omega = 2\pi f$$

Harmonic motion

- Phase angle: Consider two vibratory motions denoted by:

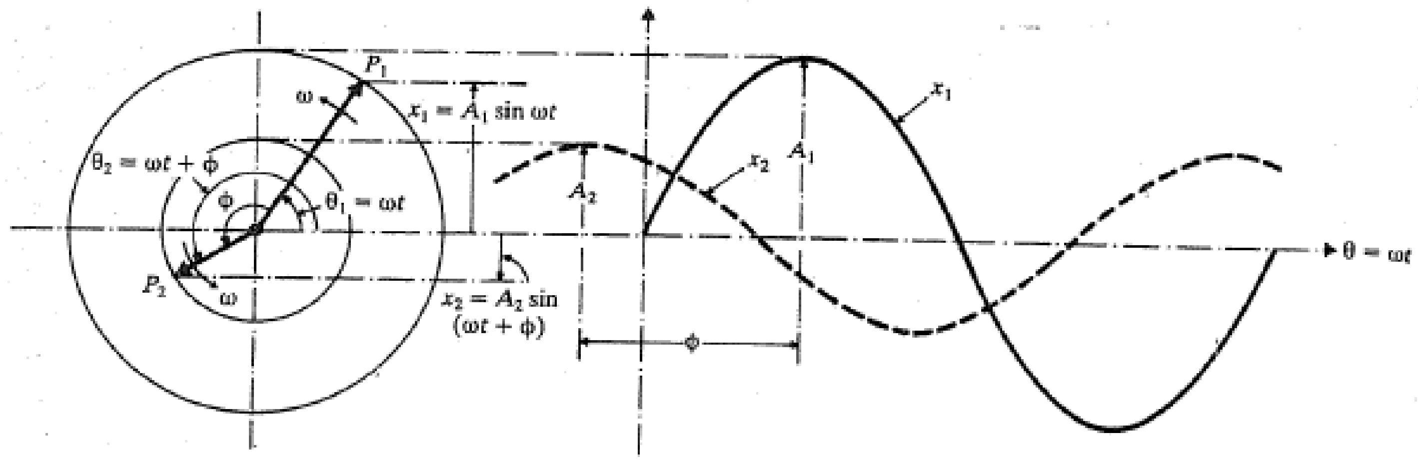
$$x_1 = A_1 \sin \omega t$$
$$x_2 = A_2 \sin(\omega t + \phi)$$

- These two harmonic motions are called **synchronous** because they have the same frequency or angular velocity ω. Two synchronous oscillations need not have the same amplitude, and they need not attain their maximum values at the same time as shown in the figure.



Harmonic motion

- In this figure, the second vector OP_2 leads the first one OP_1 by an angle ϕ known as the **phase angle**. This means that the maximum of the second vector would occur ϕ radians earlier than that of the first vector. These two vectors are said to have a phase difference of ϕ.



Harmonic motion

- From introductory physics and dynamics, the fundamental kinematical quantities used to describe the motion of a particle are displacement, velocity and acceleration vectors.
- The acceleration of a particle is given by:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x}$$

- Thus, displacement, velocity, and acceleration have the following relationships in harmonic motion:

$$x = A\sin\omega t$$

$$v = \dot{x} = A\omega\cos\omega t$$

$$a = \ddot{x} = -A\omega^2\sin\omega t$$

Operations on harmonic functions

- Using complex number representation, the rotating vector $\vec{X}$ can be written as:

$$\vec{X} = A\,e^{i\omega t}$$

where ω denotes the circular frequency (rad/sec) of rotation of the vector $\vec{X}$ in counterclockwise direction. The differentiation of the harmonics given by the above equation gives:

$$\frac{d\vec{X}}{dt} = \frac{d}{dt}(Ae^{i\omega t}) = i\omega Ae^{i\omega t} = i\omega\vec{X}$$

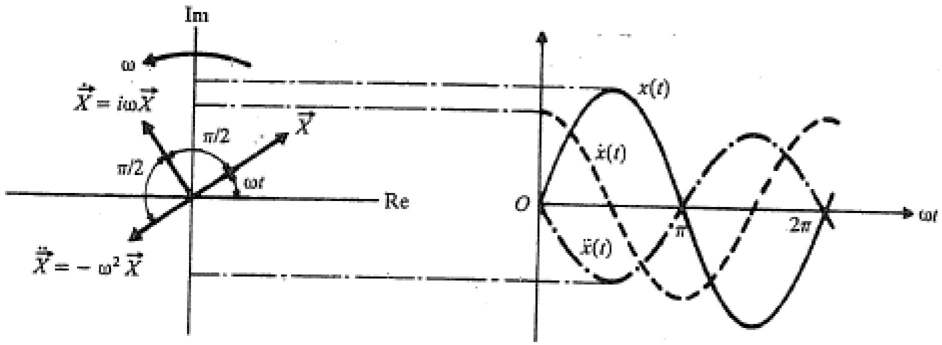
$$\frac{d^2\vec{X}}{dt^2} = \frac{d}{dt}(i\omega Ae^{i\omega t}) = -\omega^2 Ae^{i\omega t} = -\omega^2\vec{X}$$

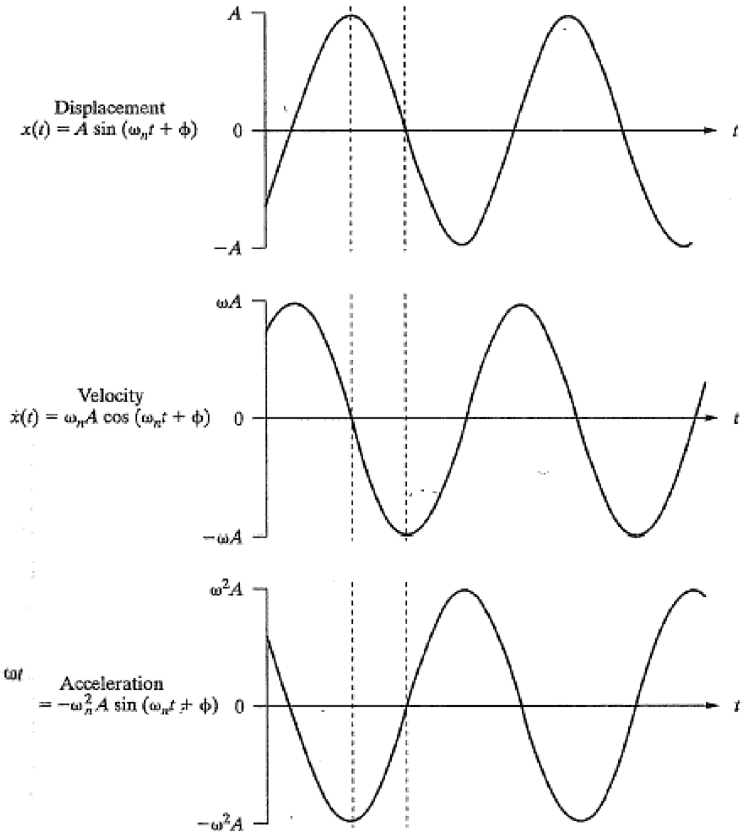
- Thus the displacement, velocity and acceleration can be expressed as:

$$\begin{aligned}
\text{displacement} &= \mathrm{Re}[Ae^{i\omega t}] &&= A\cos\omega t \\
\text{velocity} &= \mathrm{Re}[i\omega Ae^{i\omega t}] &&= -\omega A\sin\omega t \\
& &&= \omega A\cos(\omega t + 90°) \\
\text{acceleration} &= \mathrm{Re}[-\omega^2 Ae^{i\omega t}] &&= -\omega^2 A\cos\omega t \\
& &&= \omega^2 A\cos(\omega t + 180°)
\end{aligned}$$

Operations on harmonic functions

- It can be seen that the acceleration vector leads the velocity vector by 90 degrees and the velocity vector leads the displacement vector by 90 degrees.





Harmonic motion

- **Natural frequency:** If a system, after an initial disturbance, is left to vibrate on its own, the frequency with which it oscillates without external forces is known as its **natural frequency**. As will be seen, a vibratory system having n degrees of freedom will have, in general, n distinct **natural frequencies of vibration**.
- **Beats:** When two harmonic motions, with frequencies close to one another, are added, the resulting motion exhibits a phenomenon known as **beats**. For example if:

$$x_1(t) = X\cos\omega t$$

$$x_2(t) = X\cos(\omega+\delta)t$$

where δ is a small quantity.

The addition of these two motions yield:

$$x(t) = x_1(t) + x_2(t) = X[\cos\omega t + \cos(\omega+\delta)t]$$

Harmonic motion

Beats:

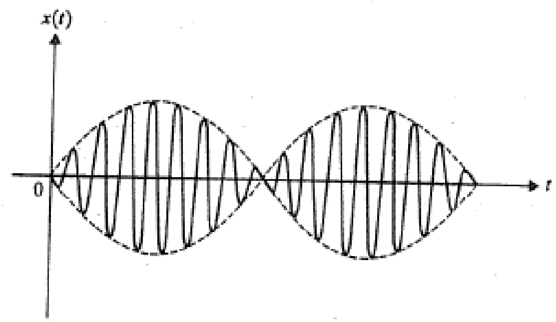
$$x(t) = x_1(t) + x_2(t) = X[\cos \omega t + \cos(\omega + \delta)t]$$

Using the relation

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

The first equation can be written as:

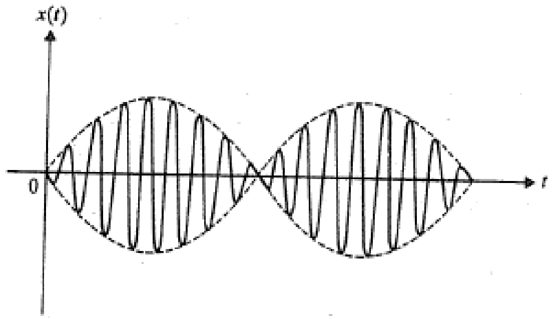
$$x(t) = 2X \cos\frac{\delta t}{2}\cos(\omega + \frac{\delta}{2})t$$



Harmonic motion

Beats:

- It can be seen that the resulting motion x(t) represents a cosine wave with frequency $\omega + \frac{\delta}{2}$ which is approximately equal to ω and with a varying amplitude $2X\cos\frac{\delta t}{2}$. Whenever, the amplitude reaches a maximum it is called a beat.
- In machines and in structures,the beating phenomenon occurs when the forcing frequency is close to the natural frequency of the system. We will later return to this topic.



Harmonic motion

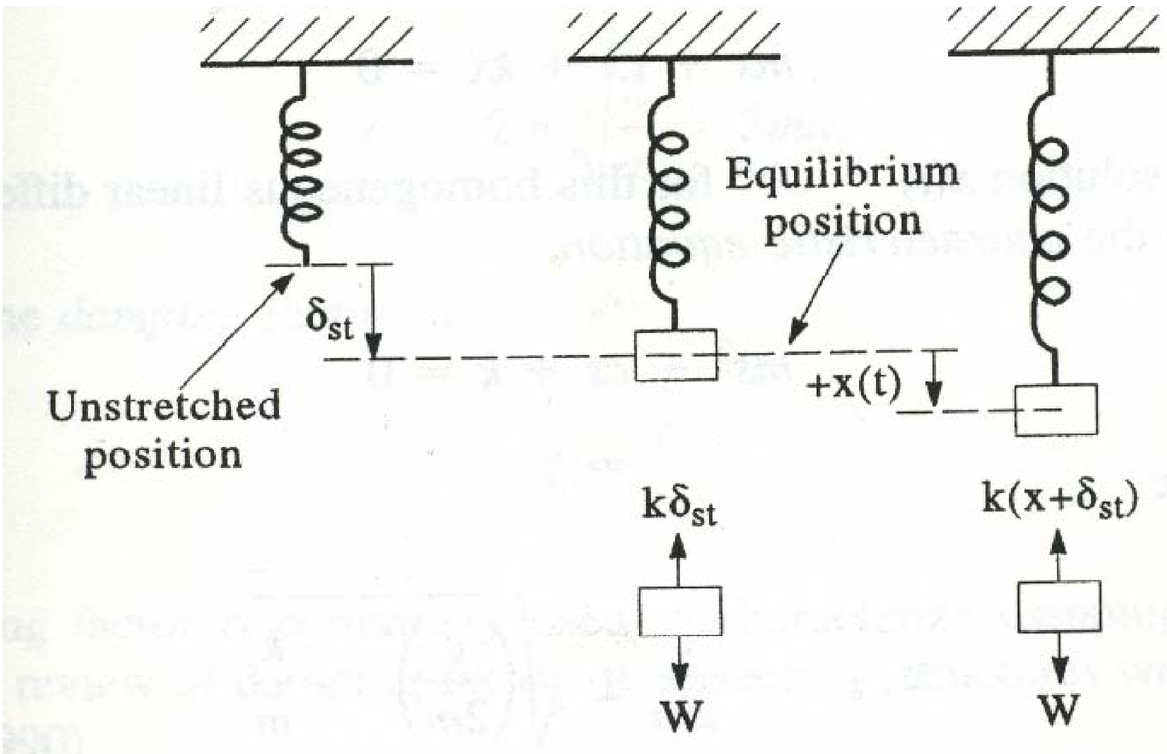
- **Octave:** When the maximum value of a range of frequency is twice its minimum value, it is known as an octave band.

- For example, each of the ranges 75-150 Hz, 150-300 Hz, and 300-600Hz can be called an octave band.

- In each case, the maximum and minimum values of frequency, which have a ratio of 2:1, are said to differ by an octave.

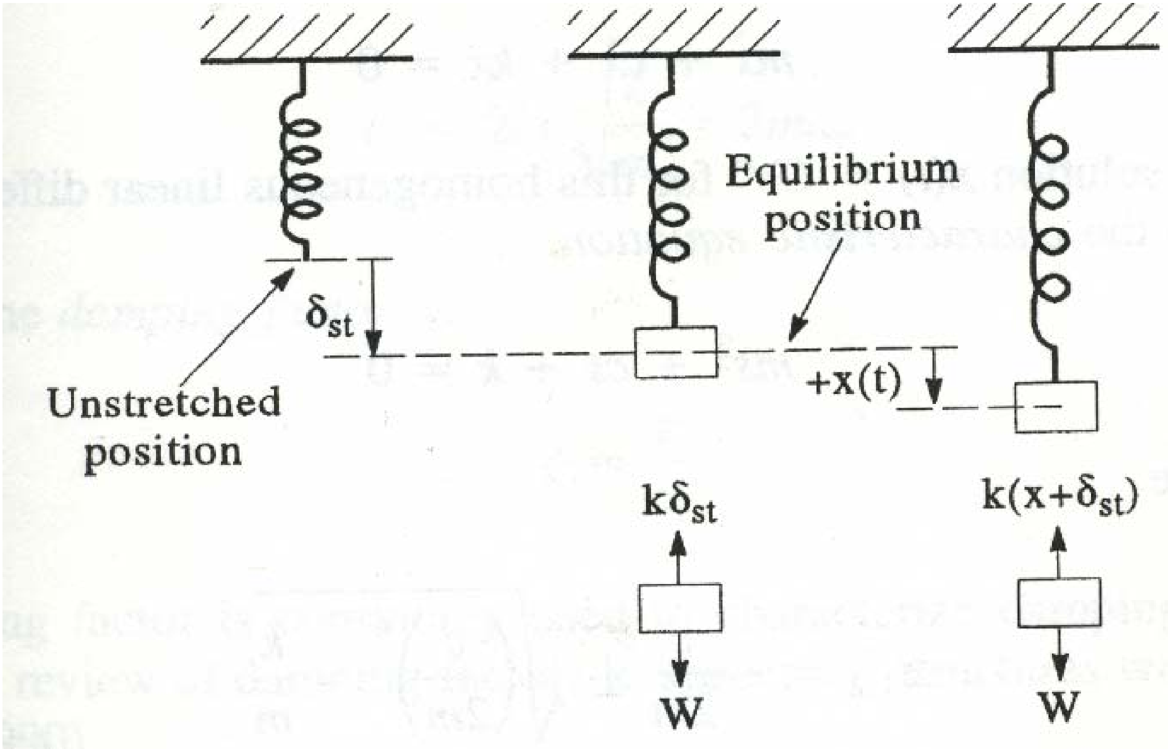
Free vibration of undamped SDOF systems

- Consider the single-degree-of-freedom (SDOF) system shown in the figure. The spring is originally in the unstretched position as shown. It is assumed that the spring obeys Hooke's law. The force in the spring is proportional to displacement with the proportionality constant (spring constant) equal to k.



Free vibration of undamped SDOF systems

- The stiffness in a spring can be related more directly to material and geometric properties of the spring. A spring like behaviour results from a variety of configurations, including longitudinal motion (vibration in the direction of the length), transverse motion (vibration perpendicular to the length), and torsional motion(vibration rotating around the length).

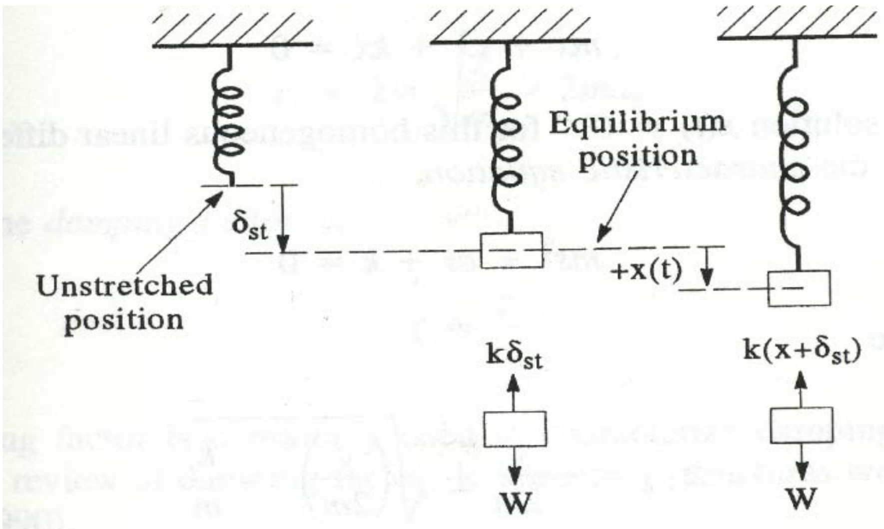


Free vibration of undamped SDOF systems

- A spring is generally made of an elastic material. For a slender elastic material of length l, cross-sectional area A and elastic modulus E (or Young's modulus), the stiffness of the bar for vibration along its length is given by:

$$k = \frac{EA}{l}$$

- The modulus E has the units of Pascal (denoted Pa) which are N/m2.



Free vibration of undamped SDOF systems

- When the mass m (weight W) is applied, the spring will deflect to a static equilibrium position δ_{st}.

- At this position, we find that:

$$W = mg = k\delta_{st}$$

- If the mass is perturbed and allowed to move dynamically, the displacement *x*, measured from the equilibrium position, will be a function of time. Here, x(t) is the absolute motion of the mass and the force in the spring can be expresssed as:

$$-k(x+\delta_{st})$$

- To determine the position as a function of time, the equations of motion are employed; the free body diagrams are drawn as shown in the figure. Note that x is measured positive downward.

