ERASMUS Teaching (2009), Technische Universität Berlin

Random vibrations-II

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Transmission of random vibration

Introduction

- We are now ready to consider how the characteristics of random signals are changed by transmission through stable linear systems.
- We shall consider the response y(t) to two separate random inputs $x_1(t)$ and $x_2(t)$. The response to a single input can then be obtained directly by putting either x_1 or x_2 zero, and the response to more than two inputs can be inferred from the form of the solution for two inputs.
- The mental picture we need is of an infinity of experiments, all proceeding simultaneously, and each with an identical linear system for which the impulse response functions are $h_1(t)$ and $h_2(t)$ and the corresponding frequency response functions are $H_1(\omega)$ and $H_2(\omega)$.

Introduction

- Each experiment is excited by sample functions from the $x_1(t)$ and $x_2(t)$ random processes and the response is a sample function from the y(t) random process.

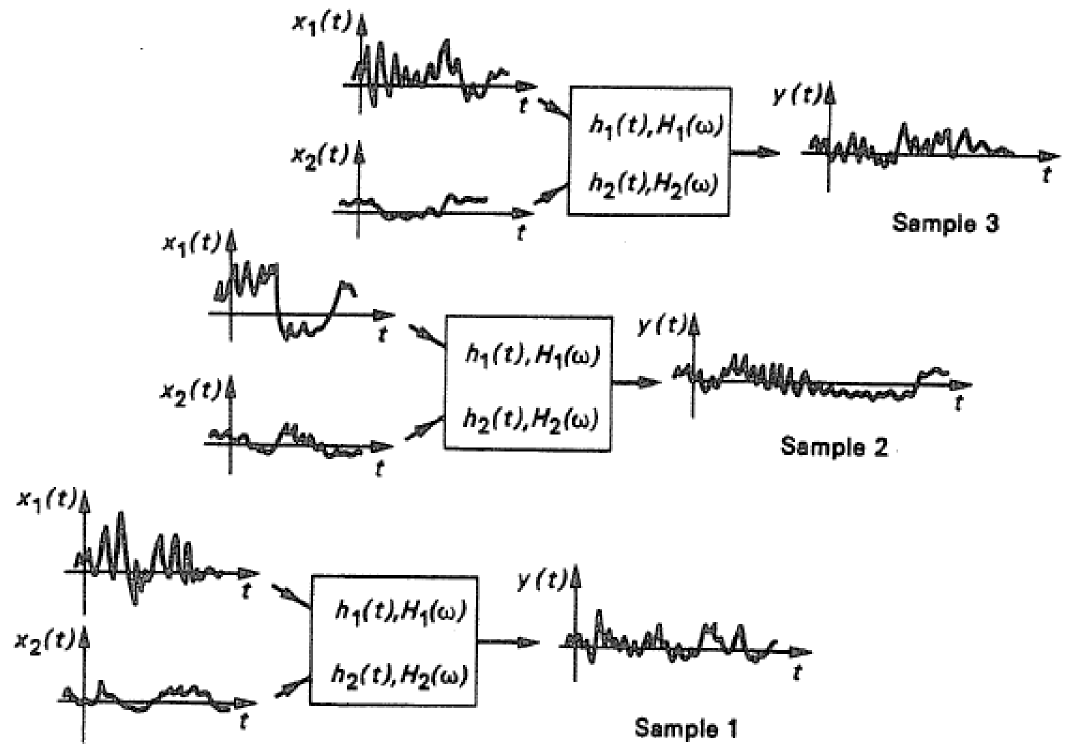
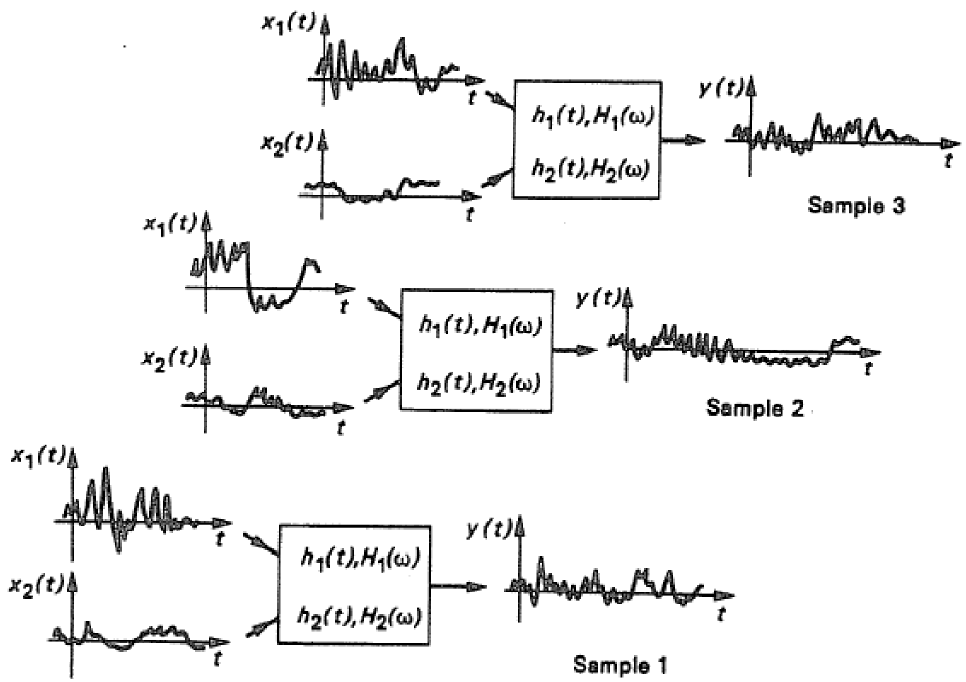


Fig. 7.1 Concept of ensemble averaging for a linear system subjected to random excitation

Introduction

- We want to know how the characteristics of the output process y(t) depend on the characteristics of the two input processes $x_1(t)$ and $x_2(t)$ and on the input-output characteristics of the system.
- The functions $h_1(t)$ and $H_1(\omega)$ give the response y(t) due to an input $x_1(t)$ and the functions $h_2(t)$ and $H_2(\omega)$ give the response y(t) due to an input $x_2(t)$.



Mean level

- According to

$$y(t)=\int_{-\infty}^{\infty} h(\theta)x(t-\theta)d\theta$$

  the response y(t) of a typical sample experiment to the inputs x_1(t) and x_2(t) may be expressed as:

$$y(t)=\int_{-\infty}^{\infty} h_1(\theta)x_1(t-\theta)\,\mathrm{d}\theta+\int_{-\infty}^{\infty} h_2(\theta)x_2(t-\theta)\,\mathrm{d}\theta.$$

- If we are now to calculate the ensemble average E[y(t)] we have to determine the average values of both the integrals on the rhs of the above equation. To do this, we need to remember that an integral is just the limiting case of a summation and that the average of a sum of numbers is the same as the sum of the average numbers, for instance:

$$E[x_1+x_2+x_3+\cdots]=E[x_1]+E[x_2]+E[x_3]+\cdots$$

Mean level

- Or using the summation sign:

$$E\left[\sum_{r=1}^{N} x_r\right] = \sum_{r=1}^{N} E[x_r].$$

- Applying this result to

$$y(t) = \int_{-\infty}^{\infty} h_1(\theta)x_1(t-\theta)\,\mathrm{d}\theta + \int_{-\infty}^{\infty} h_2(\theta)x_2(t-\theta)\,\mathrm{d}\theta.$$

gives:

$$E[y(t)] = \int_{-\infty}^{\infty} h_1(\theta)E[x_1(t-\theta)]\,\mathrm{d}\theta + \int_{-\infty}^{\infty} h_2(\theta)E[x_2(t-\theta)]\,\mathrm{d}\theta.$$

- Provided that both the random inputs are stationary, then their mean levels $E[x_1]$ and $E[x_2]$ are independent of the time of ensemble averaging $(t-\theta)$, and so we obtain:

$$E[y(t)] = E[x_1]\int_{-\infty}^{\infty} h_1(\theta)\,\mathrm{d}\theta + E[x_2]\int_{-\infty}^{\infty} h_2(\theta)\,\mathrm{d}\theta$$

Mean level

- From

$$E[y(t)] = E[x_1]\int_{-\infty}^{\infty} h_1(\theta)\,\mathrm{d}\theta + E[x_2]\int_{-\infty}^{\infty} h_2(\theta)\,\mathrm{d}\theta$$

- We see that E[y(t)] is independent of time, so that finally,

$$E[y] = E[x_1]\int_{-\infty}^{\infty} h_1(\theta)\,\mathrm{d}\theta + E[x_2]\int_{-\infty}^{\infty} h_2(\theta)\,\mathrm{d}\theta.$$

- Most engineers tend to think in terms of frequency response rather than impulse response, and we can express the above equation in these terms by using the basic result:

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

  which for ω=0 becomes

$$H(\omega = 0) = \int_{-\infty}^{\infty} h(t)\,\mathrm{d}t$$

On substitution to the second equation above, we get:

$$E[y] = E[x_1]H_1(0) + E[x_2]H_2(0)$$

Mean level

$$E[y] = E[x_1]H_1(0) + E[x_2]H_2(0)$$

- Where

$$H_1(0) = \frac{\text{constant level of } y}{\text{constant level of } x_1}$$

$$H_2(0) = \frac{\text{constant level of } y}{\text{constant level of } x_2}$$

- Or in electrical engineering terms,

$$H_1(0) = \frac{\text{d.c. (direct current) level of } y}{\text{d.c. level of } x_1}$$

$$H_2(0) = \frac{\text{d.c. level of } y}{\text{d.c. level of } x_2}.$$

- The mean levels of stationary random vibration are therefore transmitted just as though they are constant non-random signals and the superimposed random excurisons have no effect on the relationships between mean levels.

Autocorrelation

- The autocorrelation function for the output process y(t) is:

$$E[y(t)y(t+\tau)].$$

- According to

$$y(t)=\int_{-\infty}^{\infty} h_1(\theta)x_1(t-\theta)\,\mathrm{d}\theta+\int_{-\infty}^{\infty} h_2(\theta)x_2(t-\theta)\,\mathrm{d}\theta.$$

we can write formal solutions for y(t) and y(t+τ) and, putting θ_1 and θ_2 instead of θ to avoid confusion, these are

$$y(t)=\int_{-\infty}^{\infty} h_1(\theta_1)x_1(t-\theta_1)\,\mathrm{d}\theta_1+\int_{-\infty}^{\infty} h_2(\theta_1)x_2(t-\theta_1)\,\mathrm{d}\theta_1$$

and

$$y(t+\tau)=\int_{-\infty}^{\infty} h_1(\theta_2)x_1(t+\tau-\theta_2)\,\mathrm{d}\theta_2+\int_{-\infty}^{\infty} h_2(\theta_2)x_2(t+\tau-\theta_2)\,\mathrm{d}\theta_2$$

- Substituting the third and fourth equation into the first equation:

Autocorrelation

- We get: $$E[y(t)y(t+\tau)] = E\left[\int_{-\infty}^{\infty} h_1(\theta_1)x_1(t-\theta_1)\,d\theta_1 \int_{-\infty}^{\infty} h_1(\theta_2)x_1(t+\tau-\theta_2)\,d\theta_2 + \right.$$ $$+ \int_{-\infty}^{\infty} h_1(\theta_1)x_1(t-\theta_1)\,d\theta_1 \int_{-\infty}^{\infty} h_2(\theta_2)x_2(t+\tau-\theta_2)\,d\theta_2 +$$ $$+ \int_{-\infty}^{\infty} h_2(\theta_1)x_2(t-\theta_1)\,d\theta_1 \int_{-\infty}^{\infty} h_1(\theta_2)x_1(t+\tau-\theta_2)\,d\theta_2 +$$ $$\left. + \int_{-\infty}^{\infty} h_2(\theta_1)x_2(t-\theta_1)\,d\theta_1 \int_{-\infty}^{\infty} h_2(\theta_2)x_2(t+\tau-\theta_2)\,d\theta_2 \right].$$

- Since we are dealing with a stable system, the integrals in the above equation converge and it is legitimate to write each product of two integrals as a single double integral so that for instance:

$$\int_{-\infty}^{\infty} h_1(\theta_1)x_1(t-\theta_1)\,d\theta_1 \int_{-\infty}^{\infty} h_1(\theta_2)x_1(t+\tau-\theta_2)\,d\theta_2 =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_1(\theta_2)x_1(t-\theta_1)x_1(t+\tau-\theta_2)\,d\theta_1\,d\theta_2.$$

Autocorrelation

- When we come to average the expression

$$E[y(t)y(t+\tau)] = E\Big[\int_{-\infty}^{\infty} h_1(\theta_1)x_1(t-\theta_1)\,\mathrm{d}\theta_1 \int_{-\infty}^{\infty} h_1(\theta_2)x_1(t+\tau-\theta_2)\,\mathrm{d}\theta_2 + $$
$$+ \int_{-\infty}^{\infty} h_1(\theta_1)x_1(t-\theta_1)\,\mathrm{d}\theta_1 \int_{-\infty}^{\infty} h_2(\theta_2)x_2(t+\tau-\theta_2)\,\mathrm{d}\theta_2 + $$
$$+ \int_{-\infty}^{\infty} h_2(\theta_1)x_2(t-\theta_1)\,\mathrm{d}\theta_1 \int_{-\infty}^{\infty} h_1(\theta_2)x_1(t+\tau-\theta_2)\,\mathrm{d}\theta_2 + $$
$$+ \int_{-\infty}^{\infty} h_2(\theta_1)x_2(t-\theta_1)\,\mathrm{d}\theta_1 \int_{-\infty}^{\infty} h_2(\theta_2)x_2(t+\tau-\theta_2)\,\mathrm{d}\theta_2\Big].$$

- We have then to average these double integrals to find for example:

$$E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_1(\theta_2)x_1(t-\theta_1)x_1(t+\tau-\theta_2)\mathrm{d}\theta_1\,\mathrm{d}\theta_2\right]$$

- which is equal to: $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_1(\theta_2)E[x_1(t-\theta_1)x_1(t+\tau-\theta_2)]\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2.$$

Autocorrelation

- Provided that the input process $x_1(t)$ is stationary, its autocorrelation function is independent of absolute time t, and so:

$$E[x_1(t-\theta_1)x_1(t+\tau-\theta_2)] = R_{x_1}(\tau-\theta_2+\theta_1).$$

- The term

$$E\left[\int_{-\infty}^{\infty} h_1(\theta_1)x_1(t-\theta_1)\,\mathrm{d}\theta_1 \int_{-\infty}^{\infty} h_1(\theta_2)x_1(t+\tau-\theta_2)\,\mathrm{d}\theta_2\right.$$

may be written:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_1(\theta_2)R_{x_1}(\tau-\theta_2+\theta_1)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2$$

and is independent of time t. The same reasoning can be applied to the other three terms and so for stationary excitation, the output autocorrelation function is independent of absolute time t and can be expressed by the following rather lengthy expression:

Autocorrelation

$$R_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_1(\theta_2)R_{x_1}(\tau - \theta_2 + \theta_1)\,d\theta_1\,d\theta_2 +$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_2(\theta_2)R_{x_1x_2}(\tau - \theta_2 + \theta_1)\,d\theta_1\,d\theta_2 +$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\theta_1)h_1(\theta_2)R_{x_2x_1}(\tau - \theta_2 + \theta_1)\,d\theta_1\,d\theta_2 +$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\theta_1)h_2(\theta_2)R_{x_2}(\tau - \theta_2 + \theta_1)\,d\theta_1\,d\theta_2$$

involving double convolutions of the input autocorrelation and crosscorrelation functions.