Free vibration of undamped SDOF systems

- Applying Newton's second law,

$$W - k(x + \delta_{st}) = m\ddot{x}$$

- But from the static condition, note that W=k$\delta_{st.}$ Thus, the equation of motion becomes:

$$m\ddot{x} + kx = 0$$

- With the standard form of: $$\ddot{x} + \frac{k}{m}x = 0$$

- This is Case III that has complex roots where the general solution was computed as:

$$x = e^{-ax/2}(A\cos\omega t + B\sin\omega t)$$

- Since in the above equation:

$$a = 0 \qquad \omega^2 = b - \frac{1}{4}a^2$$

Free vibration of undamped SDOF systems

- It is shown that

$$x(t) = A\cos\omega_n t + B\sin\omega_n t$$

where A and B are constants of integration and

$$\omega_n = \sqrt{\frac{k}{m}} \qquad (rad/\sec)$$

Here ω_n defines the natural frequency of the mass. This is the frequency at which the mass will move regardless of the amplitude of the motion as long as the spring in the system continues to obey Hooke's law. The natural frequency in Hertz is:

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (Hz)$$

Free vibration of undamped SDOF systems

- The initial conditions $x = x_o$ at t=0 and $\dot{x} = \dot{x}_o$ at $t = 0$ are used to evaluate the constants of integration A and B. When substituted into the equation

$$x(t) = A\cos\omega_n t + B\sin\omega_n t$$

we get:

$A = x_o$ and $B = \frac{\dot{x}_o}{\omega_n}$ and consequently $x(t) = x_o \cos\omega_n t + \frac{\dot{x}_o}{\omega_n}\sin\omega_n t$

- The sum in the equation

$$x(t) = A\cos\omega_n t + B\sin\omega_n t$$

can also be combined to a phase shifted cosine with amplitude $C = \sqrt{A^2 + B^2}$ and phase angle ϕ=arctan(B/A). For this purpose let:

$$A = C\cos\phi \text{ and } \mathrm{B} = C\sin\phi$$

Free vibration of undamped SDOF systems

- Introducing the new values of A and B into

$$x(t) = A\cos\omega_n t + B\sin\omega_n t$$

we get:

$$x(t) = C\cos\phi\cos\omega_n t + C\sin\phi\sin\omega_n t$$

- Since

$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

- x(t) can be expressed as:

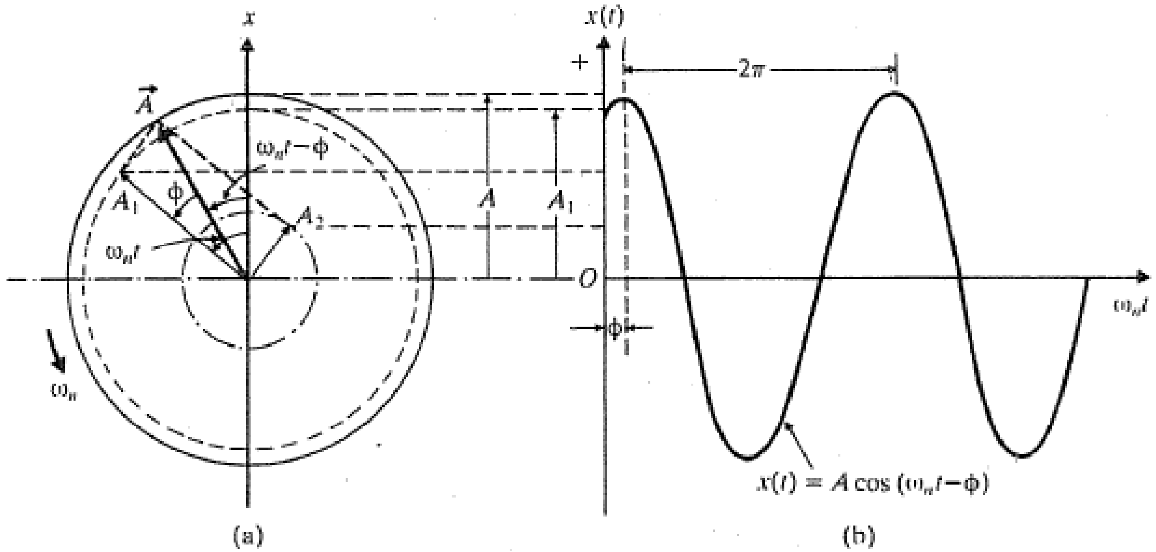
$$x(t) = C\cos(\omega_n t - \phi)$$

- Where ϕ=arctan(B/A) and consequently:

$$\phi = \arctan\left(\frac{\dot{x}_o}{\omega_n x_o}\right) \text{ and } C = \sqrt{A^2 + B^2} = \sqrt{x_o^2 + \left(\frac{\dot{x}_o}{\omega_n}\right)^2}$$

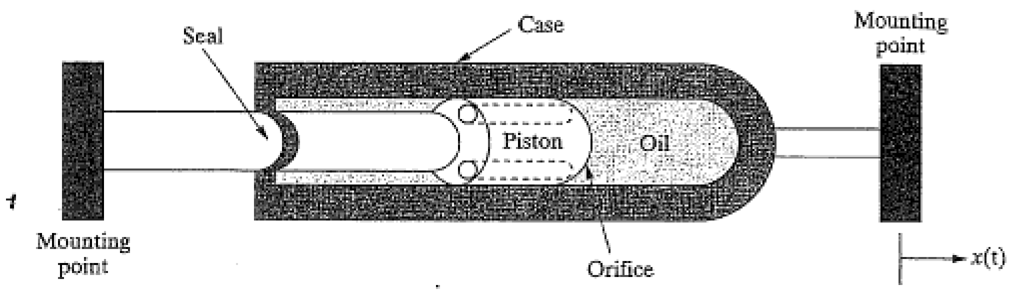
Free vibration of undamped SDOF systems

- The equation $x(t) = x_o \cos \omega_n t + \frac{\dot{x}_o}{\omega_n} \sin \omega_n t$ is a harmonic function of time. Thus, the spring-mass system is called a harmonic oscillator. The nature of harmonic oscillation is shown in the figure. If C denotes a vector of magnitude C, which makes an angle $\omega_n t - \phi$ with respect to the vertical x axis, then the solution $x(t) = C\cos(\omega_n t - \phi)$ can be seen to be the projection of vector C on the x axis.



Damping

- **Undamped and damped vibration:** The response of a spring-mass model predicts that the system will oscillate indefinitely. However, everyday observation indicates that most freely oscillating systems eventually die out and reduce to zero motion.
- The choice of representative model for the observed decay in an oscillating system is based partially on physical observation and partially on mathematical convenience. The theory of differential equations suggests that adding a term to equation $m\ddot{x}(t)+kx(t)=0$ of the form $c\dot{x}$, where c is a constant, will result in a solution x(t) that dies out.
- Physical observation agrees fair well with this model and it is used very successfully to model the damping or decay in a variety of mechanical systems.
- This type of damping is called the **viscous damping**.

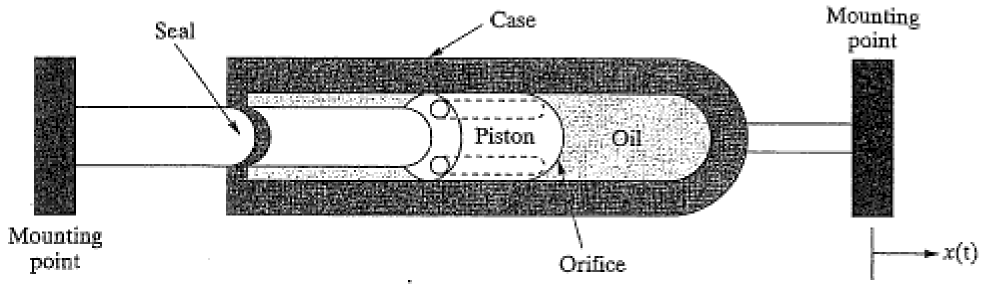


Damping

- The laminar flow of the oil through the perforations as the piston moves causes a damping force on the piston.
- The force is proportional to the velocity of the piston, in a direction opposite that of the piston motion. This damping force has the form:

$$f_c = c\dot{x}(t)$$

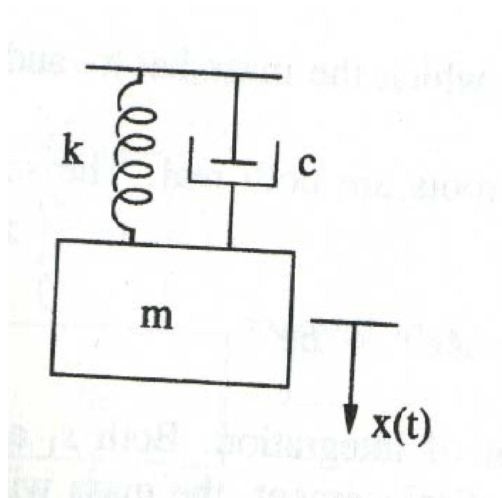
where c is a constant of proportionality related to the oil viscosity. The constant c, called the damping coefficient, has units of Ns/m, or kg/s.

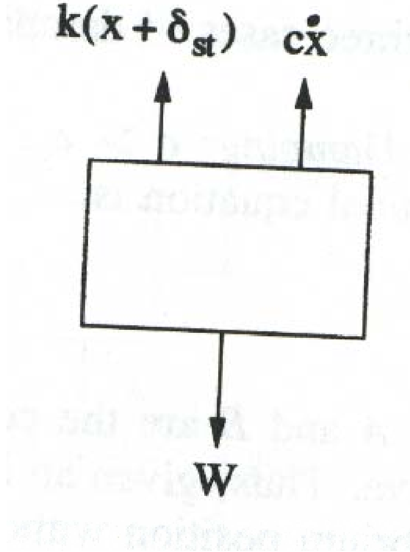


Damped free vibration of SDOF system

- Consider the spring-mass system with an energy dissipating mechanism described by the damping force as shown in the figure. It is assumed that the damping force F_D is proportional to the velocity of the mass, as shown; the damping coefficient is *c*. When Newton's second law is applied, this model for the damping force leads to a linear differential equation,

$$m\ddot{x} + c\dot{x} + kx = 0$$





Damped free vibration of SDOF system

- The ODE is homogeneous linear and has constant coefficients. The characteristic equation is found by dividing the below equation by m:

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0$$

- By the roots of a quadratic equation, we obtain:

$$s_1 = -\alpha + \beta, \quad s_2 = -\alpha - \beta,$$

where

$$\alpha = \frac{\mathrm{c}}{2\mathrm{m}} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}$$

- It is now most interesting that depending on the amount of damping (much, medium or little) there will be three types of motion corresponding to the three cases I, II and III.

Damped free vibration of SDOF system

Case		Roots	Definition
I	$c^2 > 4mk$	Distinct real roots s_1, s_2	Overdamping
II	$c^2 = 4mk$	Real double root	Critical damping
III	$c^2 < 4mk$	Complex conjugate roots	Underdamping

Damped free vibration of SDOF system

- Define the critical damping coefficient c_c as that value of c that makes the radical equal to zero,

$$c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

- Define the damping factor as:

$$\xi = \frac{c}{c_c} = \frac{c}{2m\omega_n}$$

- Introducing the above equation into

$$s_{1,2} = -\frac{\mathrm{c}}{2\mathrm{m}} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

- We find: $$s_{1,2} = \left(-\xi \pm \sqrt{\xi^2 - 1}\right)\omega_n$$
- Then the solution can be written as:

$$x(t) = Ae^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + Be^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

Three cases of damping

- Heavy damping when $c > c_c$
- Critical damping $c = c_c$
- Light damping $0 < c < c_c$

Heavy damping (c > cc or ζ>1)

- The roots are both real. The solution to the differential equation is:

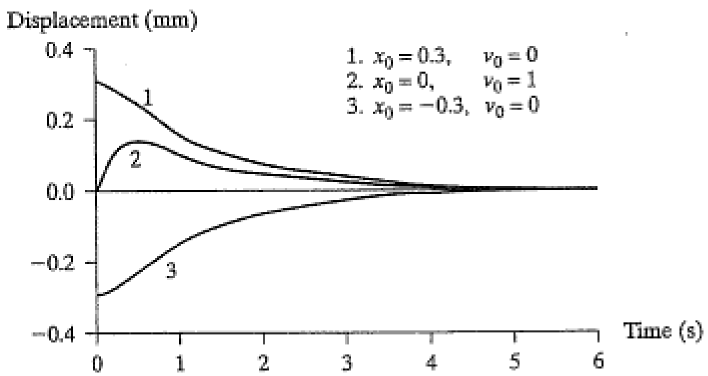
$$x(t) = Ae^{s_1 t} + Be^{s_2 t}$$

where A and B are the constants of integration. Both s_1 and s_2 will be negative because $\alpha > 0$, $\beta > 0$, and $\beta^2 = \alpha^2 - k/m < \alpha^2$. Since

$$s_1 = -\alpha + \beta, \quad s_2 = -\alpha - \beta, \text{where } \alpha = \frac{\mathrm{c}}{2\mathrm{m}} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}$$

Thus, given any initial displacement, the mass will decay to the equilibrium position without vibratory motion. An overdamped system does not oscillate but rather returns to its rest position exponentially.

$$x(t) = Ae^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + Be^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$



Critical damping (c = c_c, or ζ=1)

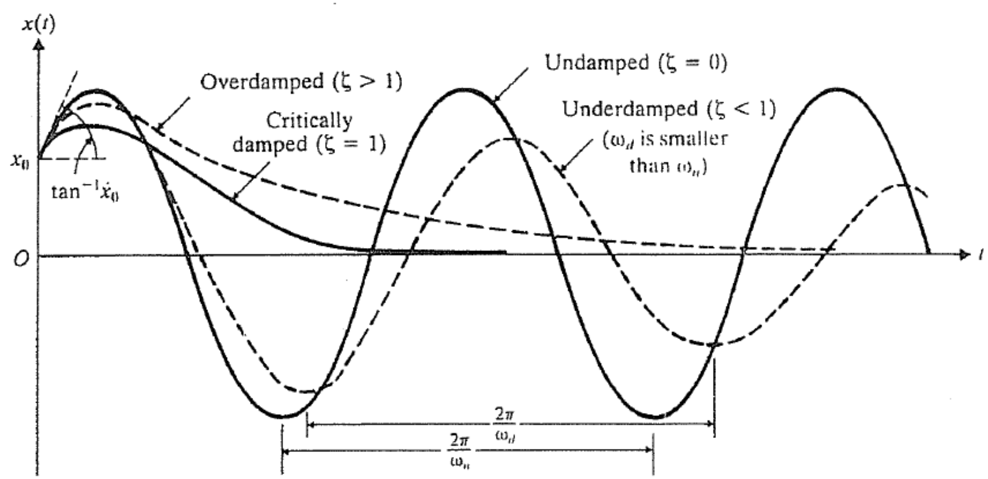
- Since $\beta = \frac{1}{2m}\sqrt{c^2 - 4mk}$ is zero in this case, $s_1 = s_2 = -\alpha = -c_c/2m = -\omega_n$.
- Both roots are equal and the general solution is: $x(t) = (A + Bt)e^{-\omega_n t}$.
  Substituting the initial conditions, $x = x_o$ at t=0 and $\dot{x} = \dot{x}_o$ at $t = 0$

$$A = x_o \text{ and } B = \dot{x}_o + \omega_n x_o$$

and the solution becomes:

$$x(t) = \left[x_o + (\dot{x}_o + \omega_n x_o)t\right]e^{-\omega_n t}$$

- The motion is again not vibratory and decays to the equilibrium position.



Light damping (0 < c < c_c or ζ<1)

- This case occurs if the damping constant c is so small that

$$c^2 < 4mk$$

- Then β is no longer real but pure imaginary.

$$\beta = i\omega^* \text{ where } \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

- The roots of the characteristic equation are now complex conjugate:

$$s_1 = -\alpha + i\omega^*, \qquad s_2 = -\alpha - i\omega^*$$

with

$$\alpha = \frac{c}{2m}$$

- Hence the corresponding general solution is:

$$x = e^{-\alpha t}(A\cos\omega^* t + B\sin\omega^* t) = Ce^{-\alpha t}\cos(\omega^* t - \phi_o)$$

where

$$\mathrm{C}^2 = A^2 + B^2 \text{ and } \tan\phi_o = B/A$$

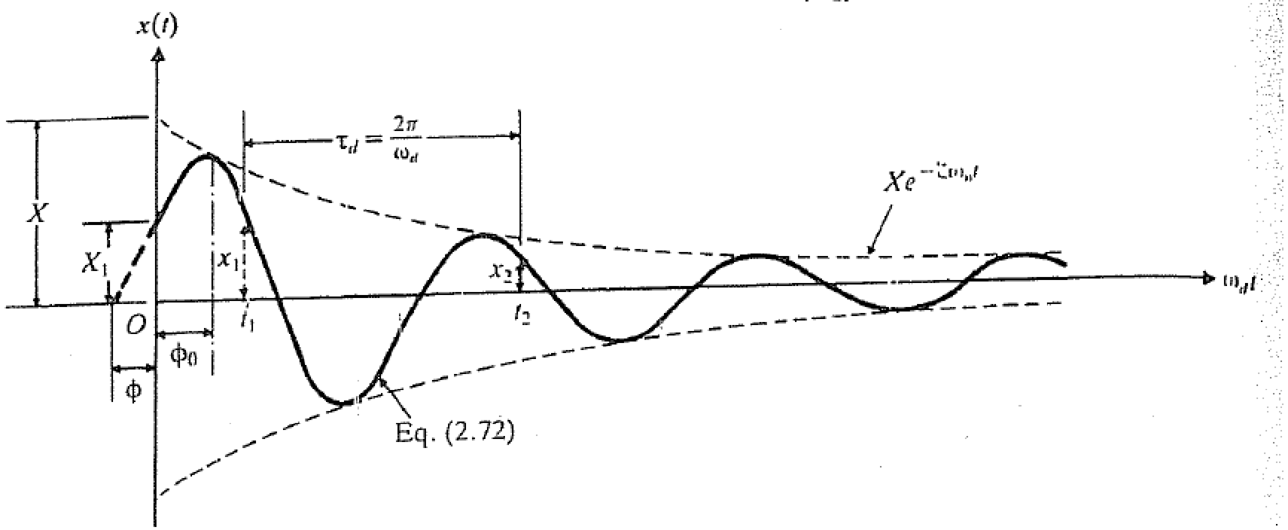
Light damping ($0 < c < c_c$)

- The solution can also be expressed as:

$$x(t) = e^{-\zeta\omega_n t}\left(A\cos\sqrt{1-\zeta^2}\,\omega_n t + B\sin\sqrt{1-\zeta^2}\,\omega_n t\right)$$

- The roots are complex. It is easily shown, using Euler's formula that the general solution is: $x(t) = \left[C\cos(\omega_d t - \phi_o)\right]e^{-\xi\omega_n t}$

  where C and ϕ are the constants of integration. The damped natural frequency ω_d is given by $\omega_d = \omega_n\sqrt{1-\xi^2}$



Light damping ($0 < c < c_c$)

- For the initial conditions

$$x(t=0) = x_o$$
$$\dot{x}(t=0) = \dot{x}_o$$

- The equation

$$x(t) = e^{-\zeta\omega_n t}\left(A\cos\sqrt{1-\zeta^2}\,\omega_n t + B\sin\sqrt{1-\zeta^2}\,\omega_n t\right)$$

can be expressed as:

$$x(t) = e^{-\zeta\omega_n t}\left(x_o\cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_o + \zeta\omega_n x_o}{\omega_d}\sin\sqrt{1-\zeta^2}\,\omega_n t\right)$$

where

$$\zeta = \frac{c}{2m\omega_n}$$

Nature of the roots in the complex plane

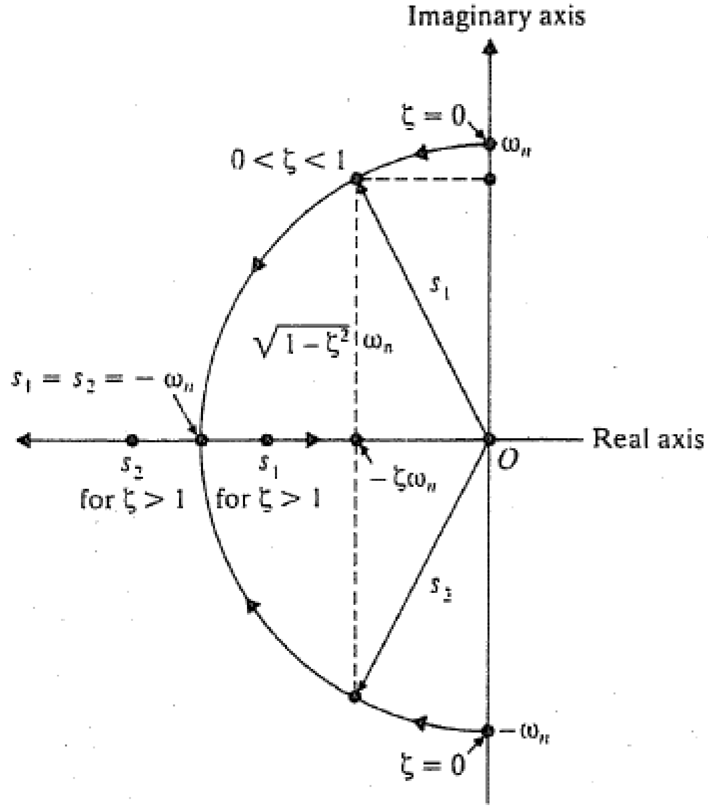
- For ζ=0, we obtain the imaginary roots $i\omega_n$ and $-i\omega_n$ and a solution of

$$x(t) = A\cos\omega_n t + B\sin\omega_n t$$

- For 0<ζ<1, the roots are complex conjugate and are located symmetrically about the real axis.