- The **autocorrelation function** for the output process $R_y(\tau)$ is **independent of absolute time t** for **stationary excitation** and this is a general result for the response of any constant parameter linear system. It turns out that all averages of the output process are time invariant for stationary excitation, and the **output process** is therefore itself **stationary**.

Spectral density

- Although the equation

$$R_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_1(\theta_2)R_{x_1}(\tau - \theta_2 + \theta_1)\,d\theta_1\,d\theta_2 +$$
$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_2(\theta_2)R_{x_1x_2}(\tau - \theta_2 + \theta_1)\,d\theta_1\,d\theta_2 +$$
$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\theta_1)h_1(\theta_2)R_{x_2x_1}(\tau - \theta_2 + \theta_1)\,d\theta_1\,d\theta_2 +$$
$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\theta_1)h_2(\theta_2)R_{x_2}(\tau - \theta_2 + \theta_1)\,d\theta_1\,d\theta_2$$

is such a complicated expression, fortunately considerable simplifications emerge if we take Fourier transforms of both sides to find $S_y(\omega)$, the spectral density of the output process.

Spectral density

- From

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_x(\tau) e^{-i\omega\tau} d\tau$$

the Fourier transform of the first double integral on the rhs of the equation

$$R_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_1(\theta_2)R_{x_1}(\tau - \theta_2 + \theta_1)\,d\theta_1\,d\theta_2 +$$
$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_2(\theta_2)R_{x_1x_2}(\tau - \theta_2 + \theta_1)\,d\theta_1\,d\theta_2 +$$
$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\theta_1)h_1(\theta_2)R_{x_2x_1}(\tau - \theta_2 + \theta_1)\,d\theta_1\,d\theta_2 +$$
$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\theta_1)h_2(\theta_2)R_{x_2}(\tau - \theta_2 + \theta_1)\,d\theta_1\,d\theta_2$$

is:

$$I_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}\left\{\int_{-\infty}^{\infty} d\theta_1 \int_{-\infty}^{\infty} d\theta_2\, h_1(\theta_1)h_1(\theta_2)R_{x_1}(\tau - \theta_2 + \theta_1)\right\}$$

Spectral density

- Changing the order of integration, this may be written:

$$I_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\theta_1\, h_1(\theta_1) \int_{-\infty}^{\infty} \mathrm{d}\theta_2\, h_1(\theta_2) \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{-i\omega\tau} R_{x_1}(\tau - \theta_2 + \theta_1)$$

- The last integral is with respect to τ with θ_1 and θ_2 constant. We can therefore legitimately write this:

$$\mathrm{e}^{i\omega(\theta_1-\theta_2)} \int_{-\infty}^{\infty} \mathrm{d}(\tau - \theta_2 + \theta_1)\, \mathrm{e}^{-i\omega(\tau-\theta_2+\theta_1)} R_{x_1}(\tau - \theta_2 + \theta_1)$$

- which from

$$S_x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_x(\tau) e^{-i\omega\tau} d\tau$$

is equal to:

$$\mathrm{e}^{i\omega(\theta_1-\theta_2)} \cdot 2\pi S_{x_1}(\omega)$$

Spectral density

- So that the equation

$$I_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta_1\, h_1(\theta_1)\int_{-\infty}^{\infty} d\theta_2\, h_1(\theta_2)\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau} R_{x_1}(\tau - \theta_2 + \theta_1)$$

can be written:

$$I_1 = \int_{-\infty}^{\infty} d\theta_1\, h_1(\theta_1)\int_{-\infty}^{\infty} d\theta_2\, h_1(\theta_2)\, e^{i\omega(\theta_1-\theta_2)} S_{x_1}(\omega)$$

$$= \int_{-\infty}^{\infty} d\theta_1 h_1(\theta_1)\, e^{i\omega\theta_1}\int_{-\infty}^{\infty} d\theta_2 h_1(\theta_2)\, e^{-i\omega\theta_2}\; S_{x_1}(\omega).$$

- The two remaining integrals can be related to the frequency response function H(ω) since from:

$$H_1(\omega) = \int_{-\infty}^{+\infty} h_1(\theta_2) e^{-i\omega\theta_2} d\theta_2$$

and the complex conjugate of $H_1(\omega)$

$$H_1^*(\omega) = \int_{-\infty}^{+\infty} h_1(\theta_1) e^{i\omega\theta_1} d\theta_1$$

Spectral density

- Hence we obtain finally,

$$I_1 = H_1^*(\omega) H_1(\omega) S_{x1}(\omega)$$

for the Fourier transform of the first double integral on the right hand side of

$$\begin{aligned} R_y(\tau) = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1) h_1(\theta_2) R_{x_1}(\tau - \theta_2 + \theta_1)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 + \\ & + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1) h_2(\theta_2) R_{x_1 x_2}(\tau - \theta_2 + \theta_1)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 + \\ & + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\theta_1) h_1(\theta_2) R_{x_2 x_1}(\tau - \theta_2 + \theta_1)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 + \\ & + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\theta_1) h_2(\theta_2) R_{x_2}(\tau - \theta_2 + \theta_1)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \end{aligned}$$

- The same procedure may be applied to the rest of the above equation and the final results of taking the Fourier transforms of both sides of the equation is the following expression for the spectral density of the output process.

Spectral density

$$S_y(\omega) = H_1^*(\omega)H_1(\omega)S_{x_1}(\omega) + H_1^*(\omega)H_2(\omega)S_{x_1x_2}(\omega) + \\ + H_2^*(\omega)H_1(\omega)S_{x_2x_1}(\omega) + H_2^*(\omega)H_2(\omega)S_{x_2}(\omega).$$

- This is a most important result. By considering more than two inputs, it is not difficult to show that, for N inputs, the corresponding expression is:

  $$S_y(\omega) = \sum_{r=1}^{N}\sum_{s=1}^{N} H_r^*(\omega)H_s(\omega)S_{x_rx_s}(\omega)$$

  when we define: $S_{x_rx_r} = S_{x_r}$

  for the spectral density of the rth input. The equation $S_y(\omega) = \sum_{r=1}^{N}\sum_{s=1}^{N} H_r^*(\omega)H_s(\omega)S_{x_rx_s}(\omega)$ is the central result of the random vibration theory and its simplicity justifies our faith in the Fourier transform and frequency response approach. In the case of response to a single input, the above equation becomes:

  $$S_y(\omega) = H^*(\omega)H(\omega)S_x(\omega)$$

  Or, since the product of a complex number and its complex conjugate is equal to the magnitude of the number squared,

  $$S_y(\omega) = |H(\omega)|^2\, S_x(\omega).$$

Spectral density

- For uncorrelated inputs for which the cross-spectral density terms are all zero, the equation $S_y(\omega) = |H(\omega)|^2 S_x(\omega).$ can be generalized to:

$$S_y(\omega) = \sum_{r=1}^{N} |H_r(\omega)|^2 S_{x_r}(\omega).$$

Mean square response

- Once the response spectral density has been determined, the mean square response can be calculated directly from:

$$E\left[y^2\right] = \int_{-\infty}^{+\infty} S_y(\omega) d\omega$$

which for a single input becomes:

$$E[y^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_x(\omega)\, d\omega$$

and for many uncorrelated inputs is:

$$E[y^2] = \sum_{r=1}^{N} \int_{-\infty}^{\infty} |H_r(\omega)|^2 S_{x_r}(\omega)\, d\omega.$$

- For uncorrelated inputs, the mean square response is therefore the sum of the mean square responses due to each input separately. However, in general, this is not the case and for correlated inputs, the mean square response is not just the sum of the separate mean square responses. In these cases, $S_y(\omega) = \sum_{r=1}^{N} \sum_{s=1}^{N} H_r^*(\omega) H_s(\omega) S_{x_r x_s}(\omega)$ must be used to find the response spectral density $S_y(\omega)$ and then the integral given below is evaluated.

$$E\left[x^2\right] = \int_{-\infty}^{+\infty} S_x(\omega) d\omega$$

Example 1

- Determine the output spectral density $S_y(\omega)$ for the single degree of freedom oscillator shown in the figure when it is excited by a forcing function x(t) whose spectral density $S_x(\omega)=S_o$.

- From $S_y(\omega) = |H(\omega)|^2 S_0$ where $H(\omega)$ is the complex frequency response function. To find $H(\omega)$, put $x(t)=e^{i\omega t}$ and $y=H(\omega)e^{i\omega t}$ in the equation of motion:

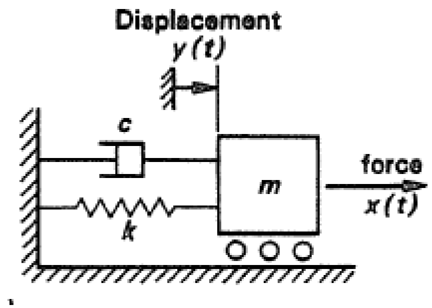
$$m\ddot{y} + c\dot{y} + ky = x(t)$$

to obtain:

$$(-m\omega^2 + ci\omega + k)H(\omega) = 1$$

and

$$H(\omega) = \frac{1}{-m\omega^2 + ic\omega + k}.$$



Example 1

- Hence the output spectral density as sketched in Figure d is:

$$S_y(\omega) = \frac{S_0}{(k - m\omega^2)^2 + c^2\omega^2}$$

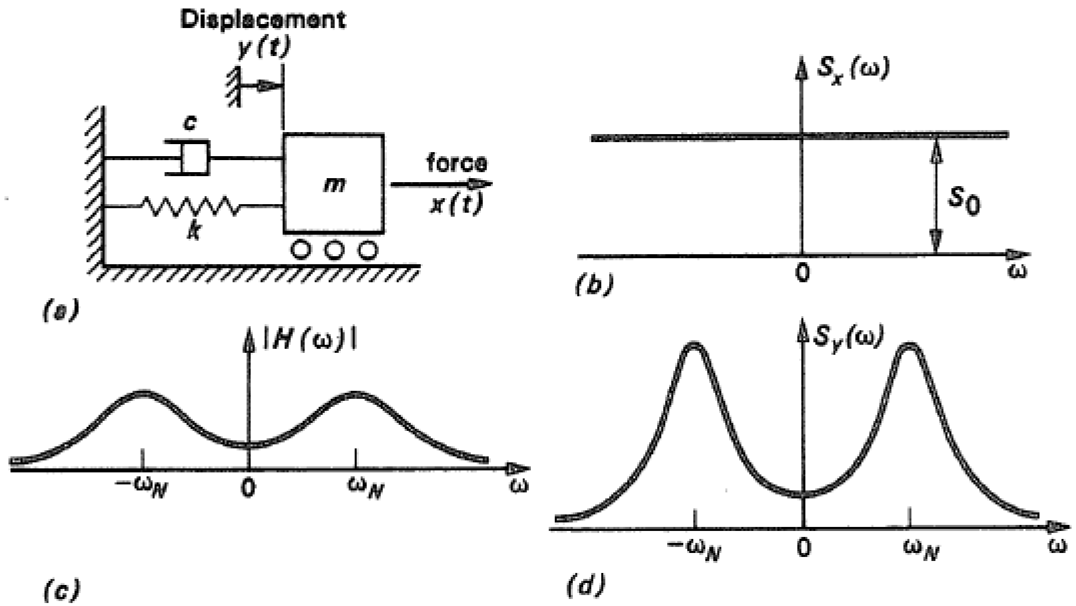
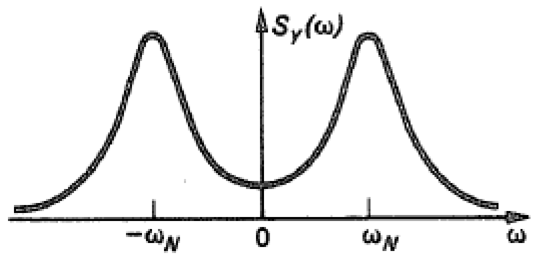


Fig. 7.2 Response spectral density $S_y(\omega)$ for a single degree-of-freedom oscillator subjected to a white noise force input $S_x(\omega) = S_0$.

Example 1

- The area under the spectral density curve is equal to the mean square, which may be written: $$E[y^2] = \int_{-\infty}^{\infty} \left| \frac{1}{-m\omega^2 + ic\omega + k} \right|^2 S_0 \, d\omega.$$
- A list of definite integrals of this form can be found in the literature and the result is: $$E[y^2] = \frac{\pi S_0}{kc}$$

  which is independent of the magnitude of the mass m. This is a surprising result as we would naturally expect m to affect the overall meansquare level of the output $E[y^2]$. The explanation can be seen by considering the height and width of the spectral peak in the figure (Remember that the peak in the left-hand half of the figure is just the mirror image of the peak in the right hand half, since $S_y(\omega)$ is an even function of ω:



Example 1

- The peak value occurs for small damping when $\omega \simeq \sqrt{\frac{k}{m}} = \omega_N$
- Its height is therefore: $S_y(\omega_N) = \frac{S_0}{c^2\omega_N^2} = \frac{S_0 m}{c^2 k}$
  which is proportional to m.
- The width of the spectral peak needs definition, but suppose that we arbitrarily define this as the difference in frequency $2\Delta\omega$ between the two points on either side of ω_N whose height is half the peak height (the so-called half power bandwidth). For small damping:

$$\Delta\omega \ll \omega_N$$

$$2\Delta\omega \simeq \frac{c}{m}$$

  which is inversely proportional to m. Hence we can see that increasing mass m increases the height of the spectral peak, but at the same time reduces its width. As we have seen, it turns out that these two opposite effects cancel out and the total area under the spectral density curve, and therefore the meansquare value, is independent of m.