- As the value of ζ approaches 1, both roots approach the point $-\omega_n$ on the real axis.

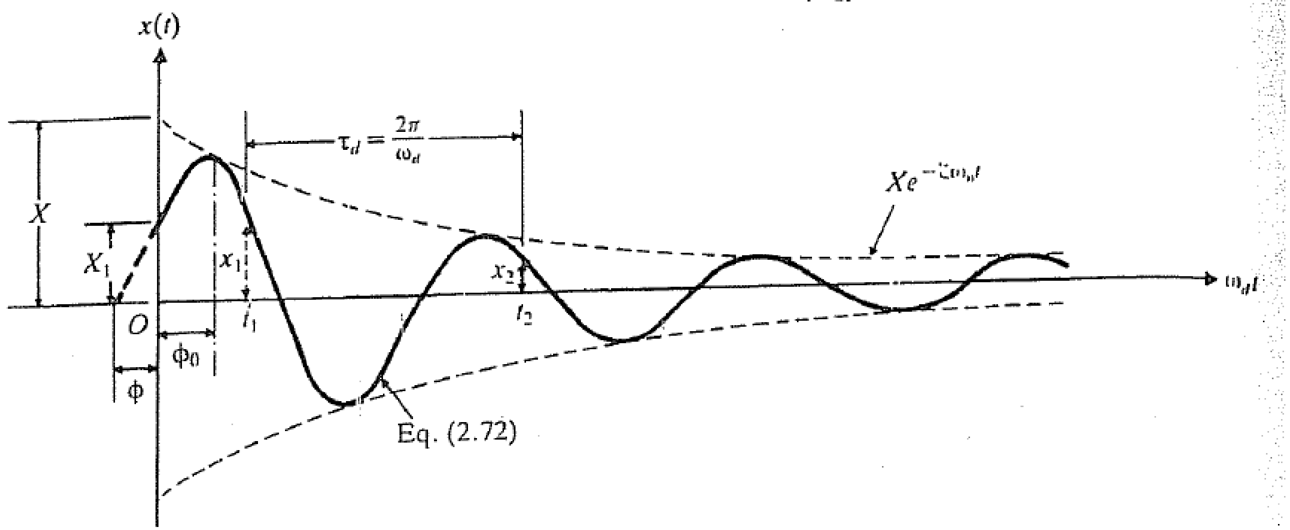
- If ζ is greater than 1, both roots lie on the real axis, one increasing and the other decreasing.



Logarithmic decrement

- The **logarithmic decrement** represents the rate at which the amplitude of a free damped vibration decreases. It is defined as the natural logarithm of the ratio of any two successive amplitudes.

- Let t_1 and t_2 denote the times corresponding to two consecutive amplitudes (displacements) measured one cycle apart for an underdamped system as shown in the figure.



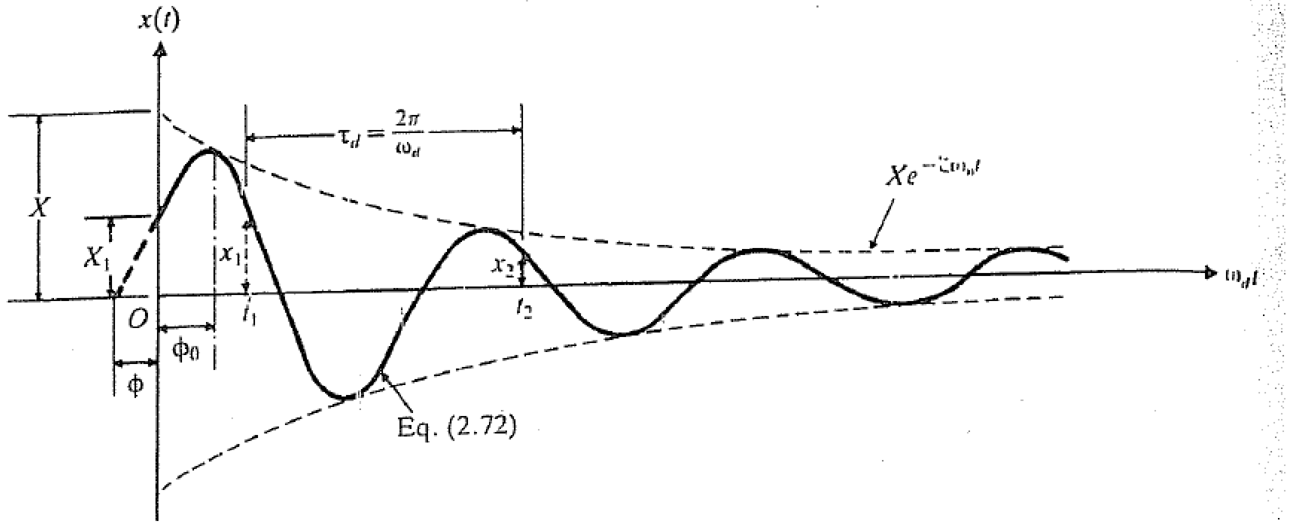
Logarithmic decrement

- Using $x(t) = \left[A\cos(\omega_d t - \phi)\right]e^{-\xi\omega_n t}$, we can form the ratio:

$$\frac{x_1}{x_2} = \frac{X_o e^{-\zeta\omega_n t_1}\cos\left(\omega_d t_1 - \phi_o\right)}{X_o e^{-\zeta\omega_n t_2}\cos\left(\omega_d t_2 - \phi_o\right)}$$

- But $t_2 = t_1 + \tau_d$, hence $\cos(\omega_d t_2 - \phi_0) = \cos(2\pi + \omega_d t_1 - \phi_0) = \cos(\omega_d t_1 - \phi_0)$.

$$\frac{x_1}{x_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n(t_1+\tau_d)}} = e^{\zeta\omega_n\tau_d}$$



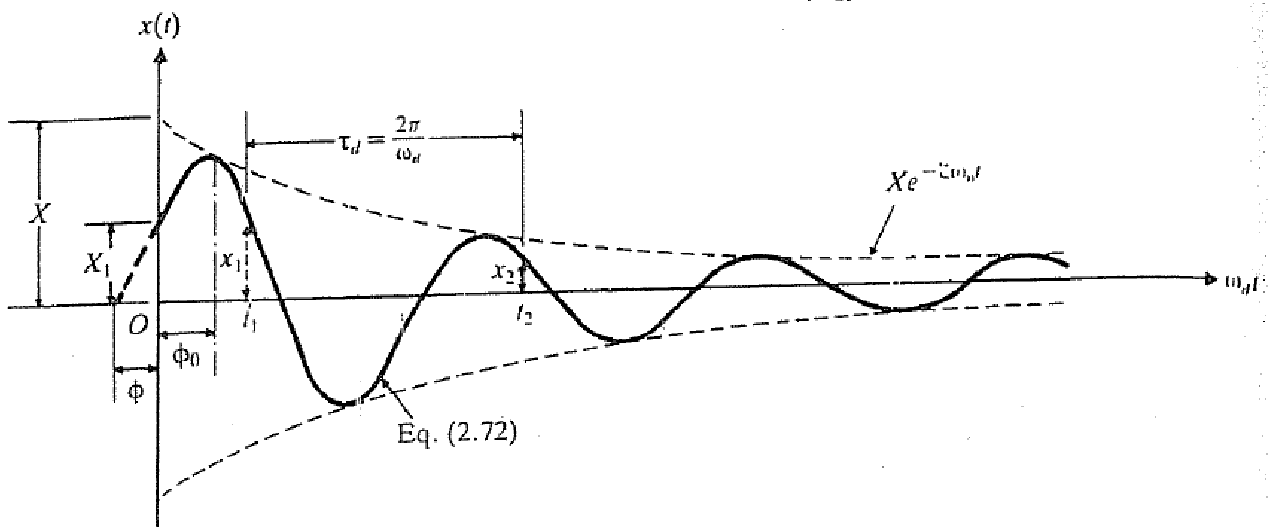
Logarithmic decrement

- The logarithmic decrement can be found from:

$$\delta = \ln\frac{x_1}{x_2} = \zeta\omega_n\tau_d = \zeta\omega_n\frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}\cdot\frac{c}{2m}$$

- For small damping, the above equation can be approximated as:

$$\delta \simeq 2\pi\zeta \qquad \text{if} \qquad \zeta << 1$$



Logarithmic decrement

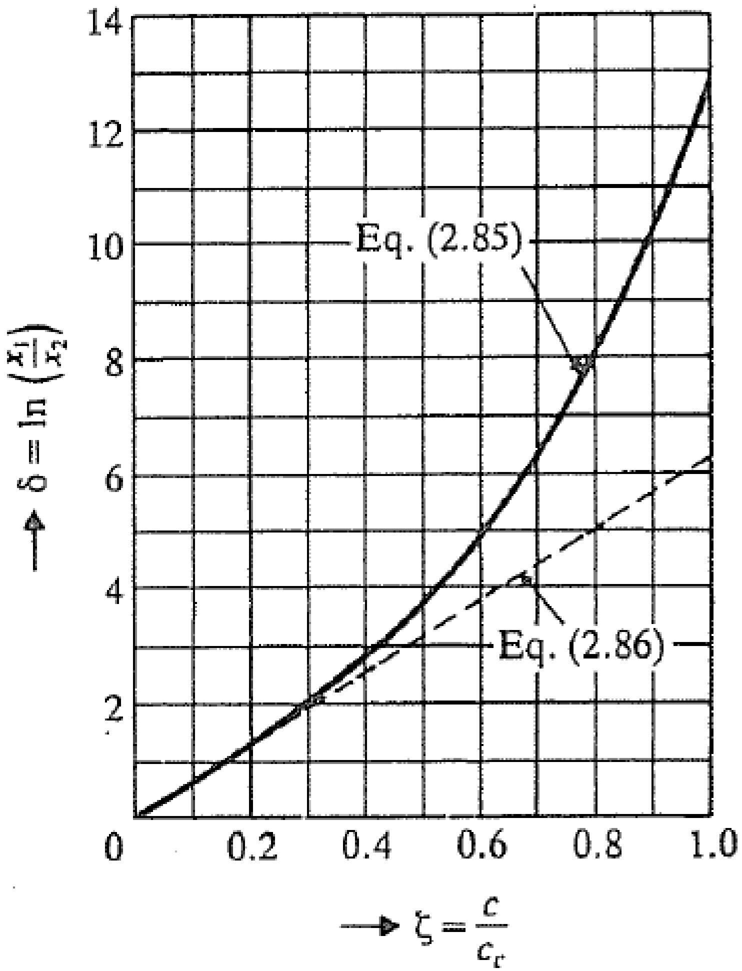
- Figure shows the variation of the logarithmic decrement δ with ζ as shown in the equations:

$$\delta = \ln\frac{x_1}{x_2} = \zeta\omega_n\tau_d = \zeta\omega_n\frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$$

$$= \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}\cdot\frac{c}{2m}$$

$$\delta \simeq 2\pi\zeta \qquad \text{if} \qquad \zeta << 1$$

- It can be noticed that for values up to ζ = 0.3, the two curves are difficult to distinguish.



Logarithmic decrement

- The logarithmic decrement is dimensionless and is actually another form of the dimensionless damping ratio ζ. Once δ is known, ζ can be found by solving:

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

- If we use

$$\delta \simeq 2\pi\zeta \qquad \text{if} \qquad \zeta << 1$$

instead of

$$\delta = \ln\frac{x_1}{x_2} = \zeta\omega_n\tau_d = \zeta\omega_n\frac{2\pi}{\sqrt{1 - \zeta^2}\omega_n}$$

we have:

$$\zeta \simeq \frac{\delta}{2\pi}$$

Logarithmic decrement

- If the damping in the given system is not known, we can determine it experimentally by measuring any two consecutive displacements x_1 and x_2. By taking the natural logarithm of the ratio x_1 and x_2, we obtain δ. By using

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

  we can compute the damping ratio ζ.
- In fact the damping ratio ζ can also be found by measuring two displacements separated by any number of complete cycles. If x_1 and x_{m+1} denote the amplitudes corresponding to times t_1 and $t_{m+1}=t_1+m\tau_d$ where m is an integer, we obtain:

$$\frac{x_1}{x_{m+1}} = \frac{x_1}{x_2}\frac{x_2}{x_3}\frac{x_3}{x_4}\cdots\frac{x_m}{x_{m+1}}$$

Logarithmic decrement

- Since any two successive displacements separated by one cycle satisfy the equation:

$$\frac{x_j}{x_{j+1}} = e^{\zeta \omega_n \tau_d}$$

the equation

$$\frac{x_1}{x_{m+1}} = \frac{x_1}{x_2}\frac{x_2}{x_3}\frac{x_3}{x_4} \cdots \frac{x_m}{x_{m+1}}$$

becomes:

$$\frac{x_1}{x_{m+1}} = (e^{\zeta \omega_n \tau_d})^m = e^{m\zeta \omega_n \tau_d}$$

- The above equation yields $\delta = \frac{1}{m} \ln \left(\frac{x_1}{x_{m+1}} \right)$

which can be substituted into the either of the equations to obtain the viscous damping ratio ζ:

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \qquad \zeta \simeq \frac{\delta}{2\pi}$$

Forced Vibration

- Harmonic excitation
- Base excitation

Harmonically excited vibration

- A mechanical or structural system is said to undergo **forced vibration** whenever external energy is supplied to the system during vibration. External energy can be supplied to the system through either an applied force or an imposed displacement excitation.
- The applied force or displacement excitation may be harmonic, nonharmonic but periodic, nonperiodic or random in nature. The response of a system to harmonic excitation is called **harmonic response**.
- The nonperiodic excitations may have a long or short duration. The response of a dynamic system to suddenly applied nonperiodic excitations is called **transient response**.
- In this part of the course, we shall consider the dynamic response of a single degree of freedom system under harmonic excitations of the form

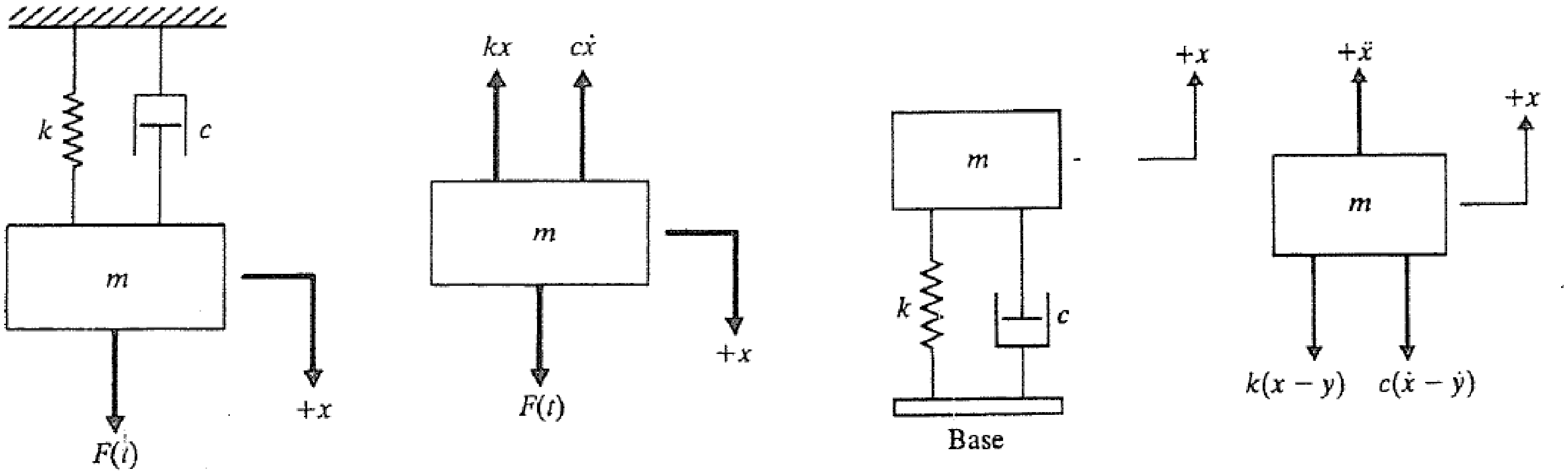
$$F(t) = F_o e^{i(\omega t+\phi)} \text{ or } F(t) = F_o \cos(\omega t + \phi) \text{ or } F(t) = F_o \sin(\omega t + \phi)$$

where

F_o is the amplitude, ω is the frequency, and ϕ is the phase angle of the harmonic excitation.

Forced-Excited System: Harmonic Excitation

- Some single degree of freedom (SDOF) systems with an external force are shown in the figure. Force can be applied both as an external force F(t), or as a base motion y(t), as shown. The coordinate x(t) is the absolute motion of the mass. The forces W and $k\delta_{st}$ are ignored in the free-body diagrams as we know they will add to zero in the equation of motion.



(a) (b) Free-body diagram

A spring-mass-damper system.

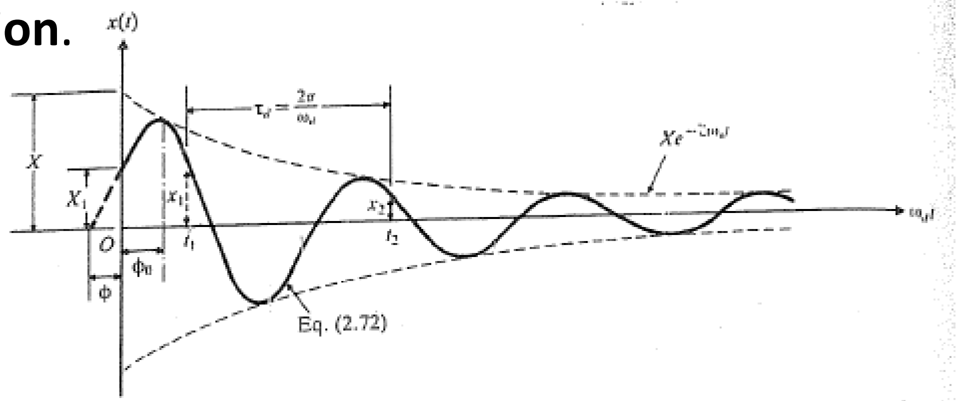
Forced-Excited System: Harmonic Excitation

- Consider the force-excited system of the figure, where the applied force is harmonic, $F(t) = F_o \sin \omega t$
- Applying Newton's second law, the equation of motion becomes:

$$m\ddot{x} + c\dot{x} + kx = F_o \sin \omega t$$

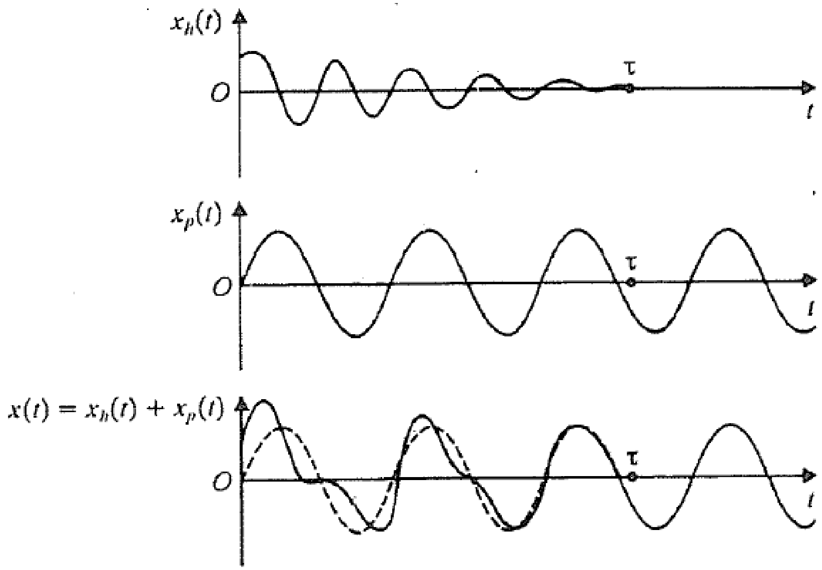
- The general solution for this second-order nonhomogeneous linear differential equation is $x(t) = x_h(t) + x_p(t)$

  where x_h is the complementary solution or solution to the homogeneous equation. But this solution dies out soon . Our interest focuses on x_p, the particular solution. In vibration theory, the particular solution is also called the **steady-state solution**.



Forced-Excited System: Harmonic Excitation

- The variations of homogeneous, particular, and general solutions with time for a typical case are shown in the figure.
- It can be seen that $x_h(t)$ dies out and $x(t)$ becomes $x_p(t)$ after some time (τ in the figure).
- The part of the motion that dies out due to damping (the free vibration part) is called **transient**. The particular solution represents the **steady state vibration** and is present as long as the forcing function is present.



Forced-Excited System: Harmonic Excitation

- The vertical motions of a mass-spring system subjected to an external force r(t) can be expressed as:

$$mx'' + cx' + kx = r(t)$$

- Mechanically, this means that at each time instant t, the resultant of the internal forces is in equilibrium with r(t). The resulting motion is called a forced motion with forcing function r(t), which is also known as the **input force** or the **driving force**, and the solution x(t) to be obtained is called the **output** or the **response of the system** to the driving force.
- Of special interest are periodic external forces, and we shall consider a driving force of the form: $r(t) = F_o \cos \omega t$
- Then we have the nonhomogeneous ODE:

$$mx'' + cx' + kx = F_o \cos \omega t$$

Forced-Excited System: Harmonic Excitation

Solving the nonhomogeneous ODE

To find y_p, we use the method of undetermined coefficients:

$$x_p(t) = a\cos\omega t + b\sin\omega t$$

$$x'_p(t) = -\omega a\sin\omega t + \omega b\cos\omega t$$

$$x''_p(t) = -\omega^2 a\cos\omega t - \omega^2 b\sin\omega t$$

Substituting the above equations into

$$mx'' + cx' + kx = F_o\cos\omega t$$

And collecting the cosine and the sine terms, we get

$$[(k - m\omega^2)a + \omega cb]\cos\omega t + [-\omega ca + (k - m\omega^2)b]\sin\omega t = F_o\cos\omega t$$

The cosine terms on both sides must be equal, and the coefficient of the sine term on the left must be zero since there is no sine term on the right.

Forced-Excited System: Harmonic Excitation

- This gives the two equations: $(k - m\omega^2)a + \omega c b = F_o$

  $-\omega c a + (k - m\omega^2)b = 0$

  for determining the unknown coefficients a and b. This is a linear system. We can solve it by elimination to find:

$$a = F_o \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}$$

$$b = F_o \frac{\omega c}{(k - m\omega^2)^2 + \omega^2 c^2}$$

- If we set $\sqrt{\frac{k}{m}} = \omega_o$, we obtain: $a = F_o \frac{m(\omega_o^2 - \omega^2)}{m^2(\omega_o^2 - \omega^2)^2 + \omega^2 c^2}$

$$b = F_o \frac{\omega c}{m^2(\omega_o^2 - \omega^2)^2 + \omega^2 c^2}$$

- We thus obtain the general solution of the nonhomogeneous ODE in the form: $x(t) = x_h(t) + x_p(t)$

Forced-Excited System: Harmonic Excitation

Case I: Undamped forced oscillations:

- If the damping of the physical system is so small that its effect can be neglected over the time interval considered, we can set c=0. Then

$$a = F_o \frac{m(\omega_o^2 - \omega^2)}{m^2(\omega_o^2 - \omega^2)^2 + \omega^2 c^2}$$

$$b = F_o \frac{\omega c}{m^2(\omega_o^2 - \omega^2)^2 + \omega^2 c^2}$$

reduces to

$$a = \frac{F_o}{m(\omega_o^2 - \omega^2)}$$

$$b = 0$$

Hence

$$x_p(t) = a \cos \omega t + b \sin \omega t$$

becomes

$$x_p(t) = \frac{F_o}{m(\omega_o^2 - \omega^2)} \cos \omega t = \frac{F_o}{k[1 - (\frac{\omega}{\omega_o})^2]} \cos \omega t$$

Forced-Excited System: Harmonic Excitation

- We thus have the general solution of the undamped system as:

$$x(t) = C\cos(\omega_o t - \delta) + \frac{F_o}{m(\omega_o^2 - \omega^2)}\cos\omega t$$

- We see that this output is a superposition of two harmonic oscillations of the natural frequency $\omega_o/2\pi$ [cycles/sec] of the system, which is the frequency of the undamped motion and the frequency $\omega/2\pi$ [cycles/sec] of the driving force.

Harmonic motion

Beats:

- As mentioned before, if the frequency of the forcing function and the frequency of the system are very close to each other, then again beating effect should be expected.

- If we for example take the particular soluton:

$$x(t) = \frac{F_o}{m(\omega_o^2 - \omega^2)}\left(\cos \omega t - \cos \omega_o t\right) \quad \left(\omega \neq \omega_o\right)$$

which can be rewritten as:

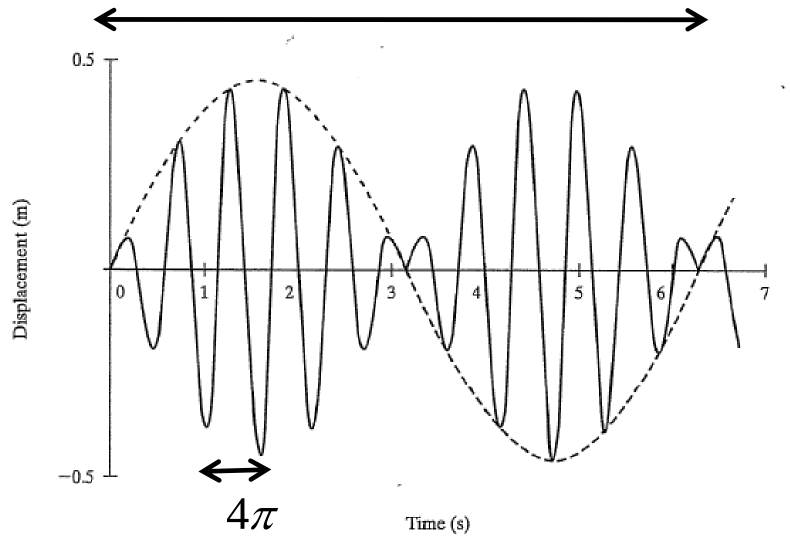
$$x(t) = \frac{2F_o}{m(\omega_o^2 - \omega^2)} \sin\left(\frac{\omega_o + \omega}{2} t\right) \sin\left(\frac{\omega_o - \omega}{2} t\right)$$

- Since ω is close to ω_o, the difference $\omega_o - \omega$ is small. Hence the period of the last sine function is large. This is because the greater the quantity under the sine, the smaller the period is.

Harmonic motion

Beats:

$$T_b = \frac{4\pi}{\omega_o - \omega} = \frac{2TT_o}{T - T_o}$$



$$\frac{4\pi}{\omega + \omega_o}$$

$$x(t) = \frac{F_o}{m(\omega_o^2 - \omega^2)} \sin\left(\frac{\omega_o + \omega}{2} t\right) \sin\left(\frac{\omega_o - \omega}{2} t\right)$$

Harmonic motion

Beats:

This phenomenon is also frequently observed in lightly damped sytems with close coupling of the torsional and translational frequencies.

A typical example is the thirteen storey steel framed Santa Clara County Office Building as reported in :

Celebi and Liu, 'Before and after retrofit- response of a building during ambient and strong motions', Journal of Wind Engineering and Industrial Aerodynamics, 77&78 (1998) 259-268.

The proximity of the torsional frequency at 0.57 Hz to the translational frequency at 0.45 Hz causes the observed coupling and beating effect in this structure.

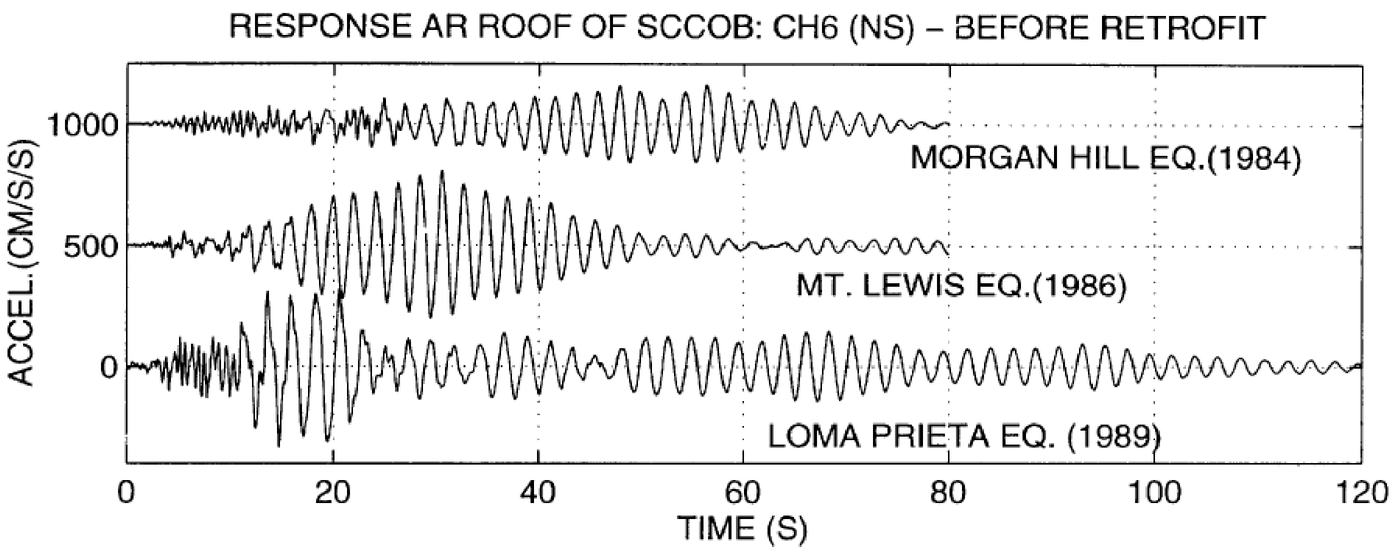


Fig. 2. Response at the roof of SCCOB during the three earthquakes.

Harmonic motion

Comparison of the frequencies of the building

Thirteen storey steel framed Santa Clara County Office Building as reported in :

Celebi and Liu, 'Before and after retrofit- response of a building during ambient and strong motions', Journal of Wind Engineering and Industrial Aerodynamics, 77&78 (1998) 259-268.

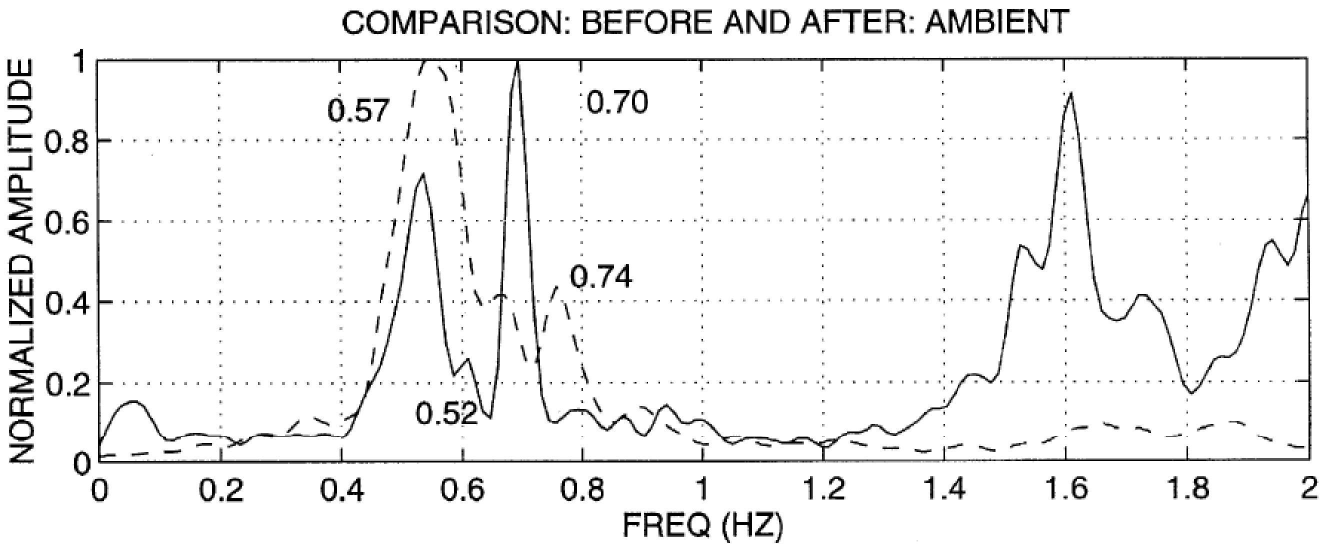


Fig. 12. Comparison of normalized amplitude spectra of motions before (solid lines) and after retrofit (dashed lines).

Resonance

- If the damping of the system is so small that its effect can be neglected over the time interval considered, we can set c=0. Then the particular response can be expressed by:

$$x_p = \frac{F_o}{m(\omega_o^2 - \omega^2)} \cos \omega t = \frac{F_o}{k\left[1 - \left(\frac{\omega}{\omega_o}\right)^2\right]} \cos \omega t$$

- Putting cosωt=1, we see that the maximum amplitude of the particular solution is:

$$a_o = \frac{F_o}{k} \rho$$

$$\text{where } \rho = \frac{1}{1 - \left(\frac{\omega}{\omega_o}\right)^2}$$

$$If \quad \omega \to \omega_o, \text{ then } \rho \text{ and } a_o \text{ tend to infinity.}$$

- This excitation of large oscillations by matching input and natural frequencies is called **resonance**.

Resonance

- In the case of resonance and no damping, the ODE $\ddot{x} + \omega_o^2 x = \dfrac{F_o}{m}\cos\omega_o t$

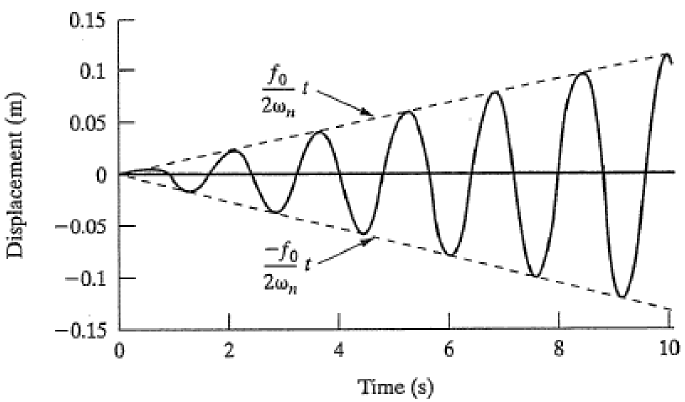
  becomes: $m\ddot{x} + c\dot{x} + kx = F_o \cos\omega t$
- Then from the modification rule, the particular solution becomes:

$$x_p(t) = t(a\cos\omega_o t + b\sin\omega_o t)$$

- By substituting this into the second equation, we find:

$$x_p(t) = \frac{F_o}{2m\omega_o} t \sin\omega_o t$$

- We see that because of the factor t, the amplitude becomes larger and larger. Practically speaking, systems with very little damping may undergo large vibrations that can destroy the system.