Example 2

- The massless trolley shown in the figure is connected to two abutments by springs and viscous dashpots as shown. If the abutments move distances $x_1(t)$ and $x_2(t)$ with spectral densities $S_{x_1}(\omega) = S_{x_2}(\omega) = S_0$ (constant) but $x_2(t + T) = x_1(t)$, so that the cross-spectra are

$$S_{x_1 x_2}(\omega) = S_0 \, e^{-i\omega T}$$

$$S_{x_2 x_1}(\omega) = S_0 \, e^{i\omega T}$$

determine the response spectral density $S_y(\omega)$ and the mean square response $E[y^2]$

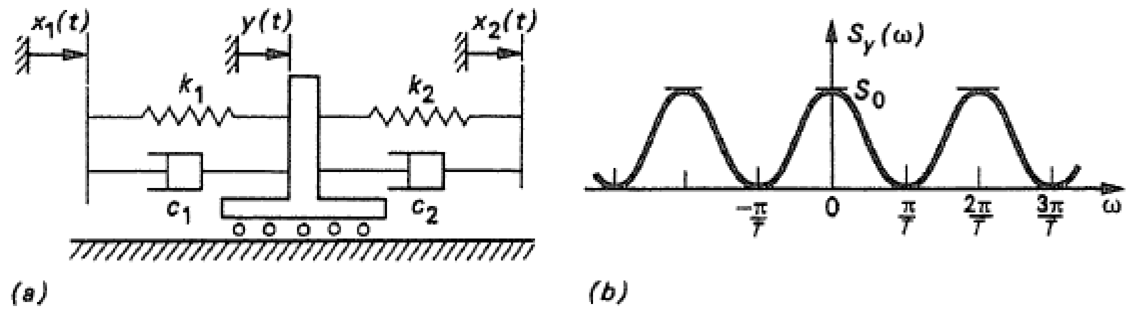


Fig. 7.3 Response spectral density for a system with two correlated inputs

Example 2

- For the equilibrium of the trolley

$$k_1(x_1 - y) + c_1(\dot{x}_1 - \dot{y}) = k_2(y - x_2) + c_2(\dot{y} - \dot{x}_2)$$

- So that the equation of motion is:

$$(c_1 + c_2)\dot{y} + (k_1 + k_2)y = k_1x_1 + c_1\dot{x}_1 + k_2x_2 + c_2\dot{x}_2.$$

- To determine the frequency response functions, first put

$$x_1 = e^{i\omega t}, x_2 = 0, \qquad y = H_1(\omega)e^{i\omega t}$$

to find:

$$H_1(\omega) = \frac{k_1 + ic_1\omega}{k_1 + k_2 + i(c_1 + c_2)\omega}$$

and then put $x_1 = 0,\ x_2 = e^{i\omega t},\ y = H_2(\omega)e^{i\omega t}$

to find:

$$H_2(\omega) = \frac{k_2 + ic_2\omega}{k_1 + k_2 + i(c_1 + c_2)\omega}.$$

Hence according to:

$$S_y(\omega) = H_1^*(\omega)H_1(\omega)S_{x_1}(\omega) + H_1^*(\omega)H_2(\omega)S_{x_1x_2}(\omega) + \\ + H_2^*(\omega)H_1(\omega)S_{x_2x_1}(\omega) + H_2^*(\omega)H_2(\omega)S_{x_2}(\omega).$$

Example 2

$$
\begin{aligned}
S_y(\omega) = {} & \frac{k_1^2 + c_1^2\omega^2}{(k_1 + k_2)^2 + (c_1 + c_2)^2\omega^2} S_0 + \\
& + \frac{(k_1 - ic_1\omega)(k_2 + ic_2\omega)}{(k_1 + k_2)^2 + (c_1 + c_2)^2\omega^2} S_0 \, e^{-i\omega T} \\
& + \frac{(k_1 + ic_1\omega)(k_2 - ic_2\omega)}{(k_1 + k_2)^2 + (c_1 + c_2)^2\omega^2} S_0 \, e^{i\omega T} \\
& + \frac{k_2^2 + c_2^2\omega^2}{(k_1 + k_2)^2 + (c_1 + c_2)^2\omega^2} S_0
\end{aligned}
$$

- which after collecting terms becomes:

$$
S_y(\omega) = S_0 \left\{ \frac{\begin{matrix} k_1^2 + k_2^2 + c_1^2\omega^2 + c_2^2\omega^2 + 2(k_1k_2 + c_1c_2\omega^2)\cos\omega T + \\ + 2(k_1c_2\omega - k_2c_1\omega)\sin\omega T \end{matrix}}{(k_1 + k_2)^2 + (c_1 + c_2)^2\omega^2} \right\}.
$$

- When the delay time T is zero, so that $x_1(t)=x_2(t)$, then we see that $S_y(\omega) = S_0$ and motion of the trolley is always equal to motion of the abutments.

Example 2

- For large ω (high frequencies):

$$S_y(\omega) \to S_0 \left\{ \frac{c_1^2 + c_2^2 + 2c_1c_2 \cos \omega T}{(c_1 + c_2)^2} \right\}$$

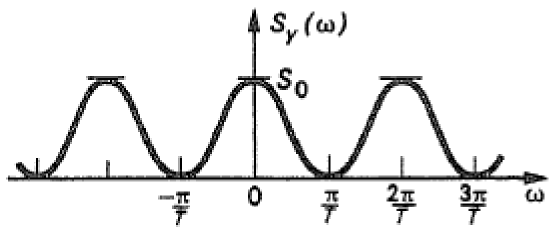
- which is finite and so:

$$E[y^2] = \int_{-\infty}^{\infty} S_y(\omega)\, d\omega \to \infty$$

on account of the characteristics of white noise excitation which has an infinite mean square value. For the case when the two spring stiffnesses and the two damper coefficients are the same,

$$S_y(\omega) = \frac{S_0}{2}(1 + \cos \omega T)$$

which has the form shown in the figure.



Cross-correlation

- In the case of a system excited by many inputs, it is sometimes helpful to determine the cross-correlation between the output and one of the inputs. Beginning with the definition

$$R_{x_1 y}(\tau) = E[x_1(t)y(t+\tau)]$$

for the case of two inputs, using

$$R_{x_1 y}(\tau) = E\left[x_1(t)\int_{-\infty}^{\infty} h_1(\theta)x_1(t+\tau-\theta)\,\mathrm{d}\theta + \right.$$
$$\left. + x_1(t)\int_{-\infty}^{\infty} h_2(\theta)x_2(t+\tau-\theta)\,\mathrm{d}\theta\right].$$

- Since $x_1(t)$ is not a function of θ, it may be moved under the integral signs and the averaging process carried out to give:

$$R_{x_1 y}(\tau) = \int_{-\infty}^{\infty} h_1(\theta)E[x_1(t)x_1(t+\tau-\theta)]\,\mathrm{d}\theta +$$
$$+ \int_{-\infty}^{\infty} h_2(\theta)E[x_1(t)x_2(t+\tau-\theta)]\,\mathrm{d}\theta$$

Cross-correlation

- In terms of the input autocorrelation and cross-correlation functions,

$$R_{x_1 y}(\tau) = \int_{-\infty}^{\infty} h_1(\theta) R_{x_1}(\tau - \theta)\, d\theta + \int_{-\infty}^{\infty} h_2(\theta) R_{x_1 x_2}(\tau - \theta)\, d\theta$$

which expresses the cross-correlation between input $x_1(t)$ and output y(t) in terms of the autocorrelation of $x_1(t)$, the cross-correlation between $x_1(t)$ and the other input $x_2(t)$, and the impulse response functions between x_1 and y and x_2 and y.

- For the special case when x_1 is white noise so that from $R_x(\tau) = 2\pi S_0 \delta(\tau)$

$$R_{x_1}(\tau - \theta) = 2\pi S_0\, \delta(\tau - \theta)$$

and x_1 and x_2 are uncorrelated so that

$$R_{x_1 x_2}(\tau - \theta) = 0$$

Cross-correlation

- Then $$R_{x_1y}(\tau) = \int_{-\infty}^{\infty} h_1(\theta) R_{x_1}(\tau - \theta)\, d\theta + \int_{-\infty}^{\infty} h_2(\theta) R_{x_1x_2}(\tau - \theta)\, d\theta$$

  gives: $$R_{x_1y}(\tau) = \int_{-\infty}^{\infty} h_1(\theta) 2\pi S_0\, \delta(\tau - \theta)\, d\theta$$ $$= h_1(\tau) 2\pi S_0 \quad \text{from (5.9).}$$

- The cross-correlation function between a white noise input $x_1(t)$ and the output y(t) is therefore the same as the impulse response at y for a unit impulse at x_1 multiplied by the factor $2\pi S_0$. This is an interesting result which is sometimes used to obtain the impulse response function experimentally.

Cross-spectral density

- Taking the Fourier transforms of both sides of

$$R_{x_1 y}(\tau) = \int_{-\infty}^{\infty} h_1(\theta) R_{x_1}(\tau - \theta)\,\mathrm{d}\theta + \int_{-\infty}^{\infty} h_2(\theta) R_{x_1 x_2}(\tau - \theta)\,\mathrm{d}\theta$$

gives:

$$S_{x_1 y}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{-i\omega\tau}\left\{\int_{-\infty}^{\infty} h_1(\theta) R_{x_1}(\tau - \theta)\,\mathrm{d}\theta + \right.$$
$$\left. + \int_{-\infty}^{\infty} h_2(\theta) R_{x_1 x_2}(\tau - \theta)\,\mathrm{d}\theta\right\}.$$

- Rearranging terms in the same way as for the previous calculation of the spectral density gives:

$$S_{x_1 y}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\theta\, h_1(\theta)\,\mathrm{e}^{-i\omega\theta}\int_{-\infty}^{\infty} \mathrm{d}\tau\, R_{x_1}(\tau - \theta)\,\mathrm{e}^{-i\omega(\tau-\theta)} +$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\theta\, h_2(\theta)\,\mathrm{e}^{-i\omega\theta}\int_{-\infty}^{\infty} \mathrm{d}\tau\, R_{x_1 x_2}(\tau - \theta)\,\mathrm{e}^{-i\omega(\tau-\theta)}.$$

- The integrals with respect to τ are with θ constant, and so if $(\tau\text{-}\theta)$ is replaced by ϕ (say) then d τ becomes d ϕ.

Cross-spectral density

- Using

$$S_x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_x(\tau) e^{-i\omega\tau} d\tau$$

$$S_{xy}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{xy}(\tau) e^{-i\omega\tau} d\tau$$

$$S_{yx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{yx}(\tau) e^{-i\omega\tau} d\tau$$

to evaluate the integrals with respect to ϕ, and $H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$ to evaluate the integrals with respect to θ which then remain, we obtain:

$$S_{x_1 y}(\omega) = H_1(\omega) S_{x_1}(\omega) + H_2(\omega) S_{x_1 x_2}(\omega).$$

- When there are N separate inputs, of which $x_r(t)$ is a typical one, the above equation becomes the summation

$$S_{x_r y}(\omega) = \sum_{s=1}^{N} H_s(\omega) S_{x_r x_s}(\omega)$$

Cross-spectral density

$$S_{x_r x_r} = S_{x_r}$$

- For uncorrelated inputs we can see that

$$S_{xy}(\omega) = H(\omega) S_x(\omega)$$

where H(ω) is the complex frequency response function relating the input x(t) to the output y(t), $S_x(\omega)$ is the spectral density of the input process, and $S_{xy}(\omega)$ is the cross-spectral density between the input and the output.

- From the properties of the cross-spectra

$$S_{yx}(\omega) = S_{xy}^*(\omega)$$
$$S_{xy}(\omega) = S_{yx}^*(\omega)$$

- It follows that

$$S_{yx}(\omega) = S_{xy}^*(\omega) = H^*(\omega) S_x(\omega)$$

- Since $S_x(\omega)$ is of course always a real quantity.

Probability distributions

- We now consider how the probability distribution for the response of a linear system depends on the probability distribution of the excitation. We can say at once that there is no simple relationship. There is no general method for obtaining the output probability distributions for a linear system except for the special case when the input probability distributions are Gaussian.

- The fact that we can calculate the output probability distributions for a linear system subjected to Gaussian excitation arises from the special properties of Gaussian processes.

- Firstly, there is a general theorem which says that if y_1 and y_2 are a pair of jointly Gaussian random variables, then if y is defined so that $y = y_1 + y_2$, the new random variable will also be Gaussian.

Probability distributions

- Secondly, this result may be applied to show that the response y(t) of a linear system will be a Gaussian process if the excitation x(t) is Gaussian. Using the convolution integral, we know that:

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,\mathrm{d}\tau$$

where h(t) is the impulse response function. It can be shown mathematically that the above integral can be thought of as the limiting case of a linear sum of a sum of random variables of which

$$y = y_1 + y_2$$

is the simplest example. Hence if x(t) is a Gaussian process, y(t) must be the same. Finally, these results can be extended to the case of more than one jointly Gaussian input to show that the output processes after transmission through a linear system will also be jointly Gaussian.

Probability distributions

- The output of a linear system subjected to Gaussian inputs is therefore Gaussian and the output probability distributions can be calculated if the respective mean values, variances and covariances are known.

$$\rho_{xy} = \frac{E\left[(x-m_x)(y-m_y)\right]}{\sigma_x \sigma_y}$$

- Also since a derivative can be expressed as the limiting case of the difference $\{y(t+\Delta t)-y(t)\}/\Delta t$ between two random variables $y(t+\Delta t)$ and y(t) if the process y(t) is Gaussian so is its derivative and so are higher derivatives such as acceleration.

Example

- A linear system is subjected to stationary Gaussian excitation as a result of which the response y(t) has a mean level m_y, standard deviation σ_y, and an autocorrelation function $R_y(\tau)$. Determine the probability density function $p(y_1,y_2)$ for the joint distribution of y at t_1 and y at t_2 where $t_2=t_1+\tau$. Referring to the definition of the second order Gaussian probability density from

$$\rho_{xy} = \frac{E[(x - m_x)(y - m_y)]}{\sigma_x \sigma_y}$$

the normalized covariance ρ_{y1y2} is given by:

$$\rho_{y_1 y_2} = \frac{E[(y_1 - m_y)(y_2 - m_y)]}{\sigma_y^2} = \frac{R_y(\tau) - m_y^2}{\sigma_y^2} = \rho$$

the equation $p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{(1-\rho_{xy}^2)}} e^{-\frac{1}{2(1-\rho_{xy}^2)}\left\{\frac{(x-m_x)^2}{\sigma_x^2} + \frac{(y-m_y)^2}{\sigma_y^2} - \frac{2\rho_{xy}(x-m_x)(y-m_y)}{\sigma_x\sigma_y}\right\}}$ gives:

$$p(y_1, y_2) = \frac{1}{2\pi\sigma_y^2\sqrt{(1-\rho^2)}} e^{-\frac{1}{2\sigma_y^2(1-\rho^2)}\{(y_1-m_y)^2 + (y_2-m_y)^2 - 2\rho(y_1-m_y)(y_2-m_y)\}}$$

Example

- When the time interval $\tau \to \infty,\ R_y(\tau) \to m_y^2$ and from

$$\rho_{y_1 y_2} = \frac{E[(y_1 - m_y)(y_2 - m_y)]}{\sigma_y^2} = \frac{R_y(\tau) - m_y^2}{\sigma_y^2} = \rho$$

$$\rho \to 0.$$

- In this case, $p(y_1, y_2) = \frac{1}{2\pi\sigma_y^2\sqrt{(1-\rho^2)}} e^{-\frac{1}{2\sigma_y^2(1-\rho^2)}\{(y_1-m_y)^2+(y_2-m_y)^2-2\rho(y_1-m_y)(y_2-m_y)\}}$ becomes

$$p(y_1, y_2) = \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-\frac{(y_1-m_y)^2}{2\sigma_y^2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-\frac{(y_2-m_y)^2}{2\sigma_y^2}} = p(y_1)\cdot(p(y_2)$$

- and $y_1=y(t_1)$ and $y_2=y(t_2)$ are then statistically independent.