Forced-Excited System: Harmonic Excitation

Case II: Damped forced oscillations: If the damping of the mass-spring system is not negligibly small, we have c>0 and a damping term cx' in

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad m\ddot{x} + c\dot{x} + kx = r(t)$$

Then the general solution y_h of the homogeneous ODE approaches zero as t goes to infinity. Practically, it is zero after a sufficiently long time. Hence the transient solution given by

$$y(t) = y_h(t) + y_p(t)$$

approaches the steady state solution y_p. This proves the following:

Steady state solution: *After a sufficiently long time, the output of a damped vibrating system under a purely sinusoidal driving force will practically be a harmonic oscillation whose frequency is that of the input.*

Response of a damped system under harmonic force

- If the forcing function is given by $F(t) = F_0 \cos \omega t$, the equation of motion becomes:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

- The particular solution is also expected to be harmonic; we assume it in the following form:

$$x_p(t) = X \cos(\omega t - \phi)$$

where X and ϕ are constants to be determined that denote the amplitude and the phase angle of the response, respectively. By substituting the second equation into the first:

$$X[(k - m\omega^2)\cos(\omega t - \phi) - c\omega \sin(\omega t - \phi)] = F_0 \cos \omega t$$

- Using the trigonometric relations below in the above equation

$$\cos(\omega t - \phi) = \cos \omega t \cos \phi + \sin \omega t \sin \phi$$

$$\sin(\omega t - \phi) = \sin \omega t \cos \phi - \cos \omega t \sin \phi$$

Response of a damped system under harmonic force

- We obtain: $$X\left[(k - m\omega^2)\cos\phi + c\omega\sin\phi\right] = F_0$$ $$X\left[(k - m\omega^2)\sin\phi - c\omega\cos\phi\right] = 0$$
- If we solve the above equation, we find:

$$X = \frac{F_0}{\left[(k - m\omega^2)^2 + c^2\omega^2\right]^{1/2}} \qquad \phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right).$$

- If we insert the above into the $x_p(t) = X\cos(\omega t - \phi)$ we find the particular solution. Using:

$$\omega_n = \sqrt{\frac{k}{m}} = \text{undamped natural frequency,}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}}; \frac{c}{m} = 2\zeta\omega_n,$$

$$r = \frac{\omega}{\omega_n}$$

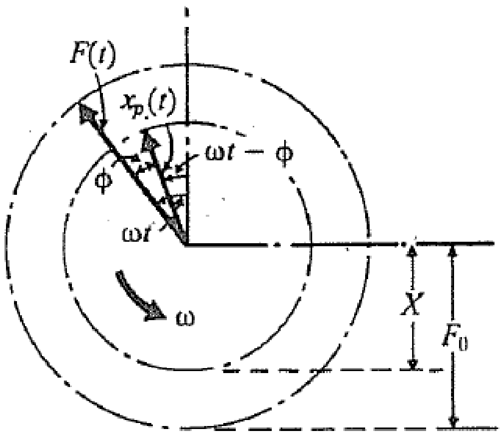
$$\delta_{st} = \frac{F_0}{k} = \text{deflection under the static force } F_0,$$

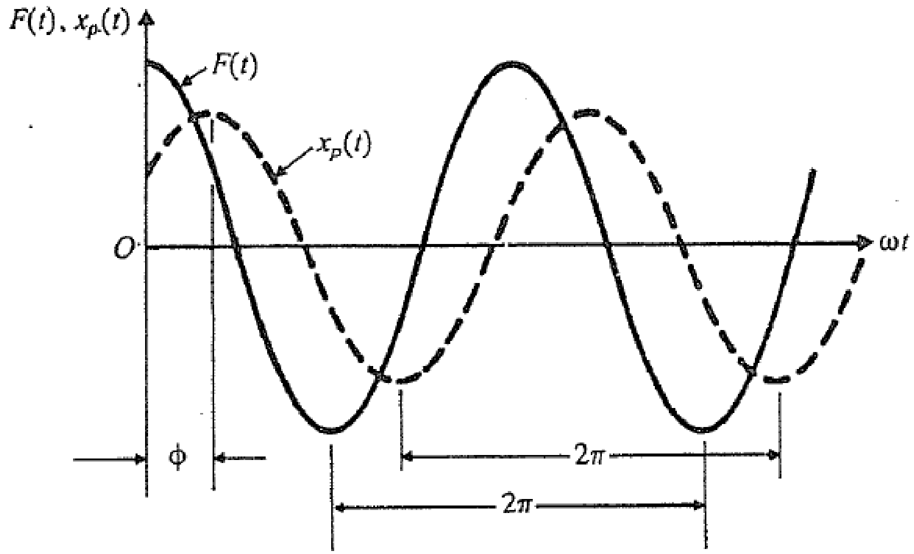
Response of a damped system under harmonic force

- We obtain:

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$\phi = \tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

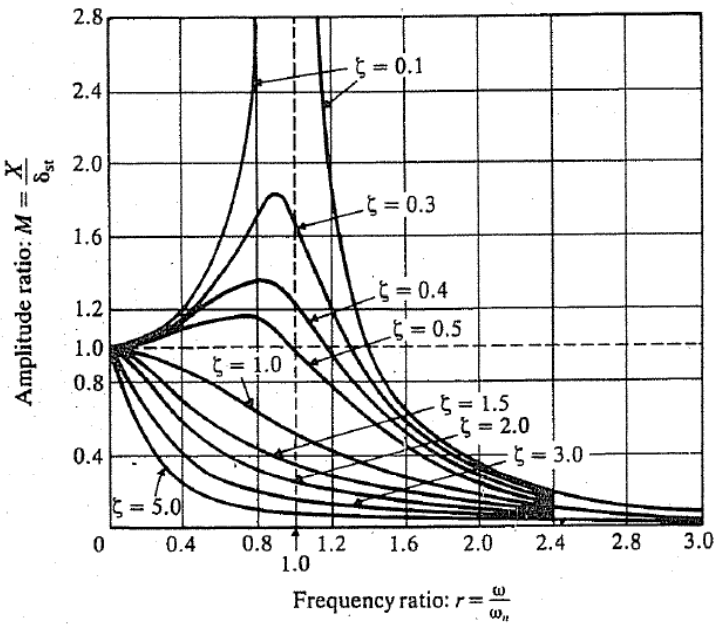


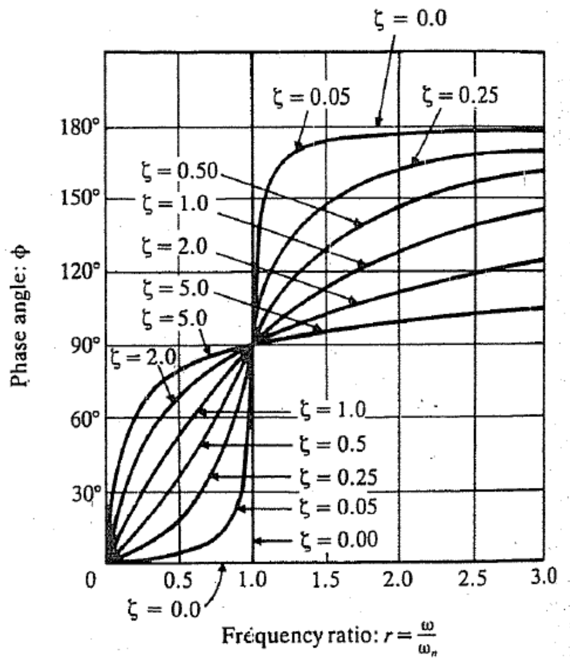


Forced-Excited System: Harmonic Excitation

- The quantity M=X/δst is known as the **magnification factor**, **amplification factor** or the **amplitude ratio**. The amplitude of the forced vibration becomes smaller with increasing values of the forcing frequency (that is, M ⟶ 0 as r ⟶ ∞)

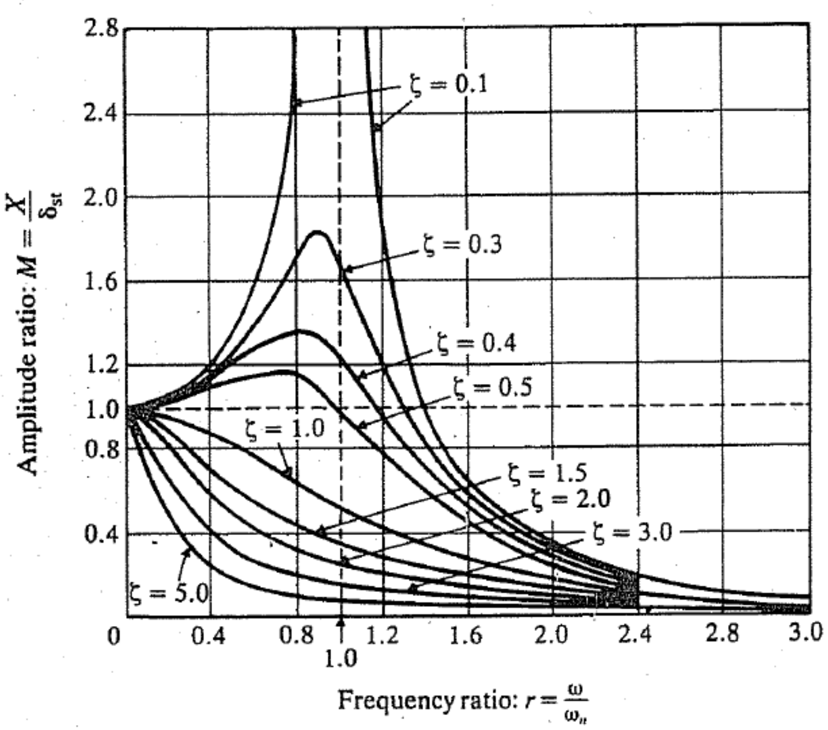
$$\frac{X_o}{F_o/k} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

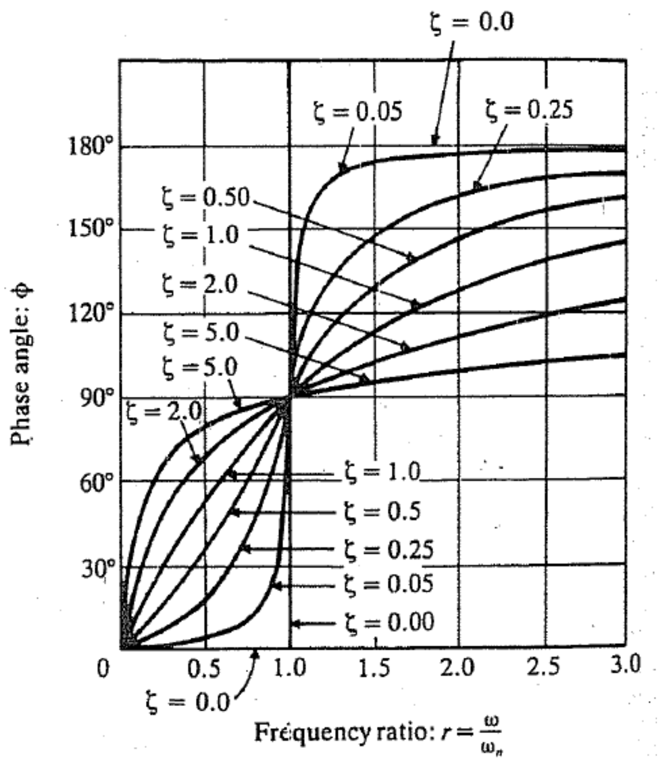




Forced-Excited System: Harmonic Excitation

- A fact that creates difficulty for designers is that the response can become large when r is close to 1 or when ω is close to ω_n. This condition is called **resonance**. The reduction in M in the presence of damping is very significant at or near resonance.

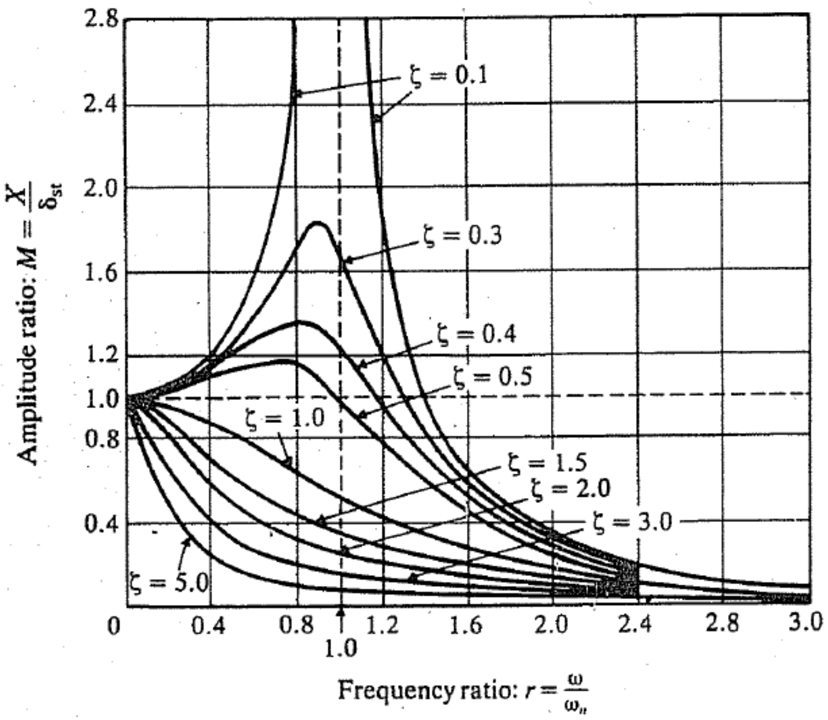


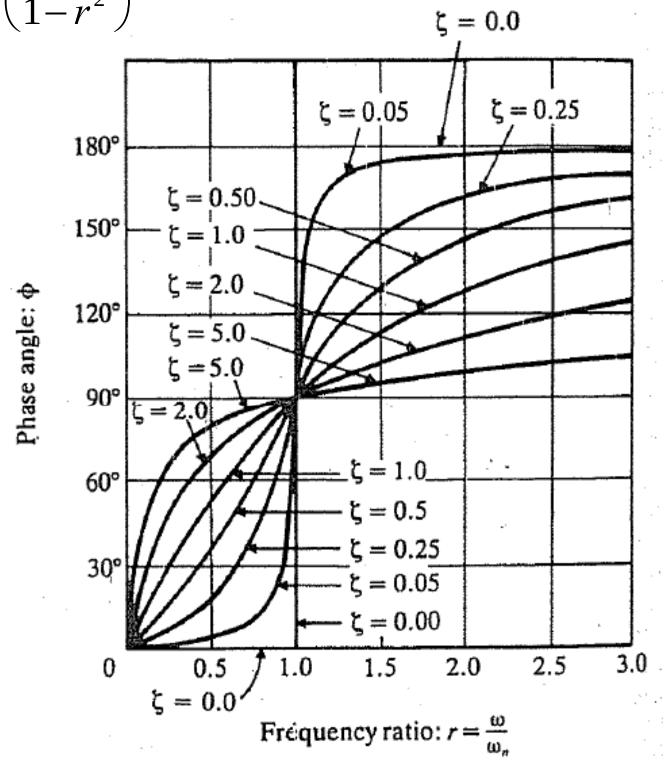


Forced-Excited System: Harmonic Excitation

- For an undamped system (ζ=0), the phase angle is zero for 0<r<1 and 180 degrees for r>1. This implies that the excitation and response are in phase for 0<r<1 and out of phase for r>1 when ζ=0.

$$\phi = \arctan\left(\frac{2\zeta r}{1-r^2}\right)$$

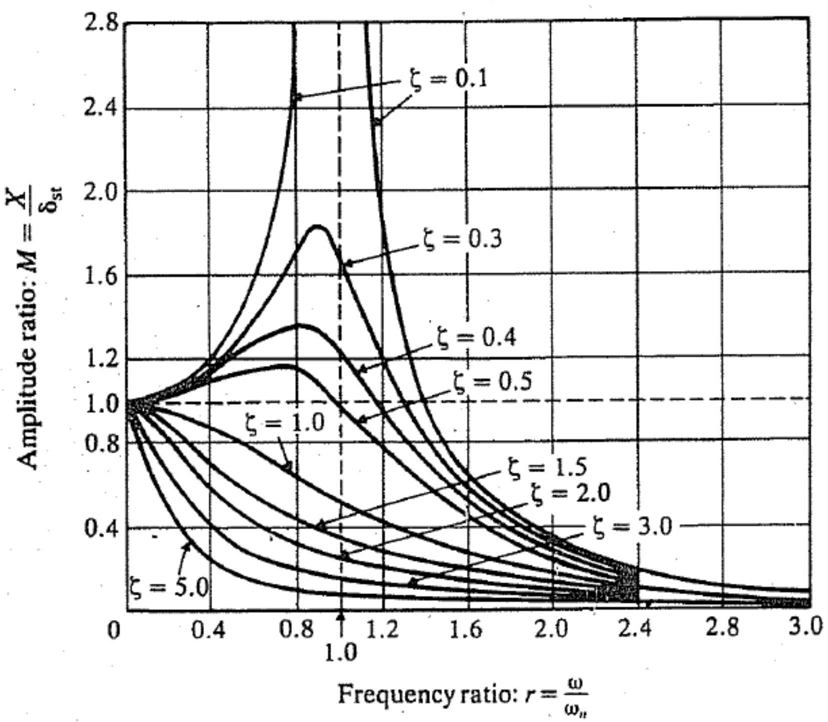


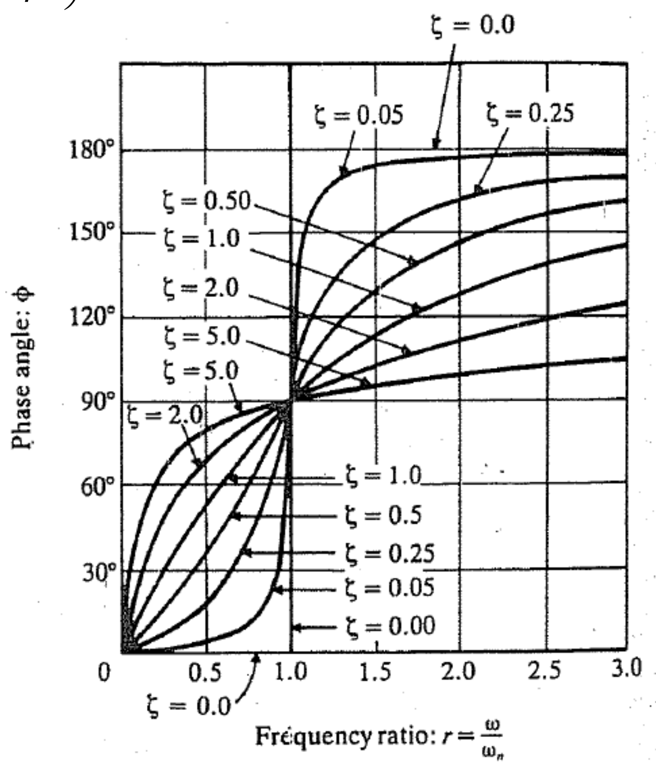


Forced-Excited System: Harmonic Excitation

- For ζ>0 and 0<r<1, the phase angle is given by 0<ϕ<90, implying that the response lags the excitation.