Probability distributions

- As a summary, for engineering calculations, the world is often assumed to be linear, stationary, ergodic and Gaussian. Actually, there is an important theorem of probability theory called “the central limit theorem” which helps to explain why the probability distributions of many naturally occurring processes should be Gaussian.

- Roughly speaking, the central limit theorem says that when a random process results from the summation of infinitely many random elementary events, then this process will tend to have Gaussian probability distributions. We therefore have good reasons to expect that, for instance, the noise generated by falling rain, or by a turbulent fluid boundary layer, or by the random emission of electrons in a thermionic device, will all have probability distributions which approximate to Gaussian.

Probability distributions

- Furthermore, even when the excitation is nonGaussian, a system's response may approximate to Gaussian if it is a narrow band response derived from broad band excitation because the convolution integral

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,\mathrm{d}\tau$$

may again be thought of as the limiting case of a linear sum of approximately independent random variables.

However, there should be one word of caution. An assumed Gaussian probability distribution may give a poor approximation at the tails of the distribution and predictions of maximum excursion of a random process, based on such an assumed distribution should be treated with suspicion.

Statistics of narrow band processes

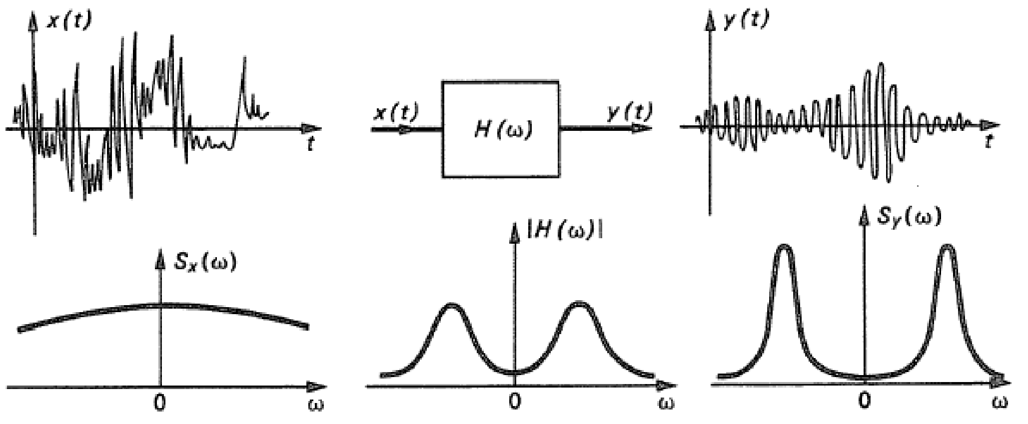
Introduction

- In the previous chapter, we considered general relations between the input and the output of a linear system subjected to random excitation. The characteristics of the excitation are modified by the response of the system, which in electrical engineering terms, acts as a filter.
- In most vibration problems, the system has at least one resonant frequency at which large amplitudes can be generated by small inputs.
- At other frequencies, transmission is reduced and, at very high frequencies, the effective mass may be so high that the output is not measurable.

Introduction

- A typical frequency response function H(ω) for such a resonant system is shown in the figure, which also shows how the characteristics of the broad band noise are changed by transmission through this system.

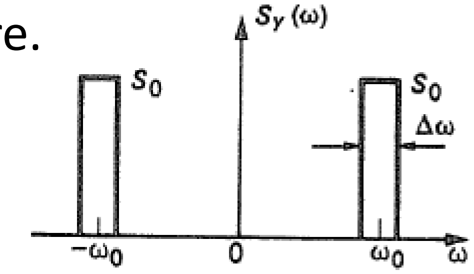
- Because the output spectrum is confined to a narrow band of frequencies in the vicinity of the resonant frequency, the response y(t) is a narrow band random process and the typical time history of y(t) resembles a sine wave of varying amplitude and phase as shown in the figure.



Narrow band response of a resonant system excited by broad band noise

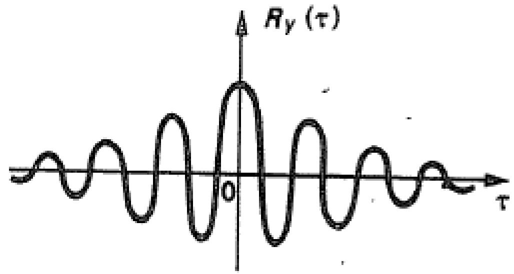
Introduction

- In order to make some simple calculations, suppose that the frequency response function has very sharp cut-offs above and below the resonant frequency so that the response spectral density Sy(ω) has the idealized form shown in the figure.



- We have already worked out the corresponding autocorrelation function in the previous chapters and this is as shown in the figure and formula below:

$$R_y(\tau) = 4S_0 \frac{\sin(\Delta\omega \ \tau/2)}{\tau} \cos \omega_0 \tau$$



Introduction

- If the excitation is Gaussian, we can find the probability distributions for y and for later reference, we shall nedd the first-order probability density function p(y) and the second-order probability density function $p(y,\dot{y})$ for the joint probability of y and its derivative. Both of these functions are given by the standard expressions

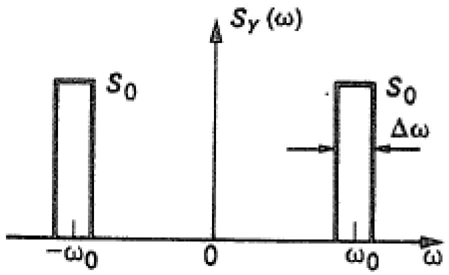
$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2}$$

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{(1-\rho_{xy}^2)}} e^{-\frac{1}{2(1-\rho_{xy}^2)}\left\{\frac{(x-m_x)^2}{\sigma_x^2}+\frac{(y-m_y)^2}{\sigma_y^2}-\frac{2\rho_{xy}(x-m_x)(y-m_y)}{\sigma_x\sigma_y}\right\}}$$

and are known provided that the statistics $m_y, m_{\dot{y}}, \sigma_y, \sigma_{\dot{y}}$ and $\rho_{y\dot{y}}$ are known.

Introduction

- First consider the mean level of y, m_y. If this were to be other than zero, the spectral density for y would have to show a delta function at ω=0, because otherwise there can not be a finite tribution to $E[y^2]$ at ω=0. Since the spectral density $S_y(\omega)$ shown in the figure does not have this delta function, the mean level is zero, m_y=0.



- Next, since the spectral density of the $\dot{y}$ process is given according to

$$S_{\dot{y}}(\omega) = \omega^2 S_y(\omega)$$

- The $S_{\dot{y}}(\omega)$ function also cannot have a delta function at ω=0, so the mean level of $\dot{y}$ is also zero, $m_{\dot{y}} = 0$

Introduction

- In order to calculate the variances σ_y^2 and $\sigma_{\dot{y}}^2$ we use

$$\sigma^2 = E\left[x^2\right] - \left(E[x]\right)^2$$

(variance)=(standard deviation)^2={Mean square-(Mean)^2}

$$E\left[x^2\right] = \int_{-\infty}^{\infty} S_x(\omega) d\omega$$

to find:

$$\sigma_y^2 = E[y^2] = \int_{-\infty}^{\infty} S_y(\omega)\,\mathrm{d}\omega = 2S_0\Delta\omega$$

$$\sigma_{\dot{y}}^2 = E[\dot{y}^2] = \int_{-\infty}^{\infty} \omega^2 S_y(\omega)\,\mathrm{d}\omega \simeq 2S_0\omega_0^2\Delta\omega$$

for $\Delta\omega \ll \omega_0$.

- Lastly from $\rho_{xy} = \dfrac{E\left[(x - m_x)(y - m_y)\right]}{\sigma_x \sigma_y}$, the normalized covariance is:

$$\rho_{y\dot{y}} = \frac{E[y\dot{y}]}{\sigma_y \sigma_{\dot{y}}}$$

Introduction

- We know that the numerator of the equation $\rho_{y\dot{y}} = \frac{E[y\dot{y}]}{\sigma_y \sigma_{\dot{y}}}$ can be expressed as:

$$E[y\dot{y}] = \frac{\mathrm{d}}{\mathrm{d}\tau} R_y(\tau) \Big|_{\tau = 0}$$

- If $R_y(\tau)$ is expressed as the Fourier integral of the corresponding spectral density, using

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega) e^{i\omega\tau} d\omega$$

- We get:

$$E[y\dot{y}] = i \int_{-\infty}^{\infty} \omega \, S_y(\omega) \, \mathrm{d}\omega.$$

- Now since $S_y(\omega)$ is a real even function of frequency ω, the integrand $\omega \, S_y(\omega)$ is a real odd function of ω. When integrated over the range minus infinity to plus infinity, the contribution from minus infinity to zero is exactly equal but opposite in sign to the contribution from zero to plus infinity. Hence the above integral must be equal to zero and we obtain:

$$E[y\dot{y}] = 0.$$

Introduction

- It is therefore a property of any stationary random process y(t) that y and its derivative $\dot{y}$ are uncorrelated and so the normalized covariance $\rho_{y\dot{y}}$ is always zero.
- We now have all the parameters we need and can substitute

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2}$$

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{(1-\rho_{xy}^2)}} e^{-\frac{1}{2(1-\rho_{xy}^2)}\left\{\frac{(x-m_x)^2}{\sigma_x^2}+\frac{(y-m_y)^2}{\sigma_y^2}-\frac{2\rho_{xy}(x-m_x)(y-m_y)}{\sigma_x\sigma_y}\right\}}$$

to obtain the probability density functions

$$p(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} \mathrm{e}^{\frac{-y^2}{2\sigma_y{}^2}} \qquad p(y,\dot{y}) = \frac{1}{2\pi\,\sigma_y\sigma_{\dot{y}}} \mathrm{e}^{-\frac{1}{2}\left(\frac{y^2}{\sigma_y{}^2}+\frac{\dot{y}^2}{\sigma_{\dot{y}}{}^2}\right)} = p(y)p(\dot{y})$$

where σ_y and $\sigma_{\dot{y}}$ are given by:

$$\sigma_y^2 = E[y^2] = \int_{-\infty}^{\infty} S_y(\omega)\,\mathrm{d}\omega = 2S_0\Delta\omega \qquad \sigma_{\dot{y}}^2 = E[\dot{y}^2] = \int_{-\infty}^{\infty} \omega^2 S_y(\omega)\,\mathrm{d}\omega \simeq 2S_0\omega_0^2\Delta\omega$$

Introduction

- These functions are sketched in the figure alongside the corresponding spectral density and autocorrelation curves.

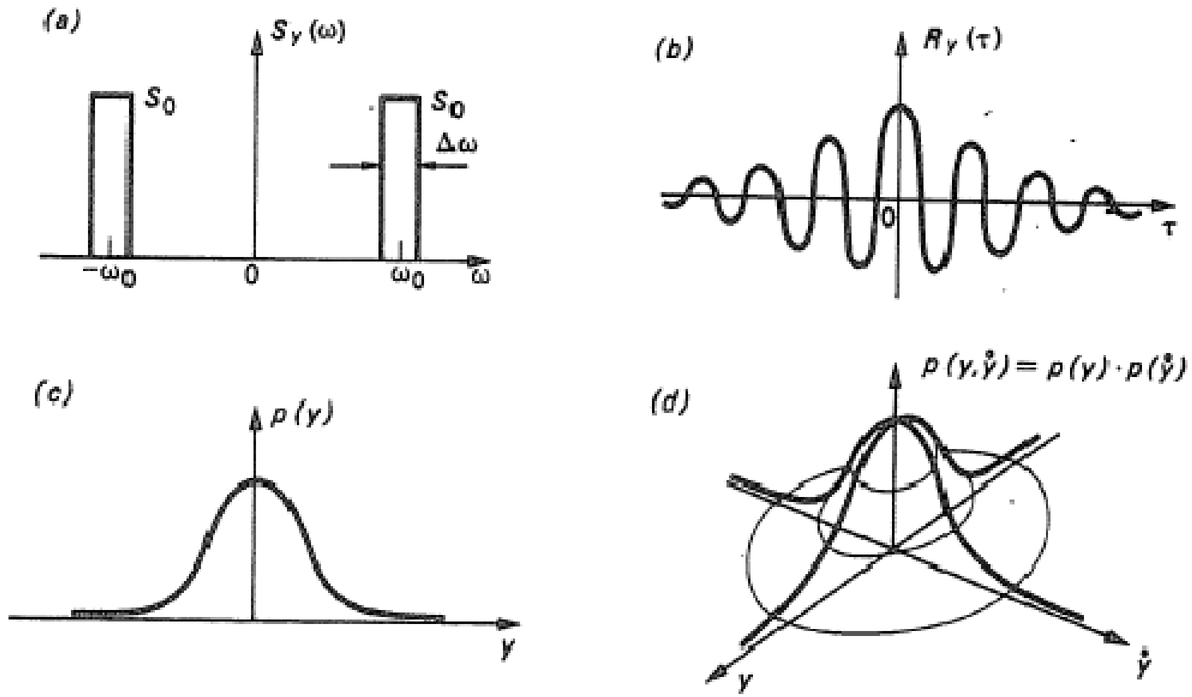


Fig. 8.2 Characteristics of a stationary, Gaussian, narrow band process

Crossing analysis

- Although the data in the previous figure says a lot about the narrow band process y(t) by describing its frequency composition and its amplitude and velocity distributions, it is possible to go further and obtain important information about the distribution of peak values, that is to say information about the amplitude of the fluctuating sine wave that makes up the process.

- Suppose that we enquire how many cycles of y(t) have amplitudes greater than the level y=a during the time period T as shown in the figure. For the sample shown, there are three cycles.

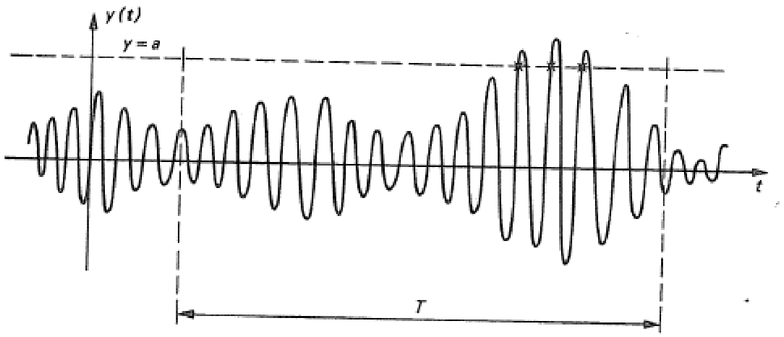


Fig. 8.3 Typical sample of a narrow band process

Crossing analysis

- Another way of saying this is that there are three crossings with positive slope of the level y=a in time T. Each of these positive slope crossings is marked with a cross in the figure.

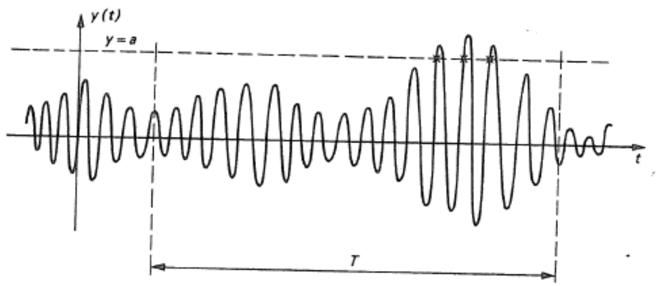


Fig. 8.3 Typical sample of a narrow band process

- Now consider the figure as one sample function of an ensemble of functions which make up the stationary random process y(t). Let $n_a^+(T)$ denote the number of positive slope crossings of y=a in time T for a typical sample and let the mean value for all the samples be $N_a^+(T)$ where

$$N_a^+(T) = E[n_a^+(T)].$$

Crossing analysis

- Since the process is stationary, if we take a second interval of duration T immediately following the first we shall obtain the same result, and for the two intervals together (total time 2T) we shall therefore obtain:

$$N_a^+(2T) = 2N_a^+(T)$$

from which it follows that, for a stationary process, the average number of crossings is proportional to the time interval T. Hence

$$N_a^+(T) \propto T$$

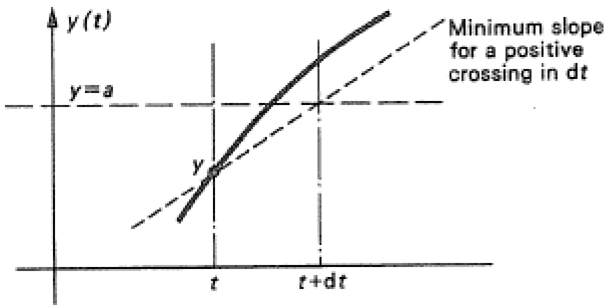
$$N_a^+(T) = \nu_a^+ T$$

where ν_a^+ is the average frequency of positive slope crossings of the level y=a. We now consider how the frequency parameter ν_a^+ can be deduced from the underlying probability distributions for y(t).

Crossing analysis

- Consider a small length of duration dt of a typical sample function as shown in the figure. Since we are assuming that the narrow band process y(t) is a smooth function of time, with no sudden ups and downs, if dt is small enough, the sample can only cross y=a with positive slope if y<a at the beginning of the interval, time t. Furthermore there is a minimum slope at time t if the level y=a is to be crossed in time dt depending on the value of y at time t. From the figure, this is: $\frac{a-y}{dt}$

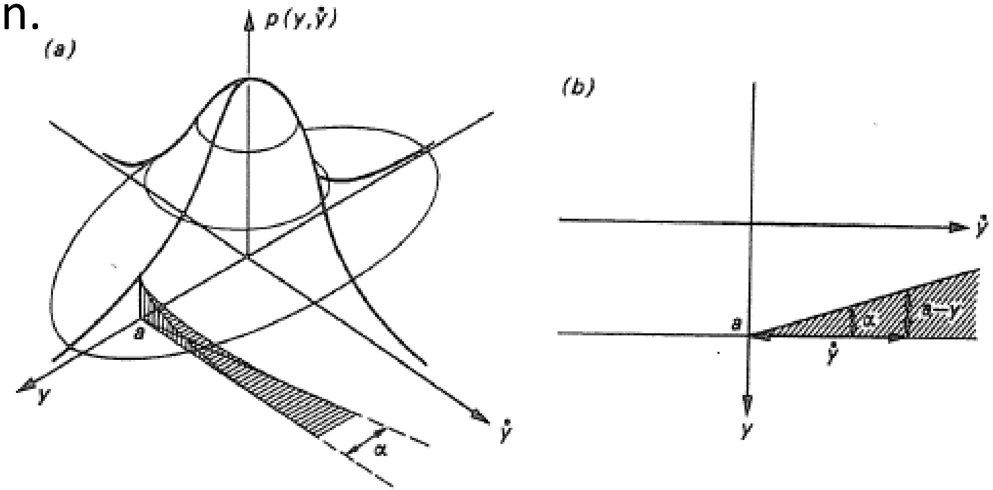
  and so there will be a positive slope crossing of y=a in the next time interval dt, if at time t, $y < a$ and $\frac{dy}{dt} > \frac{a-y}{dt}.$



Conditions for a positive slope crossing of $y = a$ in time interval dt

Crossing analysis

- In order to determine whether the conditions $y < a$ and $\frac{dy}{dt} > \frac{a-y}{dt}$. are satisfied at any arbitrary time t, we must find how the values of y and $\dot{y}$ are distributed by considering their joint probability density $p(y, \dot{y})$. . Suppose that the level y=a and time interval dt are specified. Then we are only interested in values of y<a and values of $\dot{y} = (dy/dt) > (a - y)/dt$, which means the shaded wedge of values y and $\dot{y}$ as shown in the figure. The wedge angle α is chosen so that $\tan\alpha = \frac{a-y}{\dot{y}} = dt$ in order to satisfy the above equation.



Calculation of the probability that there will be a positive slope crossing of $y = a$ in time interval dt

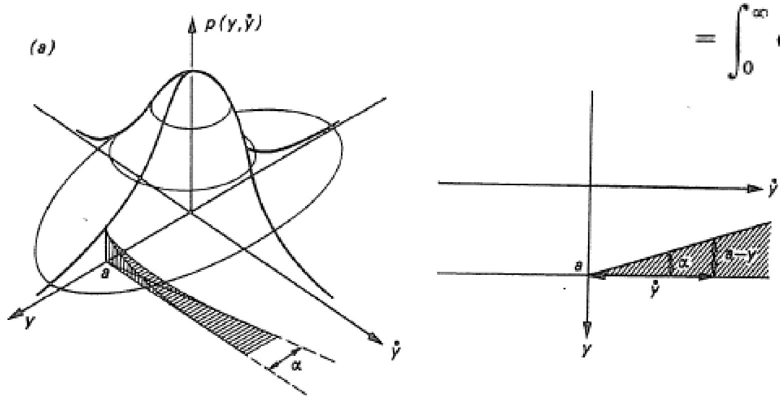
Crossing analysis

- If the values of y and $\dot{y}$ lie within this shaded wedge, then there will be a positive slope crossing of y=a in time dt. If they do not lie in the shaded wedge, then there will not be a crossing. The probability that they do lie in the shaded wedge can be calculated from the joint probability density function $p(y,\dot{y})$ and is just the shaded volume shown in the figure, i.e., the volume under the probability surface above the shaded wedge of acceptable values of y and $\dot{y}$. Hence,

$$\text{Prob}\begin{pmatrix}\text{Positive slope crossing} \\ \text{of } y = a \text{ in time d}t\end{pmatrix} = \iint p(y,\dot{y})\,\mathrm{d}y\,\mathrm{d}\dot{y}$$

over the shaded wedge in Fig. 8.5(a)

$$= \int_0^\infty \mathrm{d}\dot{y} \int_{a-\dot{y}\tan\alpha}^{a} \mathrm{d}y\, p(y,\dot{y}).$$



Calculation of the probability that there will be a positive slope crossing of $y = a$ in time interval dt

Crossing analysis

- When dt $\longrightarrow$ 0, the angle of the wedge α=dt $\longrightarrow$ 0 and in this case it is legitimate to put

$$p(y,\dot{y}) = p(y = a, \dot{y})$$

since at large values of y and $\dot{y}$, the probability density function approaches zero fast enough. Hence the equation

$$\text{Prob}\begin{pmatrix}\text{Positive slope crossing}\\ \text{of } y = a \text{ in time } dt\end{pmatrix} = \iint p(y,\dot{y})\,dy\,d\dot{y}$$

over the shaded wedge in Fig. 8.5(a)

$$= \int_0^\infty d\dot{y} \int_{a-\dot{y}\tan\alpha}^{a} dy\, p(y,\dot{y}).$$

may be written as:

$$\text{Prob}\begin{pmatrix}\text{Positive slope crossing}\\ \text{of } y = a \text{ in time } dt\end{pmatrix} = \int_0^\infty d\dot{y} \int_{a-\dot{y}\tan\alpha}^{a} dy\, p(y = a, \dot{y})$$

in which the integrand is no longer a function of y so that the first integral is just:

$$\int_{a-\dot{y}\tan\alpha}^{a} dy\, p(y = a,\dot{y}) = p(y = a,\dot{y})\,\dot{y}\tan\alpha$$

Crossing analysis

- Hence with tanα=dt

$$\text{Prob}\begin{pmatrix}\text{Positive slope crossing}\\ \text{of } y = a \text{ in time } dt\end{pmatrix} = \int_0^\infty p(y = a, \dot{y})\, \dot{y}\, dt\, d\dot{y}$$

$$= dt \int_0^\infty p(a, \dot{y})\, \dot{y}\, d\dot{y}$$

when the term $p(a, \dot{y})$ is understood to mean the joint probability density $p(y, \dot{y})$ evaluated at y=a. Now we have said that the average number of positive slope crossings in time T is $\nu_a^+ T$. The average number of crossings in time dt is therefore $\nu_a^+ dt$. Suppose that dt=0.01 s and that the frequency $\nu_a^+ = 2{\cdot}0$ crossings/s. In this case, the average number of crossings in 0.01 s would be $\nu_a^+ dt = 0{\cdot}02$ crossings. Next imagine that we were considering an ensemble of 500 samples. Each sample would either show one crossing in dt or no crossings, as dt is assumed very small compared with the average period of the narrow band process. The number of samples with a positive slope crossing must therefore be 10 since 10/500=0.02. But 10/500 is also the probability that any one sample chosen at random has a crossing in time dt. So we arrive at the following result:

$$\begin{pmatrix}\text{Average no. of positive crossings}\\ \text{of } y = a \text{ in time } dt\end{pmatrix} = \text{Prob}\begin{pmatrix}\text{Positive slope crossing}\\ \text{of } y = a \text{ in time } dt\end{pmatrix},$$

Crossing analysis

- The result $\begin{pmatrix}\text{Average no. of positive crossings} \\ \text{of } y = a \text{ in time } dt\end{pmatrix} = \text{Prob}\begin{pmatrix}\text{Positive slope crossing} \\ \text{of } y = a \text{ in time } dt\end{pmatrix}$ is only true

  because dt is small and the process y(t) is smooth so that there can not be more than one crossing of y=a in time dt. Accepting the above equation and substituting from $N_a^+(T) = \nu_a^+ T$ and

$$\text{Prob}\begin{pmatrix}\text{Positive slope crossing} \\ \text{of } y = a \text{ in time } dt\end{pmatrix} = \int_0^\infty p(y = a, \dot{y})\, \dot{y}\, dt\, d\dot{y}$$
$$= dt \int_0^\infty p(a, \dot{y})\, \dot{y}\, d\dot{y}$$

gives:

$$\nu_a^+ \, dt = dt \int_0^\infty p(a, \dot{y})\, \dot{y}\, d\dot{y}$$

from which dt cancels to give the following result for the frequency parameter ν_a^+ in terms of the joint probability density function $p(y\,\dot{y})$

$$\nu_a^+ = \int_0^\infty p(a, \dot{y})\, \dot{y}\, d\dot{y}.$$

Crossing analysis

- This is a general result which applies for any probability distribution, but for the special case of a Gaussian process we know from

$$p(y,\dot{y}) = \frac{1}{2\pi\,\sigma_y\sigma_{\dot{y}}}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{y^2}{\sigma_y{}^2}+\frac{\dot{y}^2}{\sigma_{\dot{y}}{}^2}\right)} = p(y)p(\dot{y})$$

that

$$p(a,\dot{y}) = \frac{1}{\sqrt{2\pi}\,\sigma_y}\mathrm{e}^{-a^2/2\sigma_y{}^2}\;\frac{1}{\sqrt{2\pi}\,\sigma_{\dot{y}}}\mathrm{e}^{-\dot{y}^2/2\sigma_{\dot{y}}{}^2}$$

which on substitution to

$$\nu_a^+ = \int_0^\infty p(a,\dot{y})\,\dot{y}\,\mathrm{d}\dot{y}.$$

gives:

$$\nu_a^+ = \frac{1}{\sqrt{2\pi}\,\sigma_y}\mathrm{e}^{-a^2/2\sigma_y{}^2}\int_0^\infty \frac{1}{\sqrt{2\pi}\,\sigma_{\dot{y}}}\mathrm{e}^{-\dot{y}^2/2\sigma_{\dot{y}}{}^2}\,\dot{y}\,\mathrm{d}\dot{y}.$$

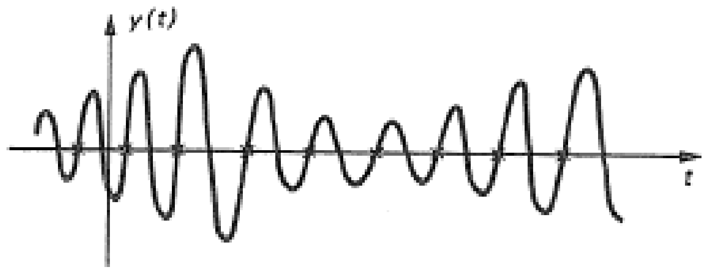
The integral is one of the standard results and its value is $\sigma_{\dot{y}}/\sqrt{2\pi}$, so that the final result is, for a Gaussian process,

$$\nu_a^+ = \frac{1}{2\pi}\frac{\sigma_{\dot{y}}}{\sigma_y}\mathrm{e}^{-a^2/2\sigma_y{}^2}.$$

Crossing analysis

- A special case occurs if we take the level a=0 because this gives a statistical average frequency for crossing the level y=0, the figure, which may be thought of as a statistical average frequency for the process.

- Notice that ν_0^+ is obtained by averaging across the ensemble and so it is not the same as the average frequency along the time axis unless the process is ergodic.



Positive slope crossings of the level $y = 0$

Example

- Calculate the frequency of positive crossings of the level y=a for the single degree of freedom oscillator shown in the figure when it is subjected to Gaussian white noise of spectral density S_o.

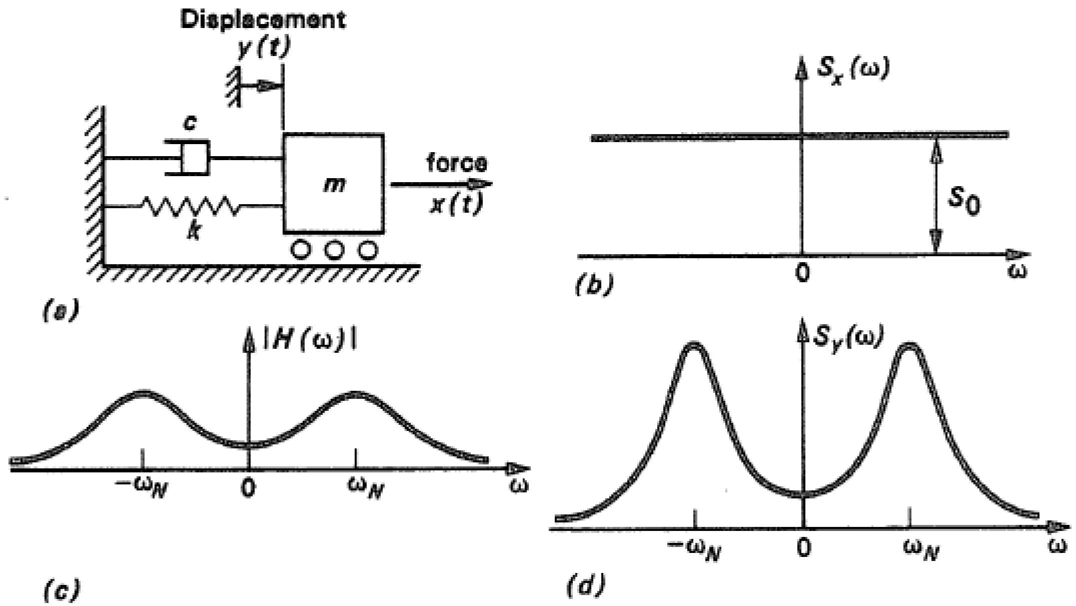


Fig. 7.2 Response spectral density $S_y(\omega)$ for a single degree-of-freedom oscillator subjected to a white noise force input $S_x(\omega) = S_0$.