$$\phi = \arctan\left(\frac{2\zeta r}{1-r^2}\right)$$

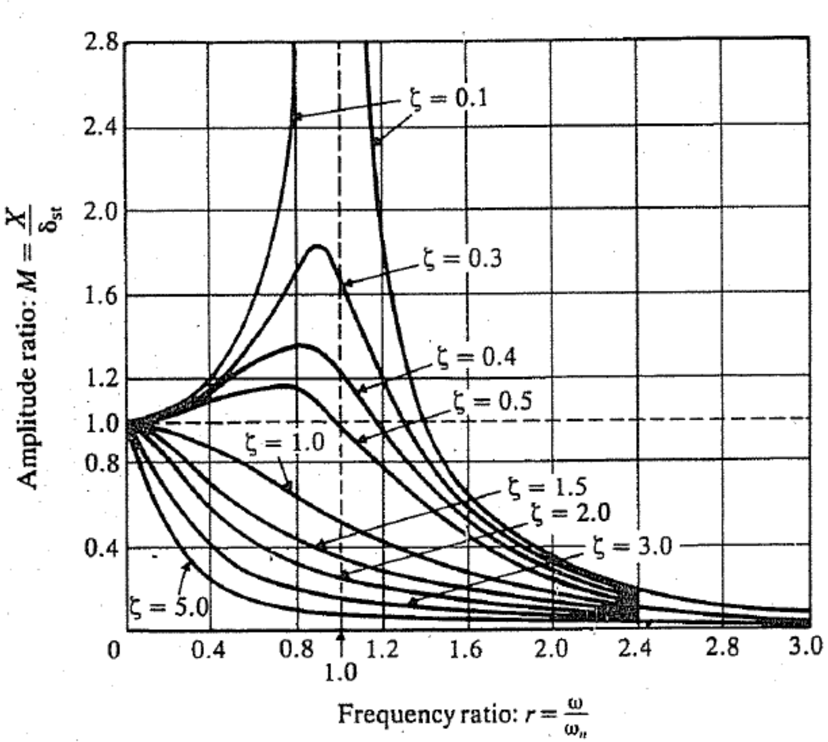


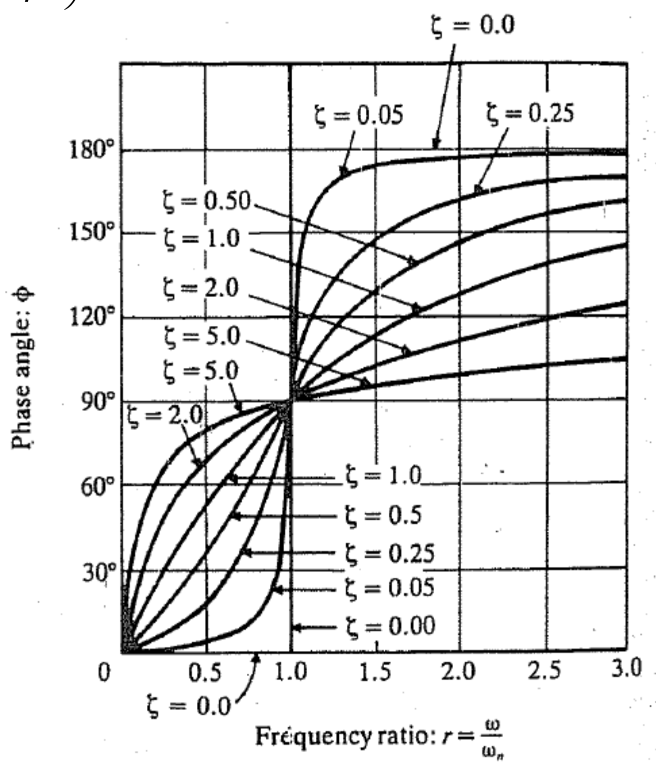


Forced-Excited System: Harmonic Excitation

- For $\zeta>0$ and $r>1$, the phase angle is given by $90<\phi<180$, implying that the response leads the excitation.

$$\phi = \arctan\left(\frac{2\zeta r}{1-r^2}\right)$$





Response of a damped system under $F(t) = F_o e^{i\omega t}$

- Using complex algebra, let the harmonic force be:

$$F(t) = F_o e^{i\omega t}$$

where F_o is a real constant and i is the imaginary unit. Assume that the response has the same frequency as the force, but is, in general, out of phase with the force

$$x(t) = X_o e^{i(\omega t+\phi)} = \tilde{X} e^{i\omega t}$$

where X_o is the amplitude of the displacement and $\tilde{X}$ is the complex displacement,

$$\tilde{X} = X_o e^{i\phi}$$

Substituting this into the differential equation of motion

$$(-m\omega^2 + ic\omega + k)\tilde{X} e^{i\omega t} = F_o e^{i\omega t}$$

Forced-Excited System: Harmonic Excitation

- Define the transfer function (or frequency response function) H(ω) as the complex displacement due to a force of unit magnitude (F_o=1). Thus,

$$H(\omega) = \frac{1}{(k - m\omega^2) + ic\omega}$$

- Rationalizing, the transfer function becomes:

$$H(\omega) = \frac{(k - m\omega^2) - ic\omega}{(k - m\omega^2)^2 + (c\omega)^2}$$

- This is also the ratio between the complex displacement response and the complex input forcing function.

Forced-Excited System: Harmonic Excitation

- Define the gain function as the modulus of the transfer function

$$|H(\omega)| = \sqrt{H(\omega)H^*(\omega)} = \sqrt{(\operatorname{Re} H)^2 + (\operatorname{Im} H)^2}$$

where H* is the complex conjugate. For the force excited system under consideration,

$$|H(\omega)| = \frac{1}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

- The gain function is the amplitude of the displacement for F_o=1. Thus,

$$\frac{X_o}{F_o} = |H(\omega)|$$

Forced-Excited System: Harmonic Excitation

- It is convenient to develop a nondimensional form of the gain function. First define the frequency ratio

$$r = \frac{\omega}{\omega_n}$$

Multiplying the equation

$$\left|H(\omega)\right| = \frac{1}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

by k and employing the definitions for ζ, c_c and ω_n, it is easily shown that

$$\frac{X_o}{F_o / k} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

and the phase angle is:

$$\phi = \arctan\left(\frac{2\zeta r}{1 - r^2}\right)$$

Forced-Excited System: Harmonic Excitation

- Transfer functions expressing the velocity and acceleration responses can be written based on equation

$$x(t) = X_o e^{i(\omega t+\phi)} = \tilde{X} e^{i\omega t}$$

by multiplying H(ω) in equation

$$\left|H(\omega)\right| = \frac{1}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}$$

by $i\omega$ and $(i\omega)^2=-\omega^2$ respectively. The resulting gain function for the velocity output would be derived by multiplying both sides of the above equation and the equation

$$\frac{X_o}{F_o/k} = \frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$$

by ω and noting that ωX_o is the amplitude of the velocity. Similarly, the gain function for the acceleration can be obtained by multiplying both sides by ω^2.

Quality factor and bandwidth

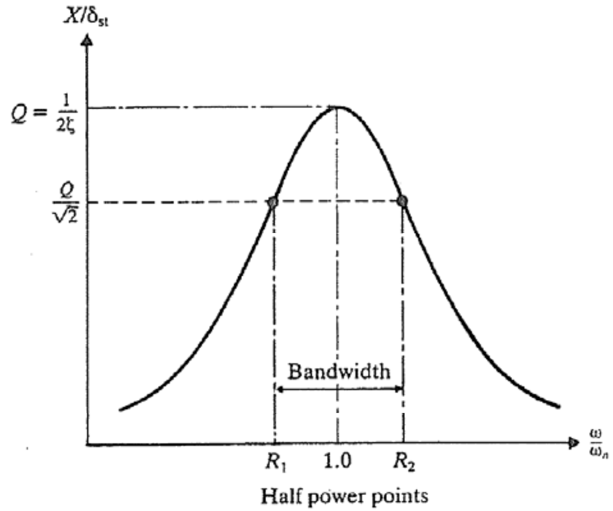
- For small values of damping we can take:

$$\left(\frac{X}{\delta_{st}}\right)_{\max} \cong \left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta} = Q$$

where

X denotes the amplitude of the response and

$\delta_{st} = \frac{F_o}{k}$ = deflection under the static force F_o

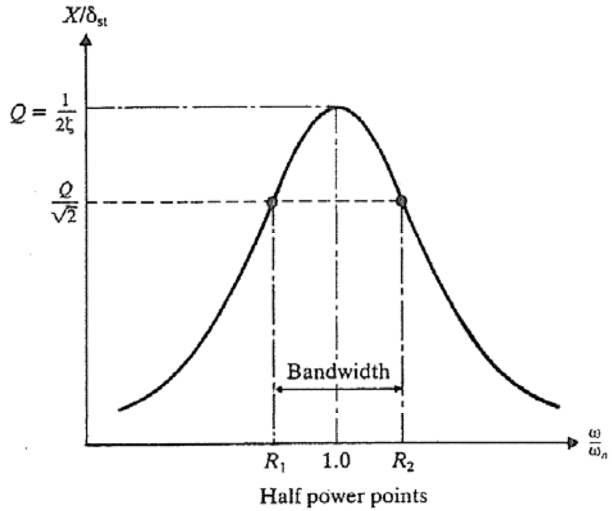


- The value of the amplitude ratio at resonance is called the Q factor or the quality factor of the system. The points R_1 and R_2, where the amplification factor falls to $Q/\sqrt{2}$, are called **half power points** because the power absorbed (ΔW) by the damper (or by the resistor in an electrical circuit), responding harmonically at a given frequency, is proportional to the square of the amplitude.

$$\Delta W = \pi c \omega X^2$$

Quality factor and bandwidth

- The differences between the frequencies associated with the half power points R_1 and R_2 is called the **bandwidth** of the system.



- To find the values of R_1 and R_2, we set $X/\delta_{st} = Q/\sqrt{2}$ so that

$$\frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = \frac{Q}{\sqrt{2}} = \frac{1}{2\sqrt{2}\zeta}$$

or

$$r^4 - r^2(2 - 4\zeta^2) + (1 - 8\zeta^2) = 0$$

- From which, we can get:

$$r_1^2 = 1 - 2\zeta^2 - 2\zeta\sqrt{1 + \zeta^2}, \qquad r_2^2 = 1 - 2\zeta^2 + 2\zeta\sqrt{1 + \zeta^2}$$

Quality factor and bandwidth

- For small values of ζ, the roots

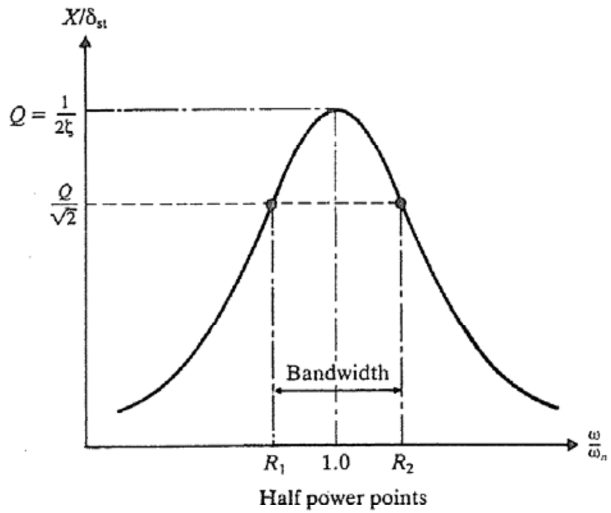
$$r_1^2 = 1 - 2\zeta^2 - 2\zeta\sqrt{1 + \zeta^2}, \qquad r_2^2 = 1 - 2\zeta^2 + 2\zeta\sqrt{1 + \zeta^2}$$

can be simplified as:

$$r_1^2 = R_1^2 = \left(\frac{\omega_1}{\omega_n}\right)^2 \simeq 1 - 2\zeta,$$

$$r_2^2 = R_2^2 = \left(\frac{\omega_2}{\omega_n}\right)^2 \simeq 1 + 2\zeta$$

where $\omega_1 = \omega|_{R1}$ and $\omega_2 = \omega|_{R2}$



Then

$$\omega_2^2 - \omega_1^2 = (\omega_2 + \omega_1)(\omega_2 - \omega_1) = (R_2^2 - R_1^2)\omega_n^2 \simeq 4\zeta\omega_n^2$$

Quality factor and bandwidth

- Using the relation $\omega_2 + \omega_1 = 2\omega_n$ in the equation

$$\omega_2^2 - \omega_1^2 = (\omega_2 + \omega_1)(\omega_2 - \omega_1) = (R_2^2 - R_1^2)\omega_n^2 \simeq 4\zeta\omega_n^2$$

we find that the bandwidth $\Delta\omega$ is given by:

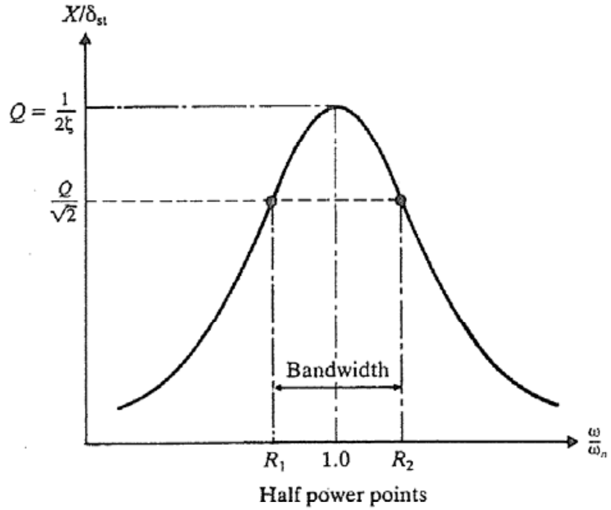
$$\Delta\omega = \omega_2 - \omega_1 \simeq 2\zeta\omega_n$$

Combining the above equation and the equation

$$\left(\frac{X}{\delta_{st}}\right)_{max} \cong \left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta} = Q$$

We obtain:

$$Q \simeq \frac{1}{2\zeta} \simeq \frac{\omega_n}{\omega_2 - \omega_1}$$



- It can be seen that the quality factor Q can be used for estimating the equivalent viscous damping in a mechanical system.

Electrical systems

- An electronic circuit is a closed path formed by the interconnection of electronic components through which an electric current can flow.
- We have just seen that linear ODEs have important applications in mechanics. Similarly, they are models of electric circuits as they occur as portions of large networks in computers and elsewhere.
- The circuits we shall consider here are basic building blocks of such networks.
- They contain three kinds of components, namely, resistors, inductors and capacitors.
- **Kirchhoff's Voltage Law (KVL):** The voltage (the electromotive force) impressed on a closed loop is equal to the sum of the voltage drops across the other elements of the loop.

Electrical systems

- Figure shows such a RLC circuit. In it a resistor of resistance R Ω (ohms), an inductor of inductance L H (Henrys) and a capacitor of capacitance C F (farads) are wired in series as shown, and connected to an electromotive force E(t) V(volts) (a generator for instance), sinusoidal as shown in the figure or some other kind.

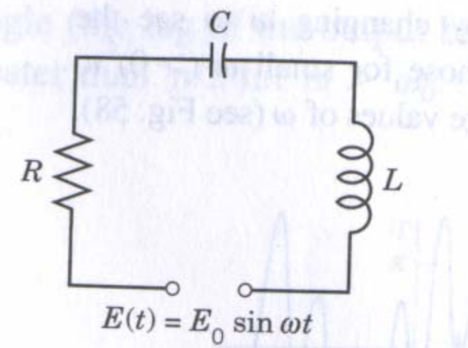


Fig. 60. *RLC*-circuit

- R, L, C, and E are given and we want to find the current I(t) A(Amperes) in the circuit.
- An ODE for the current I(t) in the RLC circuit in the figure is obtained from the Kirchhoff's Voltage Law.
- In the figure, the circuit is a closed loop and the impressed voltage E(t) equals the sum of the voltage drops across the three elements R,L,C, of the loop.

Electrical systems

- **Voltage drops:** Experiments show that the current I flowing through a resistor, inductor and capacitor causes a voltage drop (voltage difference, measured in volts) at the two ends. These drops are:

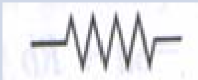
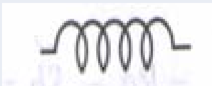
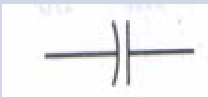
- RI (Ohm's law) Voltage drop for a resistor of resistance R ohms Ω
- $LI' = L\frac{dI}{dt}$ Voltage drop for an inductor of inductance L henrys (H)
- $\frac{Q}{C}$ Voltage drop for a capacitor of capacitance C farads (F)

- Here Q coulombs is the charge on the capacitor, related to the current by

$$I(t) = \frac{dQ}{dt} \text{ equivalently } Q(t) = \int I(t)dt$$

Electrical systems

- Table Elements in an RLC circuit

Name	Symbol	Notation	Unit	Voltage Drop
Ohm's resistor		Ohm's resistance, R	Ohms (Ω)	RI
Inductor		Inductance, L	henrys (H)	LdI/dt
Capacitor		Capacitance, C	farads (F)	Q/C

- According to Kirchhoff' voltage law we thus have an RLC circuit with electromotive force $E(t)=E_o \sin\omega t$ (E_o constant) as a model for the 'integro differential equation'.

$$LI' + RI + \frac{1}{C}\int I dt = E(t) = E_o \sin \omega t$$

Electrical systems

- To get rid of the integral in

$$LI' + RI + \frac{1}{C}\int I dt = E(t) = E_o \sin \omega t$$

- We differentiate the above equation with repect to t, obtaining:

$$LI'' + RI' + \frac{1}{C} I = E'(t) = E_o \omega \cos \omega t$$

- This shows that the current in an RLC circuit is obtained as the solution of this nonhomogeneous second-order ODE with constant coefficients.
- Using $LI' + RI + \frac{1}{C}\int I dt = E(t) = E_o \sin \omega t$ and noting that I=Q' and I'=Q'', we have directly:

$$LQ'' + RQ' + \frac{1}{C} Q = E_o \sin \omega t$$

- But in most practical problems, the current I(t) is more important than the charge Q(t) and for this reason, we shall concentrate on the below equation rather than the above.

$$LI'' + RI' + \frac{1}{C} I = E'(t) = E_o \omega \cos \omega t$$

Solving ODE for the current

- A general solution of

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_o \omega \cos \omega t$$

is the sum $I=I_h+I_p$, where I_h is a general solution of the homogeneous ODE corresponding to the above equation and I_p is a particular solution. We first determine Ip by the method of undetermined coefficients. We substitute:

$$I_p = a \cos \omega t + b \sin \omega t$$

$$I'_p = \omega(-a \sin \omega t + b \cos \omega t)$$

$$I''_p = \omega^2(-a \cos \omega t - b \sin \omega t)$$

into the first equation. Then we collect the cosine terms and equate them to $E_o\omega\cos\omega t$ on the right, and we equate the sine terms into zero because there is no sine term on the right.

$$L\omega^2(-a) + R\omega b + \frac{a}{C} = E_o \omega \qquad \text{(Cosine terms)}$$

$$L\omega^2(-b) + R\omega(-a) + \frac{b}{C} = 0 \qquad \text{(Sine terms)}$$

Solving ODE for the current

- To solve this system for a and b, we first introduce a combination of L and C, called the reactance:

$$S = \omega L - \frac{1}{\omega C}$$

- Dividing the previous two equations by ω, ordering them and substituting S gives:

$$-Sa + Rb = E_o$$
$$-Ra - Sb = 0$$

- We now eliminate b by multiplying the first equation by S and the second by R, and adding. Then we eliminate a by multiplying the first equation by R and second by –S, and adding. This gives:

$$a = \frac{-E_o S}{R^2 + S^2} \qquad b = \frac{E_o R}{R^2 + S^2}$$

$$I_p = a\cos\omega t + b\sin\omega t$$
$$I'_p = \omega(-a\sin\omega t + b\cos\omega t)$$
$$I''_p = \omega^2(-a\cos\omega t - b\sin\omega t)$$

- Equation for Ip with coefficients a and b as given above is the desired particular solution of the nonhomogeneous ODE governing the current I in an RLC circuit with sinusoidal electromotive force.

Solving ODE for the current

- Using $a = \frac{-E_o S}{R^2 + S^2}$ $b = \frac{E_o R}{R^2 + S^2}$ we can write I_p in terms of physically

  visible quantities, namely, amplitude I_o and phase lag θ of the current behind the electromotive force, that is,

$$I_p(t) = I_o \sin(\omega t - \theta)$$

where

$$I_o = \sqrt{a^2 + b^2} = \frac{E_o}{\sqrt{R^2 + S^2}}$$

$$\tan\theta = -\frac{a}{b} = \frac{S}{R}$$

The quantity $\sqrt{R^2 + S^2}$ is called the impedance. Our formula shows that the impedance equals the ratio $\frac{E_o}{I_o}$.

This is somewhat analogous to E/I = R (Ohm' s law)

Solving ODE for the current

- A general solution of the homogeneous equation corresponding to

$$LI'' + RI' + \frac{1}{C} I = E'(t) = E_o \omega \cos \omega t$$

is:

$$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

where λ_1 and λ_2 are the roots of the characteristic equation:

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

- We can write the roots in the form $\lambda_1 = -\alpha + \beta$ and $\lambda_2 = -\alpha - \beta$, where

$$\alpha = \frac{R}{2L}, \quad \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}$$

- Now in an actual circuit, R is never zero (hence R>0). From this, it follows that Ih approaches zero, theoretically as $t \to \infty$, but practically after a short time.

Solving ODE for the current

- Hence the transient current $I=I_h+I_p$ tends to the steady state current I_p and after some time the output will practically be a harmonic oscillation, which is given by:

$$I_p(t) = I_o \sin(\omega t - \theta)$$

and whose frequency is that of the input (of the electromotive force)

Analogy of electrical and mechanical quantities

- Entirely different physical or other systems may have the same mathematical model. For instance, the ODE of a mechanical system and the ODE of an electric RLC circuit can be expressed by:

$$LQ'' + RQ' + \frac{1}{C}Q = E_o \sin \omega t \qquad\qquad my'' + cy' + ky = F_o \cos \omega t$$

- The **inductance L** corresponds to the **mass**, and indeed an inductor opposes a change in current, having an inertia effect similar to that of a mass.
- The **resistance R** corresponds to the **damping constant c** and a resistor causes loss of energy, just as a damping dashpot does.
- This analogy is strictly quantitative in the sense that to a given mechanical system we can construct an electrical circuit whose current will give the exact values of the displacement in the mechanical system when suitable scale factors are used.

Analogy of electrical and mechanical quantities

- The practical importance of this analogy is almost obvious. The analogy may be used for constructing an 'electrical model' of a given mechanical model, resulting in substantial savings of time and money because electric circuits are easy to assmeble, and electric quantities can be measured much more quickly and accurately than mechanical ones.

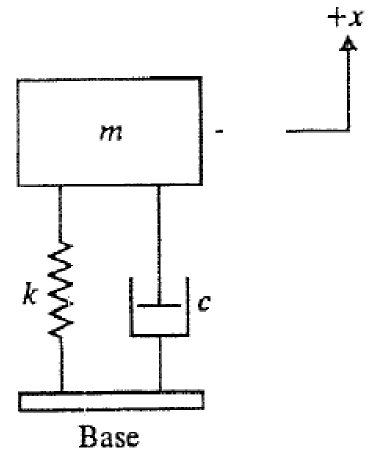
$$LQ'' + RQ' + \frac{1}{C}Q = E_o \sin \omega t \qquad my'' + cy' + ky = F_o \cos \omega t$$

- Table: Analogy of electrical and mechanical quantities

Electrical system	Mechanical system
Inductance, L	Mass m
Resistance, R	Damping c
Reciprocal of capacitance, 1/C	Spring modulus k
Electromotive force $E_o \sin\omega t$	Driving force $F_o \cos\omega t$
Current, I(t)=dq/dt	Velocity , v(t)=dy/dt
Charge, Q(t)	Displacement, y(t)

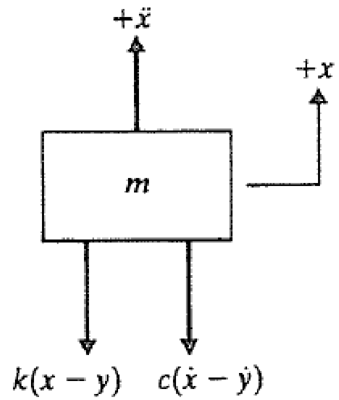
Base excited systems: absolute motion

- Consider the base-excited system of the figure. The goal of the analysis will be to determine the absolute response x(t) (typically acceleration or displacement of the mass) given the base motion y(t).



- From the free-body diagram, application of Newton's second law leads directly to the differential equation:

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$$



Base excited systems: absolute motion

- Assume that the base motion is harmonic,

$$y(t) = Y_o e^{i\omega t}$$

- And assume that the response will be harmonic,

$$x(t) = \tilde{X} e^{i\omega t}$$

where $\tilde{X}$ is the complex response. The transfer function and the gain function are derived in the same manner as for the force excited system. The transfer function is:

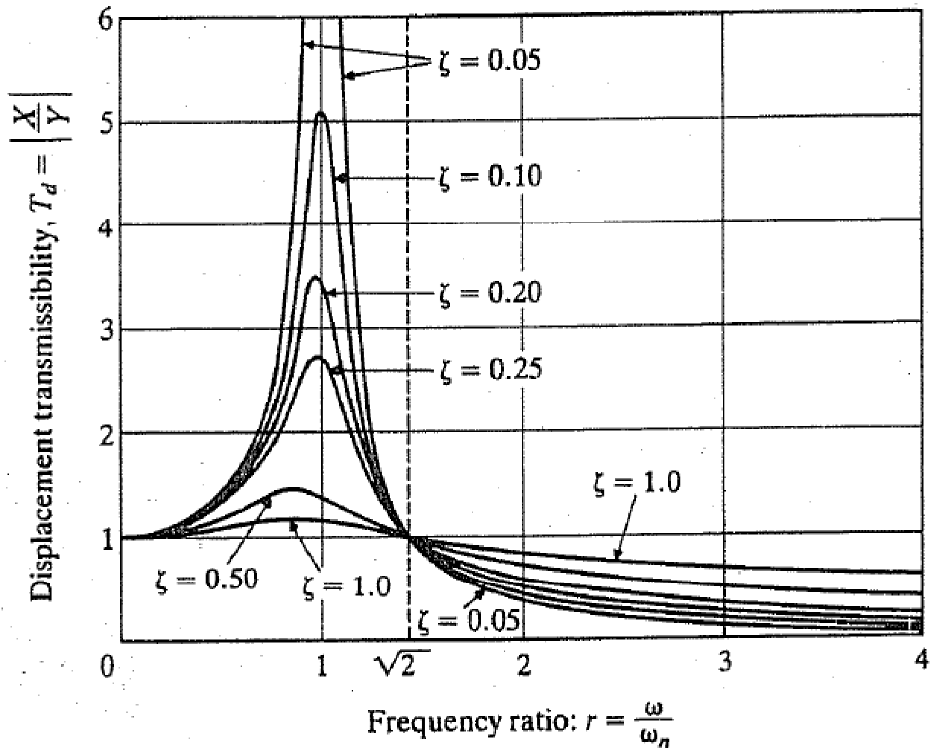
$$H(\omega) = \frac{k + ic\omega}{(k - m\omega^2) + ic\omega}$$

- The gain function is

$$\frac{X_o}{Y_o} = |H(\omega)| = \sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$$

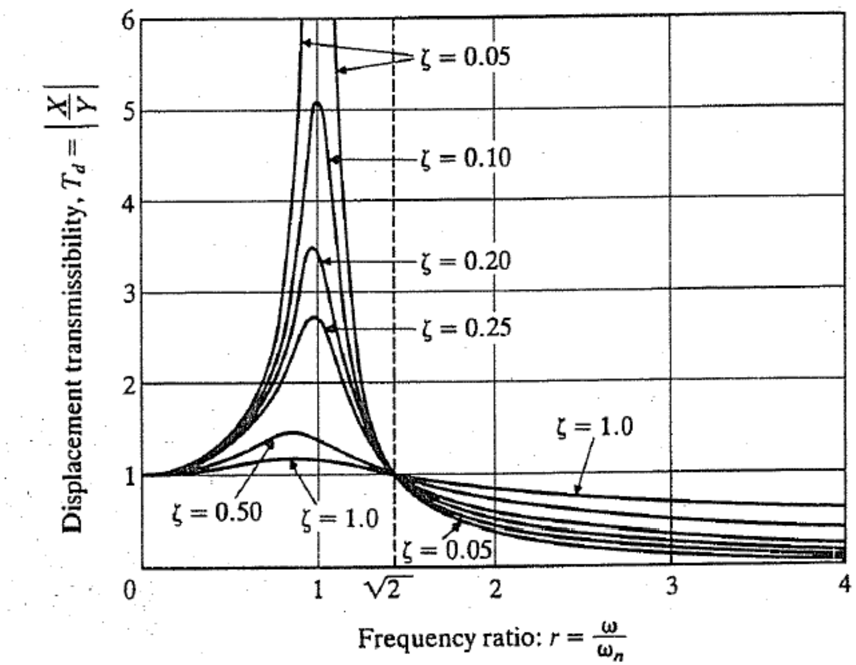
Base excited systems: absolute motion

- In nondimensional form $\frac{X_o}{Y_o} = \sqrt{\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}$
- The gain function for the absolute displacement for the base-excited system is shown in the figure.



Base excited systems: absolute motion

- The value of T_d is unity at r=0 and close to unity for small values of r.
- For an undamped system ζ=0, $T_d \rightarrow \infty$ at resonance (r=1).
- The value of T_d is less than unity (T_d<1) for values of r >$\sqrt{2}$ (for any amount of damping ζ)
- The value of T_d is equal to unity (T_d=1) for all values of ζ at r=$\sqrt{2}$



$$\frac{X_o}{Y_o} = \sqrt{\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}$$

Base excited systems: absolute motion

- The equations

$$\frac{X_o}{Y_o} = \sqrt{\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}$$

$$\frac{X_o}{Y_o} = |H(\omega)| = \sqrt{\frac{k^2+(c\omega)^2}{(k-m\omega^2)^2+(c\omega)^2}}$$

can be interpreted as the gain functions for acceleration output given acceleration input. And again note that the transfer function for velocity and acceleration responses can be derived by multiplying the equation

$$H(\omega) = \frac{k+ic\omega}{(k-m\omega^2)+ic\omega}$$

by iω and -ω^2, respectively.

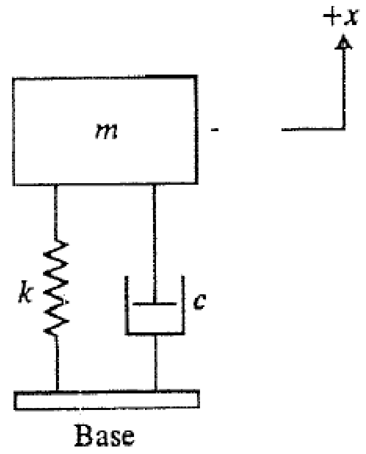
Base-excited system: Relative motion

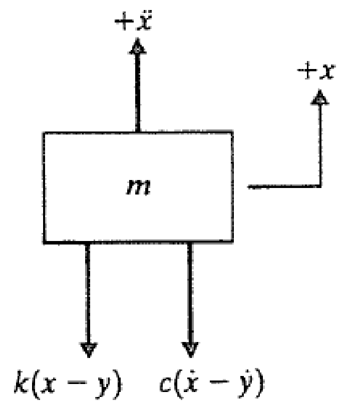
- Consider the free body diagram in the figure. Now the response variable under consideration will be the relative displacement,

$$z(t) = x(t) - y(t)$$

- In the model, the spring represents a structural element. The stress in that element will be proportional to z. Thus, this problem would be relevant to designers of structures subjected to base motions, for example, earthquakes.

- Letting $z=x-y$ in the equation of motion leads directly to $m\ddot{z} + \dot{z} + kz = -m\ddot{y}$





Base-excited system: Relative motion

- Assuming that the base motion is harmonic,

$$y(t) = Y_o e^{i\omega t}$$

- And assuming the response is also harmonic,

$$z(t) = \tilde{Z} e^{i\omega t}$$

- Following the procedure as described above, the transfer function is:

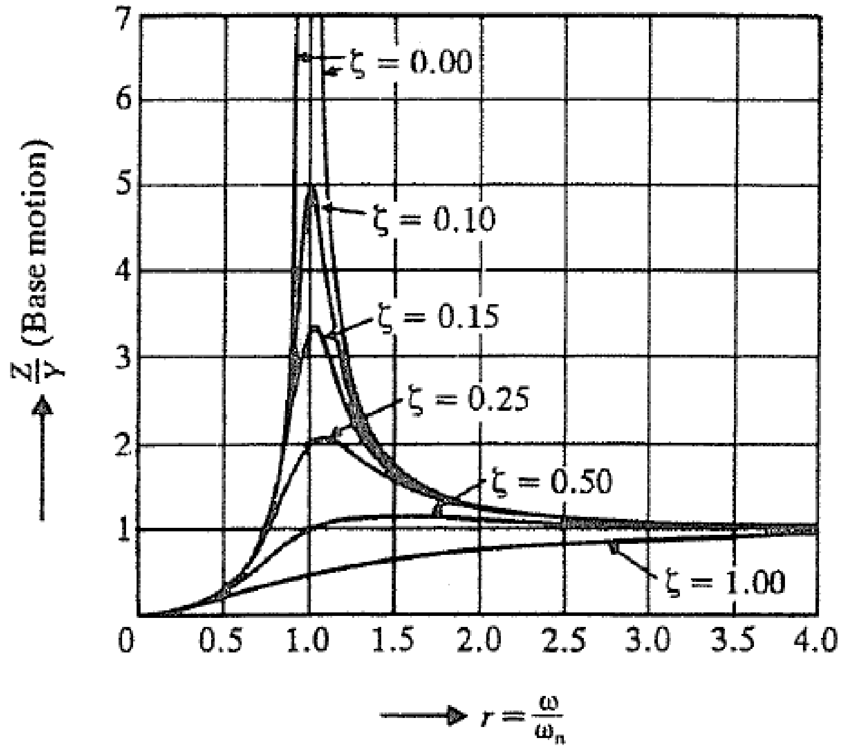
$$H(\omega) = \frac{m\omega^2}{(k - m\omega^2) + ic\omega}$$

- And the gain function is

$$\frac{Z_o}{Y_o} = \left| H(\omega) \right| = \frac{m\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

Base-excited system: Relative motion

- In nondimensionless form, $\frac{Z_o}{Y_o} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$
- The gain function for the relative motion for the base-excited system is shown in the figure:



Base-excited system: Relative motion

- Again note that the transfer function for relative velocity and acceleration responses can be derived by multiplying

$$H(\omega) = \frac{m\omega^2}{(k - m\omega^2) + ic\omega}$$

by iω and -ω^2, respectively.

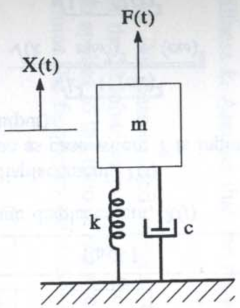
- The gain functions for velocity and acceleration responses can be obtained by multiplying both sides of the equations

$$\frac{Z_o}{Y_o} = \left|H(\omega)\right| = \frac{m\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \qquad \frac{Z_o}{Y_o} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

by ω and ω^2, respectively.

Gain functions for force excited systems

Table 7.1 Gain Functions for Force-Excited Systems



Equation of Motion

$$m\ddot{X} + c\dot{X} + kX = F(t)$$

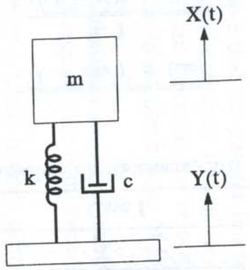
$$r = \omega/\omega_n = f/f_n \qquad \zeta = c/2m\omega_n$$

Input → Force, $F(t)$

	Case I	Case II	Case III
Output	Absolute displacement, $X(t)$	Absolute acceleration, $\ddot{X}(t)$	Force transmitted to base, $F_{TR} = kX(t) + c\dot{X}(t)$
$\lvert H(\omega)\rvert$	$\dfrac{1}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$	$\dfrac{\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$	$\dfrac{\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$
$\lvert H(r)\rvert$	$\dfrac{1}{k}\dfrac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$	$\dfrac{r^2}{m\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$	$\dfrac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$

Gain functions for base-excited systems

Table 7.2 Gain Functions for Base-Excited Systems



Absolute displacement = $X(t)$

Relative Displacement

$Z(t) = X(t) - Y(t)$

	Case I	Case II	Case III
Output	Absolute displacement, $X(t)$	Relative displacement, $Z(t)$	Relative acceleration, $\ddot{Z}(t)$
Input	Base displacement, $Y(t)$ (same as case where $\ddot{Y}$ is input, $\ddot{X}$ is output)	Base acceleration, $\ddot{Y}(t)$	Base accleration, $\ddot{Y}(t)$ (same as case where Y is input, Z is output)
$\lvert H(\omega)\rvert$	$\dfrac{\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$	$\dfrac{m}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$	$\dfrac{m\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$
$\lvert H(r)\rvert$	$\dfrac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$	$\dfrac{1}{\omega_n^2}\dfrac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$	$\dfrac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$

Example

- A fixed bottom offshore structure is subjected to oscillatory storm waves. In a first approximation, it is estimated that the waves produce a harmonic force F(t) having amplitude F=122 kN. The period of these waves is τ=8 sec. The structure is modeled as having a lumped mass of 110 tons concentrated in the deck. The weight of the structure itself is assumed to be negligible. The natural period of the structure was measured as being τ_n= 4.0 sec. It is assumed that the damping factor is ζ=5%. It is required to determine the steady state amplitude of the response of the structure.
- **Solution:** As modeled, this will be a force-excited system, and the response can be computed from the gain function of

$$\frac{X_o}{F_o / k} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

The problem reduces to one of finding the frequency ratio r and the stiffness k.

Example

- Because r is the ratio of the forcing frequency to the natural frequency, it follows that r will also be the ratio of the natural period to the forcing period. Thus,

$$r = \frac{\tau_n}{\tau} = \frac{4}{8} = 0.5$$

- To compute k, first note that the natural frequency is

$$f_n = \frac{1}{\tau_n} = 0.25\ Hz.$$

- Then noting that the expression for the natural frequency is

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{W/g}}$$

- We compute k=27667 N/m

Example

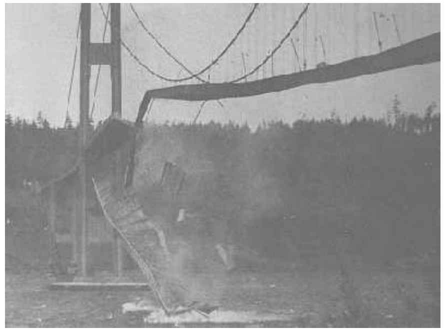
- Finally substituting into

$$\frac{X_o}{F_o / k} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

- Xo=5.87 m.

Resonance

- For resonance please download the Millenium bridge and Tacoma Bridge videos in the website of the course.