Example

- In Example 1 of the first chapter, we have worked out that the frequency response function is:

$$H(\omega) = \frac{1}{-m\omega^2 + ic\omega + k}$$

so that

$$\sigma_y^2 = \int_{-\infty}^{\infty} \left| \frac{1}{-m\omega^2 + ic\omega + k} \right|^2 S_0 \, d\omega = \frac{\pi S_0}{kc}$$

as already calculated. The frequency response function relating $\dot{y}(t)$ to the excitation x(t) is obtained by multiplying H(ω) by i ω to obtain:

$$H'(\omega) = \frac{i\omega}{-m\omega^2 + ic\omega + k}$$

so that

$$\sigma_{\dot{y}^2} = \int_{-\infty}^{\infty} \left| \frac{i\omega}{-m\omega^2 + ic\omega + k} \right|^2 S_0 \, d\omega.$$

- To result of this integral is:

$$\sigma_{\dot{y}^2} = \frac{\pi S_0}{mc}.$$

Example

- Finally substituting in $\nu_a^+ = \frac{1}{2\pi}\frac{\sigma_{\dot{y}}}{\sigma_y} e^{-a^2/2\sigma_y^2}$. gives:

$$\nu_a^+ = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\, e^{-a^2/(2\pi S_0/kc)}.$$

- The average frequency for the process is obtained by putting a=0 to give

$$\nu_0^+ = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{\omega_N}{2\pi}$$

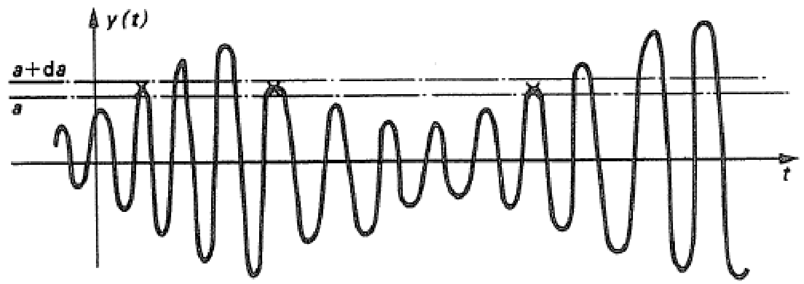
where ω_N is the natural frequency of the oscillator in rad/s.

Distribution of peaks

- Having obtained the frequency of crossings of y=a, it is not difficult to extend this calculation to determine the probability distribution of peaks. Let $p_p(a)\,\mathrm{d}a$ be the probability that the magnitude of a peak, chosen at random, lies in the range a to a+da. The probability that any peak is greater than a is therefore

$$\mathrm{Prob}\left(\text{Peak value exceeds } y = a\right) = \int_a^\infty p_p(a)\,\mathrm{d}a.$$

- Now in time T, we know that on average, there will be $\nu_0^+ T$ cycles (since one positive crossing of y=0 occurs for each full cycle of the narrow band process) of which only $\nu_a^+ T$ will have peak values exceeding y=a.



Identification of peaks in the band $y = a$ to $y = a+\mathrm{d}a$

Distribution of peaks

- The proportion of cycles whose peak value exceeds y=a is therefore $\frac{\nu_a^+}{\nu_0^+}$ and this must be the probability that any peak value , chosen at random exceeds y=a. Hence we obtain:

$$\int_a^\infty p_p(a)\,\mathrm{d}a = \frac{\nu_a^+}{\nu_0^+}$$

and this equation may be differentiated with respect to a to give:

$$-p_p(a) = \frac{1}{\nu_0^+}\frac{\mathrm{d}}{\mathrm{d}a}(\nu_a^+)$$

which is a general result for the probability density function for the occurrence of peaks. It applies for any narrow band process provided that this is a smooth process with each cycle crossing the mean level y=0 so that all the maxima occur above y=0 and all the minima occur below y=0.

Distribution of peaks

- The equation $-p_p(a) = \frac{1}{\nu_0^+}\frac{\mathrm{d}}{\mathrm{d}a}(\nu_a^+)$ applies for any probability distribution, but

 if y(t) is Gaussian, then there is a simple and important result for $p_p(a)$. Substituting

$$\nu_a^+ = \frac{1}{2\pi}\frac{\sigma_{\dot{y}}}{\sigma_y}\mathrm{e}^{-a^2/2\sigma_y{}^2}.$$

 into

$$-p_p(a) = \frac{1}{\nu_0^+}\frac{\mathrm{d}}{\mathrm{d}a}(\nu_a^+)$$

 gives:

$$-p_p(a) = \frac{\mathrm{d}}{\mathrm{d}a}(\mathrm{e}^{-a^2/2\sigma_y{}^2}) = -\frac{a}{\sigma_y^2}\mathrm{e}^{-a^2/2\sigma_y{}^2}$$

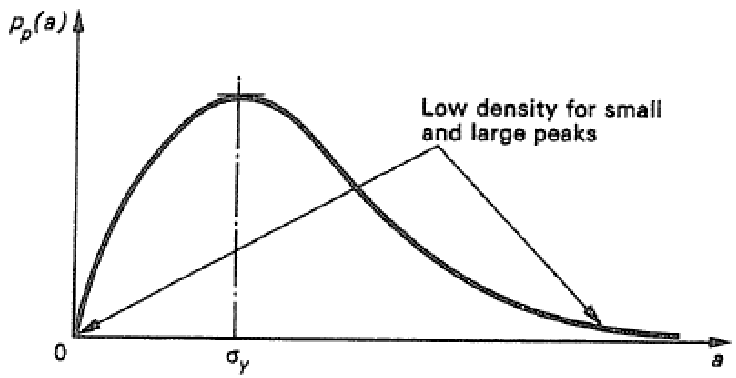
 or

$$p_p(a) = \frac{a}{\sigma_y^2}\mathrm{e}^{-a^2/2\sigma_y{}^2} \qquad 0 \leqslant a \leqslant \infty$$

which is the well known **Rayleigh distribution**.

Distribution of peaks

- The function pp(a) has its maximum value at a=σy, the standard deviation of the y process, and it is clear from the figure that the majority of the peaks have about this magnitude.



Rayleigh distribution of peaks for a Gaussian narrow band process

- The probability of finding very small or very large peaks is small and the probability that any peak, chosen at random, exceeds a is from $\int_a^\infty p_p(a)\,\mathrm{d}a = \frac{\nu_a^+}{\nu_0^+}$

$$\text{Prob}(\text{Peak value exceeds } a) = \mathrm{e}^{-a^2/2\sigma_y^2}.$$

Example

- Calculate the probability that any peak value of a Gaussian narrow band process y(t) exceeds $3\sigma_y$, From where $3\sigma_y$ is its standard deviation. From

$$\text{Prob(Peak value exceeds } a) = e^{-a^2/2\sigma_y^2}.$$

the required probability is

$$e^{-a^2/2\sigma_y^2} = e^{-4\cdot 5} = 0\cdot 011$$

so on average only about 1 peak in a 100 exceeds the $3\sigma_y$ level.

Frequency of maxima

- Our analysis of peaks leading to the Rayleigh distribution for a Gaussian process is based on the assumption that the narrow band process y(t) resembles a sine wave of varying amplitude and phase. We can investigate the validity of this assumption by calculating the distribution of the local maxima of y(t) by another approach. We know that y(t) is an extremum when dy/dt=0 and that this extremum is a maximum if, at the same time, d^2y/dt^2 is negative. Therefore the frequency of maxima of y(t) must be the frequency of the negative zero crossings of the derived process $\dot{y}(t)$ and, since there is one negative crossing for each positive crossing, this is the same as the frequency of positive zero crossings of $\dot{y}(t)$. Hence if μ_y is the frequency of maxima of y(t), and $\nu^+_{\dot{y}=0}$ is the frequency of zero crossings of $\dot{y}(t)$, we have

$$\mu_y = \nu^+_{\dot{y}=0}$$

where $\nu^+_{\dot{y}=0}$ can be calculated from $\nu_a^+ = \frac{1}{2\pi}\frac{\sigma_{\dot{y}}}{\sigma_y} e^{-a^2/2\sigma_y^2}$ by substituting

Frequency of maxima

$$\sigma_{\ddot{y}} \text{ for } \sigma_{\dot{y}} \text{ and } \sigma_{\dot{y}} \text{ for } \sigma_y$$

- And putting the level $\dot{y} = a = 0$ to obtain $\mu_y = \frac{1}{2\pi}\frac{\sigma_{\dot{y}}}{\sigma_{\ddot{y}}}$.
- This is a general expression for the frequency of maxima of the process y(t). For a theoretical narrow band process whose spectral density is shown in the figure and for which $\Delta\omega \ll \omega_0$, we have:

$$\sigma_{\dot{y}}^2 = E[\dot{y}^2] = \int_{-\infty}^{\infty} \omega^2 S_y(\omega)\,\mathrm{d}\omega \simeq 2S_0\omega_0^2\Delta\omega$$

and similarly,

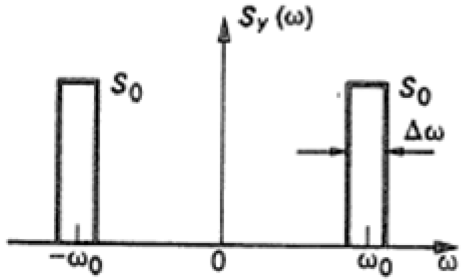
$$\sigma_{\ddot{y}}^2 = E[\ddot{y}^2] = \int_{-\infty}^{\infty} \omega^4 S_y(\omega)\,\mathrm{d}\omega \simeq 2S_0\omega_0^4\Delta\omega$$

in which case

$$\mu_y = \frac{1}{2\pi}\frac{\sigma_{\dot{y}}}{\sigma_{\ddot{y}}}.$$

gives:

$$\mu_y \simeq \frac{\omega_0}{2\pi}$$



for the frequency of maxima.

Frequency of maxima

- If the frequency of maxima μ_y is compared with $\nu_a^+ = \frac{1}{2\pi}\frac{\sigma_{\dot{y}}}{\sigma_y} e^{-a^2/2\sigma_y^2}$, we obtain:

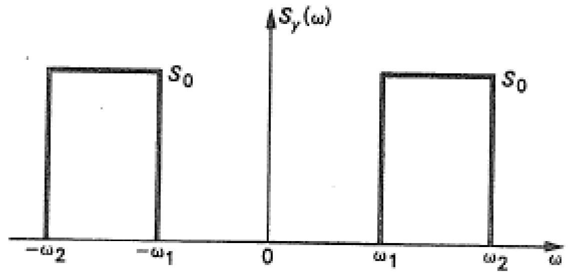
$$\nu_{y=0}^+ = \frac{1}{2\pi}\frac{\sigma_{\dot{y}}}{\sigma_y} \simeq \frac{\omega_0}{2\pi}$$

  for the frequency of zero crossings of y(t). The frequency of maxima and the frequency of zero crossings are therefore the same and our assumption that there is only one peak for each zero crossing is justified.
- If the bandwidth of the narrow band process is not narrow enough to assume $\Delta\omega \ll \omega_0$, this conclusion is however modified. Suppose that the spectral density of y(t) has the form shown in the figure. In this case, the variances of y, $\dot{y}$ and $\ddot{y}$ are:

$$\sigma_y^2 = E[y^2] = \int_{-\infty}^{\infty} S_y(\omega)\,d\omega = 2S_0(\omega_2 - \omega_1)$$

$$\sigma_{\dot{y}}^2 = E[\dot{y}^2] = \int_{-\infty}^{\infty} \omega^2 S_y(\omega)\,d\omega = \tfrac{2}{3}S_0(\omega_2^3 - \omega_1^3)$$

$$\sigma_{\ddot{y}}^2 = E[\ddot{y}^2] = \int_{-\infty}^{\infty} \omega^4 S_y(\omega)\,d\omega = \tfrac{2}{5}S_0(\omega_2^5 - \omega_1^5)$$



Spectral density of a theoretical band limited process

Frequency of maxima

- The frequency of maxima is now:

$$\mu_y = \frac{1}{2\pi}\frac{\sigma_{\ddot{y}}}{\sigma_{\dot{y}}} = \frac{1}{2\pi}\sqrt{\left\{\frac{3(\omega_2^5 - \omega_1^5)}{5(\omega_2^3 - \omega_1^3)}\right\}}$$

compared with the frequency of zero crossings which is:

$$\nu_{y=0}^{+} = \frac{1}{2\pi}\frac{\sigma_{\dot{y}}}{\sigma_y} = \frac{1}{2\pi}\sqrt{\left\{\frac{\omega_2^3 - \omega_1^3}{3(\omega_2 - \omega_1)}\right\}}$$

and these clearly are not the same.

Example

- Calculate the frequency of maxima and the frequency of zero crossings for a Gaussian process whose spectrum is flat and covers an octave bandwidth from $\omega_1/2\pi$=70.7 Hz (c/s) to $\omega_2/2\pi$=141.4 Hz (c/s) , as shown in the figure, with a center frequency of 100Hz. Note that for an octave bandwidth the upper cut-off frequency is twice the lower cut-off frequency. Substituting numbers into

$$\mu_y = \frac{1}{2\pi}\frac{\sigma_{\ddot{y}}}{\sigma_{\dot{y}}} = \frac{1}{2\pi}\sqrt{\left\{\frac{3(\omega_2^5 - \omega_1^5)}{5(\omega_2^3 - \omega_1^3)}\right\}}$$

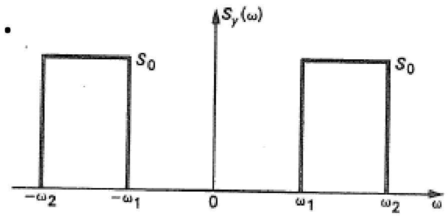
gives μ_y=115 Hz and into

$$\nu_{y=0}^{+} = \frac{1}{2\pi}\frac{\sigma_{\dot{y}}}{\sigma_y} = \frac{1}{2\pi}\sqrt{\left\{\frac{\omega_2^3 - \omega_1^3}{3(\omega_2 - \omega_1)}\right\}}$$

gives

$$\nu_{y=0}^{+} = 108 \text{ Hz}$$

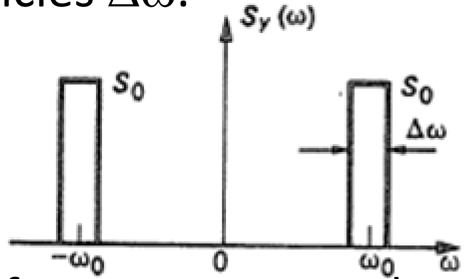
approximately. There are therefore about 6.5 percent more local maxima than there are zero crossings.



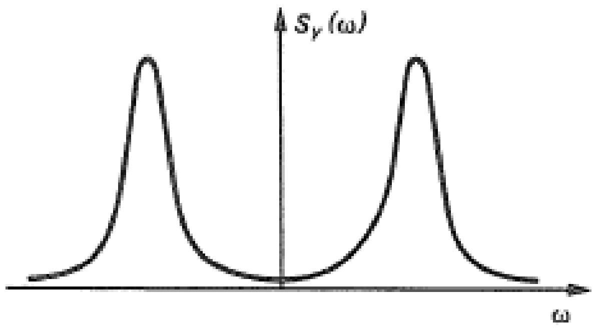
Spectral density of a theoretical band limited process

Frequency of maxima

- The reason for the difference is due to the fact that y(t) can only be represented by a sine wave of slowly varying amplitude and phase if its spectrum has the form shown in the figure and includes only a very narrow band of frequencies $\Delta\omega$.

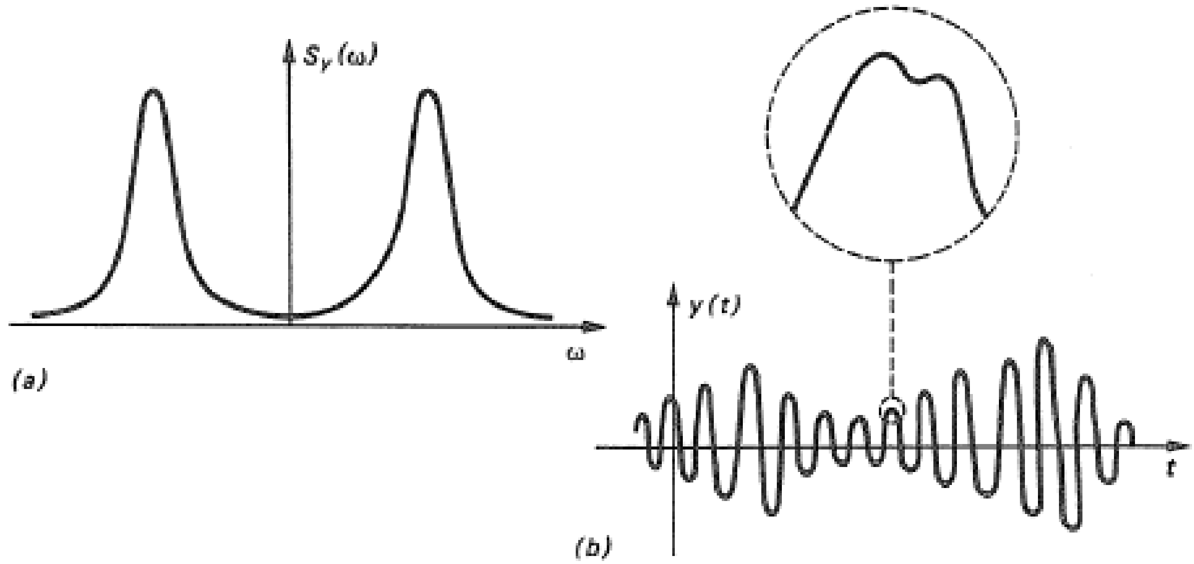


- It has been shown that for a more general narrow-band spectrum, as shown in the figure below, the high frequency components present introduce irregularities in the smooth form of the sine wave approximation and it is these irregularities which cause the additional maxima.



Frequency of maxima

- In many applications, we are concerned only with the large amplitude excursions of a narrow band process and the Rayleigh distribution of peaks, which assumes only one maximum for each zero crossing, is then a valuable guide to the probability of occurrence of large peak values



Illustrating how local irregularities in a narrow band process give more than one maximum per zero crossing.

Accuracy of measurements

Accuracy of measurements

- So far, we have been concerned with the basic mathematical theory of random processs analysis. We turn now to a more practical aspect of the subject: experimental measurements. We shall concentrate almost entirely on measuring the **spectral density** of a random process or the **cross-spectral density** between two random processes. This emphasis on spectral measurements is justified by the central role which spectra occupy in the theory of random vibrations. Their importance comes from the simple form of the input-output relations for spectral density for a linear system subjected to random excitation. In this chapter, we shall describe the operation of an analogue spectrum analyser and discuss at length the factors which affect the accuracy of any measurement of spectral density.

- The fundamental experimental problem is that our measurements must be made on one sample function of a theoretically infinite ensemble, or at the most, on only a few sample functions from the infinite ensemble.

Accuracy of measurements

- Furthermore, we are only able to analyze a limited length of any given sample function because we can not go on taking measurements for ever.
- The fact that we are limited to a single sample and that we can only analyze a finite length of it means that, automatically, we shall introduce errors into the measured spectrum.
- Even assuming that the random process we are sudying is ergodic, in which any one sample function completely represents the infinity of functions which make up the ensemble, we are still introducing errors when we only deal with a finite length of a sample function.
- Remember that a sample average for an ergodic process is only the same as an ensemble average when the sample averaging time is infinite. Obviously, this is not a practical proposition.

Accuracy of measurements

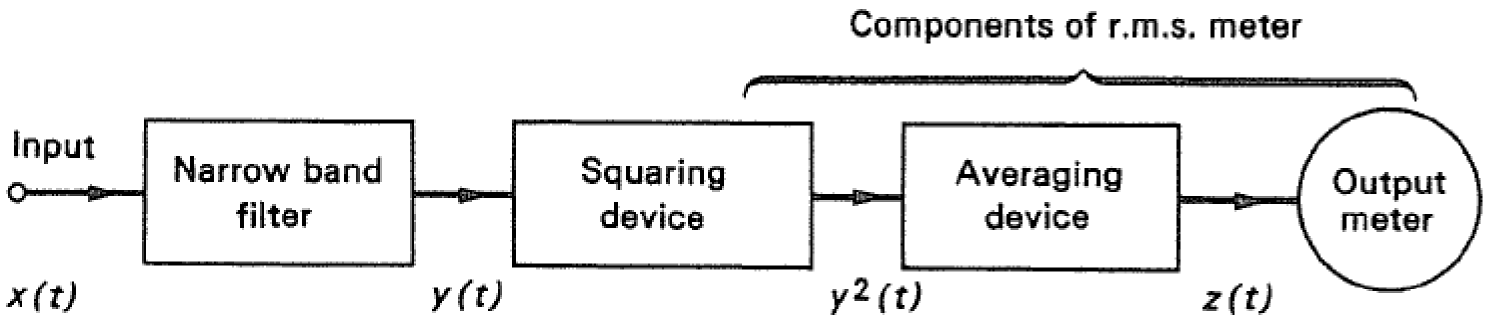
- If we follow the mathematical definition of spectral density, we must begin by measuring the autocorrelation for the process being studied and then devise a way of calculating the Fourier transform of the autocorrelation function. However, in practice, this is not the best way to proceed.

- It turns out that it is easier to measure spectra by a procedure which does not involve first calculating correlation functions. Although the experimental procedure follows a route which is not mathematically rigorous, and therefore cannot be used to define the spectral density functions, we shall see that we can obtain approximations for the true spectra which are correct to any stated accuracy; furthermore the measurements are considerably simpler and quicker than they would otherwise be.

Analogue spectrum analysis

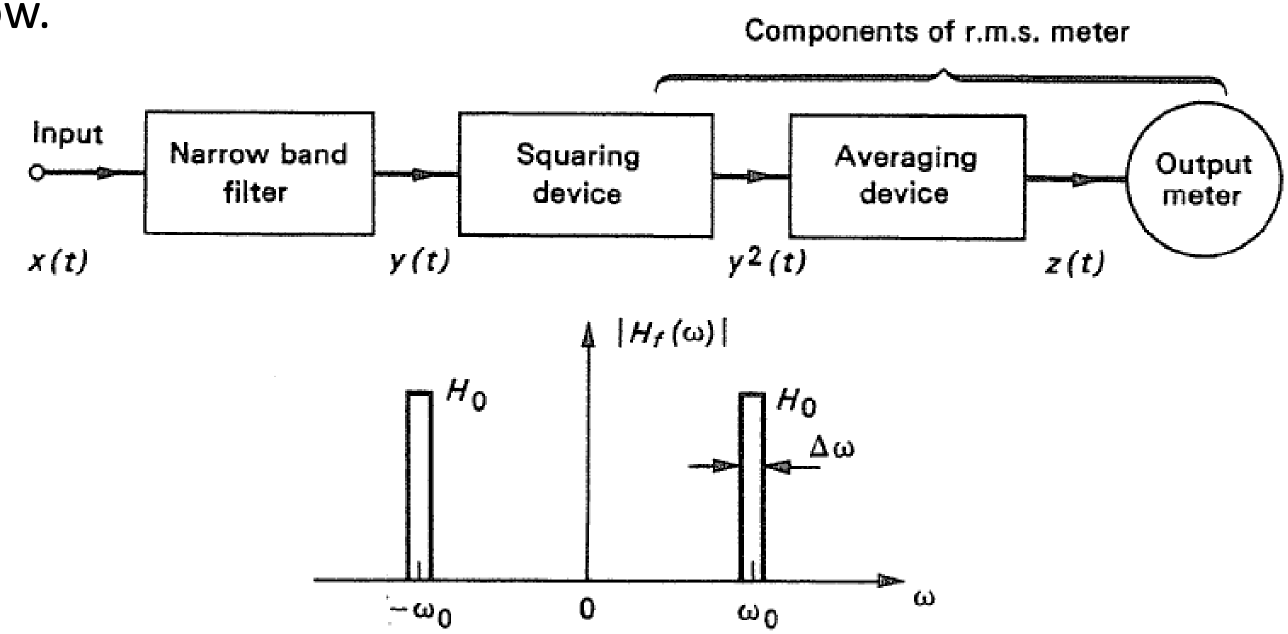
- Most engineers will be familiar with an instrument called a frequency analyzer. The output of an accelerometer, or other vibration transducer, is fed into the instrument which is essentially a variable frequency narrow-band filter with an r.m.s. meter to display the filter output.

- Usually, the filter center frequency is continuously variable and the experimenter adjusts this as he searches for the predominant frequencies present in a vibration signal. An analogue spectrum analyzer is a similar instrument except that it has more accurate filters and precisely calibrated bandwidths.



Analogue spectrum analysis

- Suppose that the input x(t) is a sample function of an ergodic (and therefore stationary) random process. The signal is filtered by a filter whose theoretical frequency response is shown in the second figure below.



Schematic of a spectrum analyser showing the theoretical filter frequency response

Analogue spectrum analysis

- The filter output y(t) is squared and then the time average z(t) calculated where

$$z(t) = \frac{1}{T}\int_0^T y^2(t)\,\mathrm{d}t.$$

- Since the averaging time T can not be infinite, z(t) is itself a function of time, and fluctuates about its true mean value (the ensemble average). However, if T is long enough, the fluctuations are small and the mean level E[z] can be approximately determined from the analyzer's output meter. From the above equation, we know that

$$E[z] = \frac{1}{T}\int_0^T E[y^2]\,\mathrm{d}t = E[y^2]$$

since y(t) is stationary, and from

$$E\left[y^2\right] = \int_{-\infty}^{\infty} \left|H(\omega)\right|^2 S_x(\omega)\,d\omega$$

Analogue spectrum analysis

$$E[y^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_x(\omega)\,d\omega$$

- Which for the filter frequency response shown in the figure, with $\Delta\omega \ll \omega_0$ can be approximated by $E[y^2] \simeq 2H_0^2 \Delta\omega\, S_x(\omega_0)$.
- Combining the above equation with

$$E[z] = \frac{1}{T}\int_0^T E[y^2]\,dt = E[y^2]$$

the average output of the spectrum analyser is proportional to the spectral density of the input process at the filter center frequency ω_o, or turning the formula round

$$S_x(\omega_0) \simeq \frac{E[z]}{2H_0^2\,\Delta\omega}.$$

The **mean output level E[z]** is therefore a direct measure of the **input spectral density**.

Analogue spectrum analysis

- We shall now investigate the accuracy of this measurement. It is clear that we are likely to improve the accuracy if we use an instrument with a long averaging time, because then the output depends on an average integrated over a long period of time as shown.

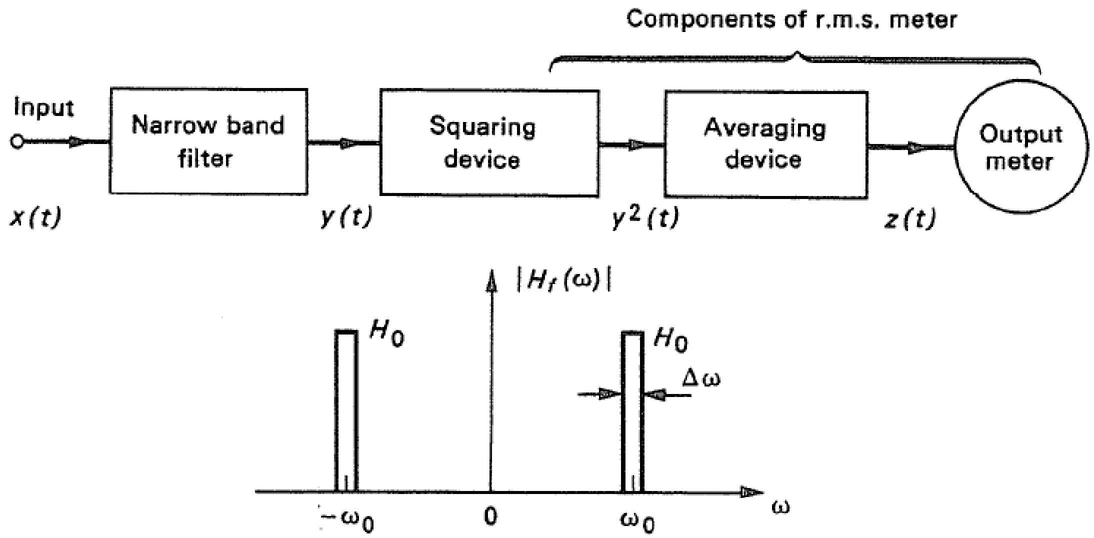
$$z(t) = \frac{1}{T}\int_0^T y^2(t)\,\mathrm{d}t.$$

- Also, if we want to distinguish sharp peaks in the curve of spectral density against frequency, we should use a sharp filter with a very narrow bandwidth $\Delta\omega$. In the next section, we shall find an expression which relates the accuracy of measurements to both these quantities.