Background for response of SDOF system to random forces

- Response of SDOF system to impulsive forces
- Response of single degree of freedom system to arbitrary loading
- Relationship between the impulse response and the transfer function
- Relationship between the Fourier transform of displacement and force

Response under a nonperiodic force

- We will see that periodic forces of any general wave form can be represented by Fourier series as a superposition of harmonic components of various frequencies.
- The response of a linear sytem is then found by superposing the harmonic response to each of the exciting forces.
- When the exciting force F(t) is nonperiodic, such as that due to blast from an explosion, a different method of calculating the response is required.
- Various methods can be used to find the response of the system to an arbitrary excitation as follows:
  - Representing the excitation by a Fourier integral
  - Using the method of convolution integral
  - Using the method of Laplace transforms
  - First approximating F(t) by a suitable interpolation model and then using a numerical procedure
  - Numerically integrating the equations of motion.

Response of single degree of freedom system to impulsive forces

- A nonperiodic exciting force usually has a magnitude that varies with time; it acts for a specified period of time and then stops.

- The simplest form is the impulsive force- a force that has a large magnitude F and acts for a very short period of time Δt.

- From dynamics, we know that impulse can be measured by finding the change in momentum of the system.

- The unit impulse F acting at t=0 is also denoted by the Dirac delta function, $\delta(t)$. The Dirac delta function at time $t=\tau$, denoted as $\delta(t-\tau)$ has the properties $\int_0^\infty \delta(t-\tau)dt = 1 \qquad \int_0^\infty \delta(t-\tau)F(t)dt = F(\tau)$

  where $0<\tau<\infty$. Thus an impulsive force acting at $t=\tau$ can be denoted as:

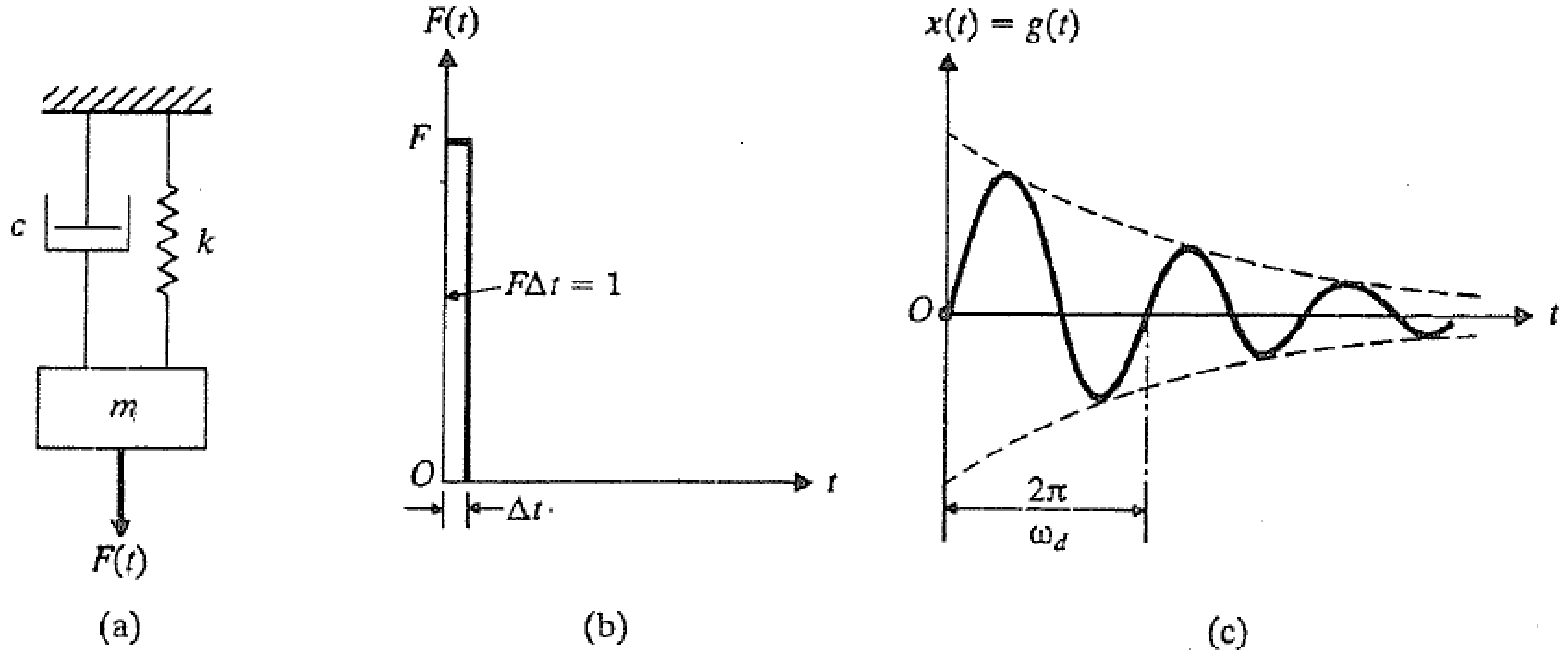
$$F(t) = F\delta(t-\tau)$$

Response of single degree of freedom system to impulsive forces

- Consider a SDOF system subjected to impulsive loading as shown in figure. The external force is:

$$F(t) = F_o \delta(t)$$

where $\delta(t)$ is the Dirac delta function.



Response of single degree of freedom system to impulsive forces

- The equation of motion of the mass will be similar to

$$m\ddot{x} + c\dot{x} + kx = F_o \sin \omega t$$

with the impulsive force of

$$F(t) = F_o \delta(t)$$

on the right hand side. The unit impulse is defined as F_o=1. The response x(t) to the unit impulse is denoted as h(t):

$$m\ddot{h} + c\dot{h} + kh = (1)\delta(t)$$

- Physically speaking, for t≈0, a radical change in the system motion takes place when the short duration high amplitude force excites an initial motion in the system. But for t>0, the response will be free vibration. Using elementary mechanics, F(Δt)=m(Δv) it can be shown that the velocity of the system just after the impulse is:

$$\dot{h}(0^+) = \frac{1}{m}$$

Response of single degree of freedom system to impulsive forces

- Using the initial conditions:

$$\dot{h}(0^+) = \frac{1}{m} \qquad h(0) = 0$$

- In the equation: $x(t) = e^{-\zeta\omega_n t}\left(x_o \cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_o + \zeta\omega_n x_o}{\omega_d} \sin\sqrt{1-\zeta^2}\,\omega_n t \right)$

- The free vibration response is:

$$h(t) = \frac{1}{m\omega_d} e^{-\xi\omega_n t} \sin\omega_d t \qquad t > 0$$

- Here h(t) is known as the force-excited absolute displacement response, **impulse response function** of the single-degree-of-freedom system. Note that h(t) characterizes a system just like the transfer function H(ω) does. The velocity and acceleration impulse response functions can also be obtained as derivatives of h(t).

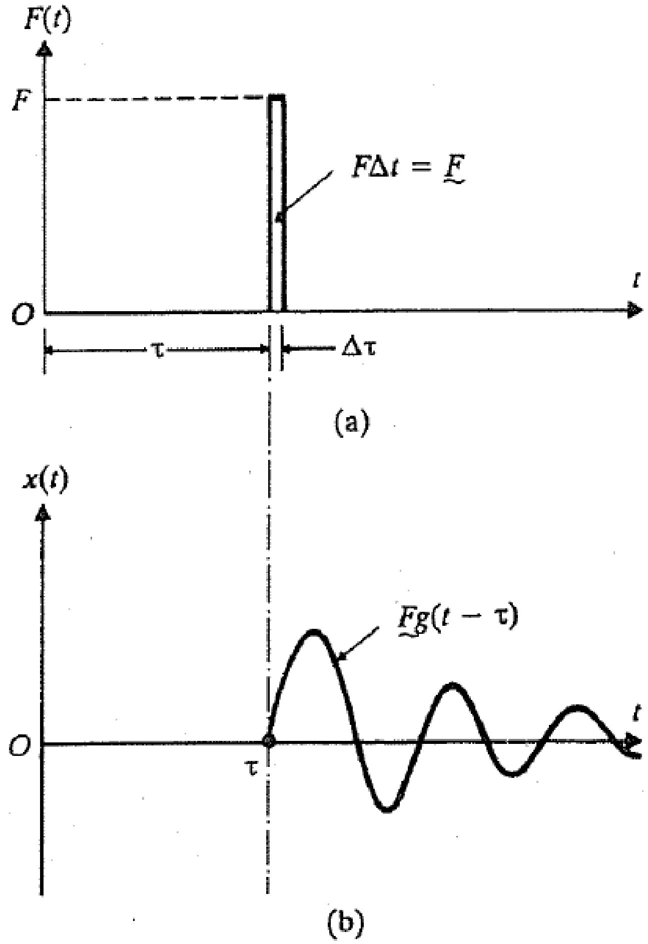
Response of single degree of freedom system to impulsive forces

- If the magnitude of the impulse is F instead of unity, the initial velocity $\dot{x}_o$ is F/m and the response of the system becomes:

$$x(t) = \frac{F}{m\omega_d} e^{-\xi\omega_n t} \sin\omega_d t = Fg(t)$$

- If the impulse F is applied at an arbitrary time t=τ by an amount F/m as shown in the figure, it will change the velocity at t=τ by an amount F/m. Assuming that x=0 until the impulse is applied, the displacement h at any subsequent time t, caused by a change in the velocity at time τ is given by the above equation with t replaced by the time elapsed after the application of the impulse, that is, t- τ. As shown in Fig.b, we obtain

$$x(t) = Fg(t-\tau)$$



Response of Single-Degree-of-Freedom System to Arbitrary Loading

- For a linear system, the impulse response function can be used to derive the response of a system under an arbitrary loading history. Consider the force shown in the figure

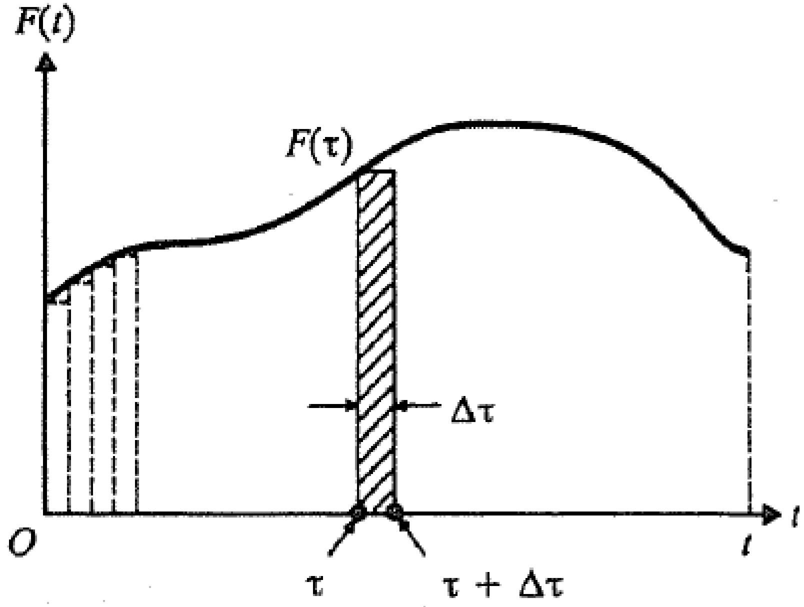


FIGURE 4.5 An arbitrary (nonperiodic) forcing function.

Response of Single-Degree-of-Freedom System to Arbitrary Loading

- The impulse during $\Delta\tau$ is $F(\tau)\,\Delta\tau$. The response to this impulse at any time $t>\tau$ is approximately $[F(\tau)\,\Delta\tau]h(t-\tau)$. Then the response at t is the sum of the responses due to a sequence of impulses. In the limit as $\Delta\tau \to 0$

$$x(t) = \int_{-\infty}^{t} F(\tau) h(t-\tau) d\tau$$

where the input F(t) is accounted for as $t \to -\infty$, for example, F(t) could be defined as zero for $t<0$. The expression for x(t) is called the convolution integral.

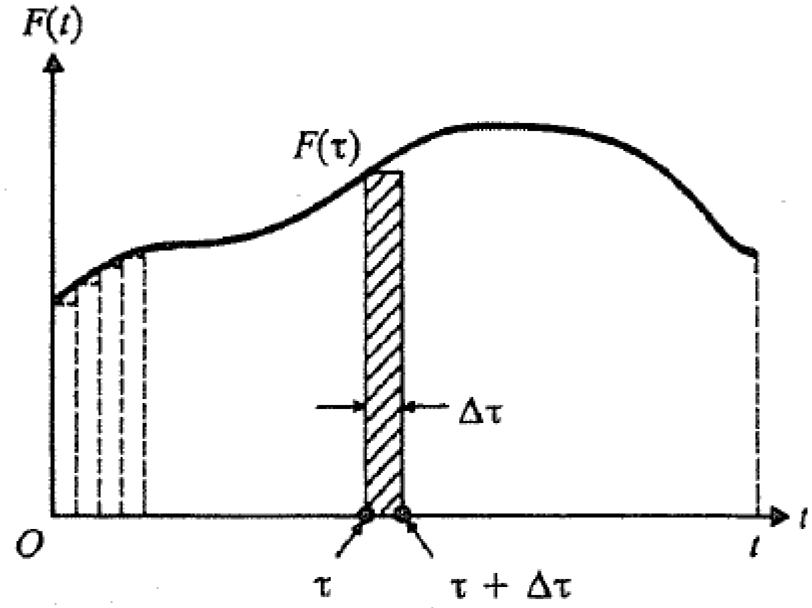


FIGURE 4.5 An arbitrary (nonperiodic) forcing function.

Response of Single-Degree-of-Freedom System to Arbitrary Loading

- Note that h(t-τ) =0 when τ>t. Thus, we can expand the limits to the interval (-∞,∞):

$$x(t) = \int_{-\infty}^{\infty} F(\tau)h(t-\tau)d\tau$$

- Another useful form is obtained by letting θ=t-τ:

$$x(t) = \int_{-\infty}^{t} F(t-\theta)h(\theta)d\theta$$

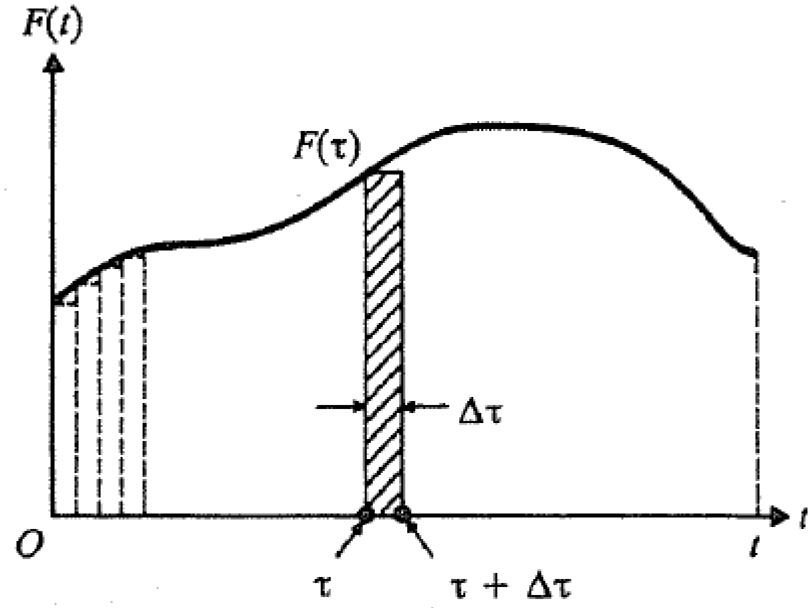


FIGURE 4.5 An arbitrary (nonperiodic) forcing function.

Response of Single-Degree-of-Freedom System to Arbitrary Loading

- By substituting the equation

$$h(t) = \frac{1}{m\omega_d} e^{-\xi\omega_n t} \sin \omega_d t \qquad t > 0$$

into

$$x(t) = \int_{-\infty}^{\infty} F(\tau) h(t-\tau) d\tau$$

we obtain:

$$x(t) = \frac{1}{m\omega_d} \int_{0}^{t} F(\tau) e^{-\zeta\omega_n (t-\tau)} \sin \omega_d (t-\tau) d\tau$$

which represents the response of an underdamped single degree of freedom system to the arbitrary excitation F(t).

- Note that the above equation does not consider the effect of initial conditions of the system.
- The integral in either of the two above equations is called the **convolution** or **Duhamel integral**.

Relationship between h(t) and H(ω)

- An important result from Fourier transform theory is that h(t) and H(ω) form a Fourier transform pair. This relationship is useful when deriving responses of dynamic systems to random vibration inputs. Let,

$$F(t) = e^{i\omega t}$$

- Then,

$$\begin{aligned} x(t) &= H(\omega)e^{i\omega t} \\ &= \int_{-\infty}^{\infty} h(t-\tau)e^{i\omega\tau}\,d\tau \\ &= \int_{-\infty}^{\infty} h(\theta)e^{i\omega(t-\theta)}\,d\theta \\ &= e^{i\omega t}\int_{-\infty}^{\infty} h(\theta)e^{-i\omega\theta}\,d\theta \end{aligned}$$

which implies that h(t) and H(ω) form a Fourier transform pair.

$$H(\omega) = \int_{-\infty}^{\infty} h(\theta)e^{-i\omega\theta}\,d\theta \qquad \rightarrow \qquad h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega t}\,d\omega$$

Relationship between h(t) and H(ω)

- Based on the Fourier transform of a convolution, the expression for the response to an arbitrary input in equation

$$x(t) = \int_{-\infty}^{\infty} F(\tau)h(t-\tau)d\tau$$

and representation in

$$H(\omega) = \int_{-\infty}^{\infty} h(\theta)e^{-i\omega\theta}d\theta \qquad \rightarrow \qquad h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega t}d\omega$$

it is clear that we can also express the response to an arbitrary input as:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)H(\omega)e^{i\omega t}d\omega$$

where F(ω) is the Fourier transform of F(t). This expression is useful for the analysis or numerical computation of system response or as the basis for random vibration computations.

Relationship between X(ω) and F (ω)

- The relationship between the Fourier transforms of x(t) and F(t) is used to derive responses of dynamic systems to random vibration input in random vibration theory. Take the Fourier transform of both sides of

$$x(t) = \int_{-\infty}^{t} F(t-\theta)h(\theta)d\theta$$

to find

$$X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} F(t-\theta)h(\theta)d\theta\right]e^{-i\omega t}dt$$

$$Let\ \tau = t-\theta,\ dt = d\tau$$

$$X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} F(\tau)h(\theta)d\theta\right]e^{-i\omega(\tau+\theta)}d\tau$$

$$\text{Rearranging:}$$

$$\mathrm{X}(\omega) = \int_{-\infty}^{\infty}\mathrm{h}(\theta)\mathrm{e}^{-\mathrm{i}\omega\theta}d\theta\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\tau)e^{-i\omega\tau}d\tau$$

Relationship between X(ω) and F (ω)

- From

$$H(\omega) = \int_{-\infty}^{\infty} h(\theta) e^{-i\omega\theta} d\theta$$

and the basic relationship of the Fourier transform:

$$u(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(t) e^{-i\omega t} dt$$

it follows that

$$\mathrm{X}(\omega) = \mathrm{H}(\omega)\mathrm{F}(\omega)$$