Variance of the measurement

- From $S_x(\omega_0) \simeq \dfrac{E[z]}{2H_0^2\,\Delta\omega}$, the spectral density $S_x(\omega)$ can be determined if

  H_o, $\Delta\omega$ and E[z] are all known. We can determine the first two to any desired accuracy by using precisely calibrated narrow band filters.



Schematic of a spectrum analyser showing the theoretical filter frequency response

Variance of the measurement

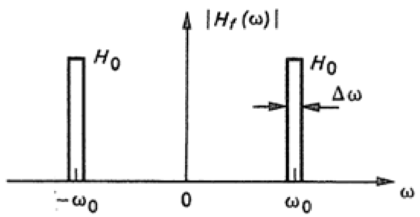
- However, E[z] cannot be precisely determined since it is an ensemble average and therefore not obtainable from measurements of finite length on a single sample. All we can do is try to make sure that z(t) never differs very much from its mean value E[z] so that a spot value of z(T) is likely to be a close approximation for E[z]. The variance of z(t) is a measure of the magnitude of its fluctuations about the mean and we define,

$$\sigma^2 = E[z^2] - (E[z])^2$$

as the variance of the measurement according to

$$\sigma^2 = E\left[x^2\right] - \left(E[x]\right)^2$$

- We shall now seek to determine σ^2. Clearly, this will depend on the characteristics of the y(T) random process, which is the output of the spectrum analyzer's narrow band filter since z(t) is a function of y(t).



$$z(t) = \frac{1}{T}\int_0^T y^2(t)\,\mathrm{d}t.$$

Variance of the measurement

- We begin by substituting for z in terms of y in

$$\sigma^2 = E[z^2] - (E[z])^2$$

- Using

$$E[z] = \frac{1}{T}\int_0^T E[y^2]\,\mathrm{d}t = E[y^2]$$

the E[z] term can be replaced by E[y^2] to obtain:

$$\sigma^2 = E[z^2] - (E[y^2])^2$$

but the E[z^2] term is more difficult. Returning to the definition of z(t),

$$z(t) = \frac{1}{T}\int_0^T y^2(t)\,\mathrm{d}t.$$

we can write:

$$z^2 = \left\{\frac{1}{T}\int_0^T y^2(t)\,\mathrm{d}t\right\}^2 = \left\{\frac{1}{T}\int_0^T y^2(t_1)\,\mathrm{d}t_1\right\}\left\{\frac{1}{T}\int_0^T y^2(t_2)\,\mathrm{d}t_2\right\}$$

where the two different time variables t_1 and t_2 are introduced so that

Variance of the measurement

- This product of two integrals may be written as the equivalent double integral

$$z^2 = \frac{1}{T^2}\int_0^T \mathrm{d}t_1 \int_0^T \mathrm{d}t_2\, y^2(t_1)y^2(t_2).$$

- Averaging the above equation for the ensemble then gives:

$$E[z^2] = \frac{1}{T^2}\int_0^T \mathrm{d}t_1 \int_0^T \mathrm{d}t_2\, E[y^2(t_1)y^2(t_2)]$$

which substituting into

$$\sigma^2 = E[z^2] - (E[y^2])^2$$

gives the following expression for the variance of the measurement σ^2.

$$\sigma^2 = \left\{\frac{1}{T^2}\int_0^T \mathrm{d}t_1 \int_0^T \mathrm{d}t_2\, E[y^2(t_1)y^2(t_2)]\right\} - (E[y^2])^2.$$

Variance of the measurement

- In order to use $\sigma^2 = \left\{ \frac{1}{T^2} \int_0^T \mathrm{d}t_1 \int_0^T \mathrm{d}t_2 \, E[y^2(t_1) y^2(t_2)] \right\} - (E[y^2])^2$.

  we first relate the fourth order average $E[y^2(t_1)y^2(t_2)]$ to the autocorrelation function for the y process Ry(τ) . This is easy to do if y(T) is a Gaussian process. Fortunately, this is likely to be a fair approximation since y(t) is the output from a narrow band filter and so we may expect its probability distribution to approach a Gaussian distribution when the input to the filter is broad band noise. For a Gaussian process with zero mean, the fourth order average $E[y_1y_2y_3y_4]$ can be expressed in terms of second order averages by the following equation

$$E[y_1 y_2 y_3 y_4] = E[y_1 y_2] \cdot E[y_3 y_4] + E[y_2 y_3] \cdot E[y_4 y_1] + E[y_1 y_3] \cdot E[y_2 y_4]$$

which for $y_3=y_1$ and $y_4=y_2$ simplifies to $E[y_1^2 y_2^2] = 2(E[y_1 y_2])^2 + (E[y^2])^2$

$$\text{if } E[y_1^2] = E[y_2^2] = E[y^2].$$

Variance of the measurement

- If now we put $y_1 = y(t_1)$ and $y_2 = y(t_2)$, we obtain:

$$E[y^2(t_1)y^2(t_2)] = 2(E[y(t_1)y(t_2)])^2 + (E[y^2])^2$$

in which, since y(t) is a stationary process,

$$E[y(t_1)y(t_2)] = R_y(t_2 - t_1)$$

and so finally for a Gaussian process,

$$E[y^2(t_1)y^2(t_2)] = 2R_y^2(t_2 - t_1) + (E[y^2])^2.$$

- Substituting

$$S_x(\omega_0) \simeq \frac{E[z]}{2H_0^2\,\Delta\omega}.$$

$$\sigma^2 = \left\{\frac{1}{T^2}\int_0^T \mathrm{d}t_1 \int_0^T \mathrm{d}t_2\, E[y^2(t_1)y^2(t_2)]\right\} - (E[y^2])^2.$$

gives the following expression for the variance of the measurement, σ^2

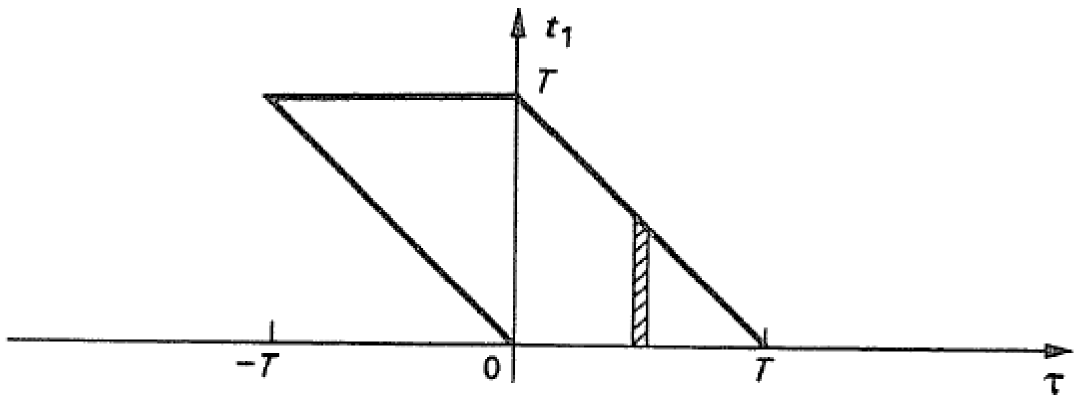
$$\sigma^2 = \frac{2}{T^2}\int_0^T \mathrm{d}t_1 \int_0^T \mathrm{d}t_2\, R_y^2(t_2 - t_1).$$

Variance of the measurement

- For a stationary input, the autocorrelation function depends only on the time difference $\tau = t_2 - t_1$, so changing one of the variables t_2 to $\tau + t_1$ (where t_1 is a constant for the integration with respect to τ) we obtain:

$$\sigma^2 = \frac{2}{T^2} \int_0^T \mathrm{d}t_1 \int_{-t_1}^{T-t_1} \mathrm{d}\tau \, R_y^2(\tau).$$

- Since the integrand $R_y^2(\tau)$ is a function of only one of the two variables of integration, we can integrate immediately with respect to the other variable. However, the limits of integration require some thought. The range of values of τ and t_1 covered by the double integral is shown in the figure:



Variance of the measurement

- Integrating with respect to t, with τ constant. i.e. Along the shaded strip in the figure gives: $R_y^2(\tau)(T-|\tau|)$ for $-T \leqslant \tau \leqslant T$

  and then integrating with respect to the other variable over its full range from –T to +T, we obtain:

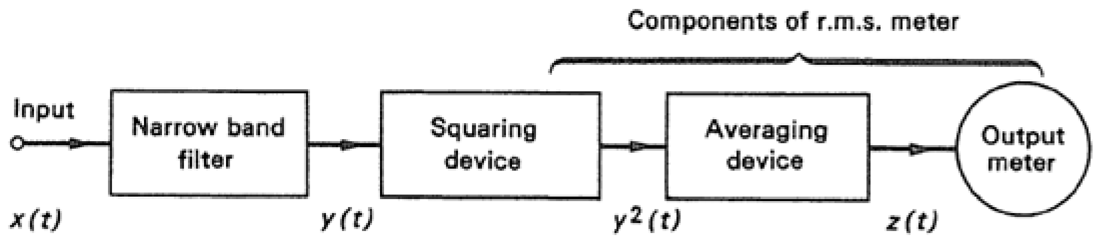
$$\sigma^2 = \frac{2}{T}\int_{-T}^{T} R_y^2(\tau)\left(1-\frac{|\tau|}{T}\right)\mathrm{d}\tau.$$

- In order to evaluate this integral, we must introduce an expression for the autocorrelation function $R_y(\tau)$. Since y(t) is the output of a narrow band filter, as shown in the figure, its autocorrelation function will have the form

$$R_y(\tau) = 4S_0 \frac{\sin(\Delta\omega\tau/2)}{\tau}\cos\omega_0\tau$$

  which may be written:

$$R_y(\tau) = R_y(0)\left\{\frac{\sin(\Delta\omega\ \tau/2)}{(\Delta\omega\ \tau/2)}\right\}\cos\omega_0\tau.$$



Variance of the measurement

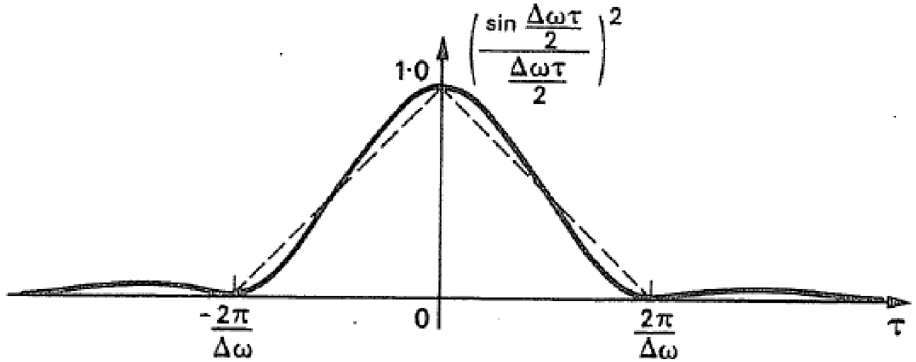
- We now want to substitute $R_y(\tau) = R_y(0)\left\{\frac{\sin(\Delta\omega \ \tau/2)}{(\Delta\omega \ \tau/2)}\right\}\cos\omega_0\tau.$ into

$$\sigma^2 = \frac{2}{T}\int_{-T}^{T} R_y^2(\tau)\left(1 - \frac{|\tau|}{T}\right)d\tau.$$

and then evaluate the integral to obtain the variance σ^2 . Fortunately, we are only looking for the order of the magnitude of σ^2 rather than an exact value, and it is sufficient to integrate only approximately. First we note that the first equation includes the quotient

$$\left\{\frac{\sin(\Delta\omega \ \tau/2)}{(\Delta\omega \ \tau/2)}\right\}$$

which when squared and plotted as a function of τ, has the form shown in the figure:

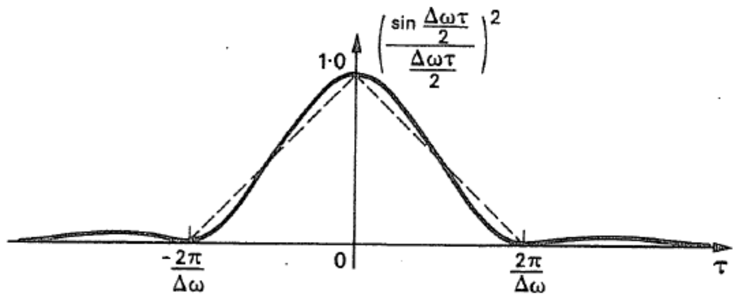


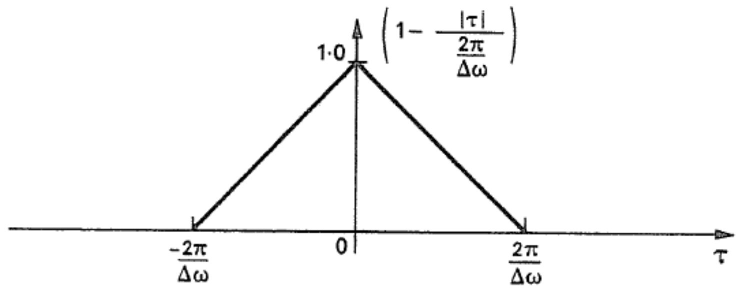
Variance of the measurement

- This may be approximated quite closely by the straight line function as shown in the figures

$$\left\{1 - \frac{|\tau|}{(2\pi/\Delta\omega)}\right\} \quad \text{for} \quad |\tau| \leqslant \frac{2\pi}{\Delta\omega}$$

$$0 \quad \text{for} \quad |\tau| > \frac{2\pi}{\Delta\omega}$$





Variance of the measurement

- With this assumption, $\sigma^2 = \frac{2}{T}\int_{-T}^{T} R_y^2(\tau)\left(1 - \frac{|\tau|}{T}\right)\mathrm{d}\tau.$ becomes, after substituting for $R_y(\tau)$

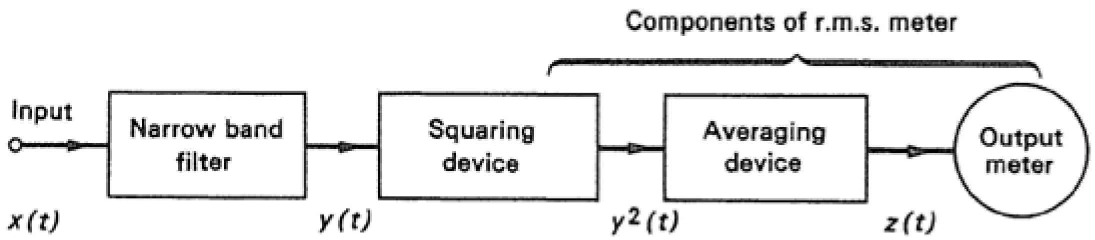
$$\sigma^2 \simeq \frac{2}{T}\int_{-2\pi/\Delta\omega}^{2\pi/\Delta\omega} R_y^2(0)\left\{1 - \frac{|\tau|}{(2\pi/\Delta\omega)}\right\}\cos^2\omega_0\tau\left(1 - \frac{|\tau|}{T}\right)\mathrm{d}\tau.$$

- Secondly, we shall assume that the averaging time T of the r.m.s. meter in figure is long so that

$$T \gg \frac{2\pi}{\Delta\omega}$$

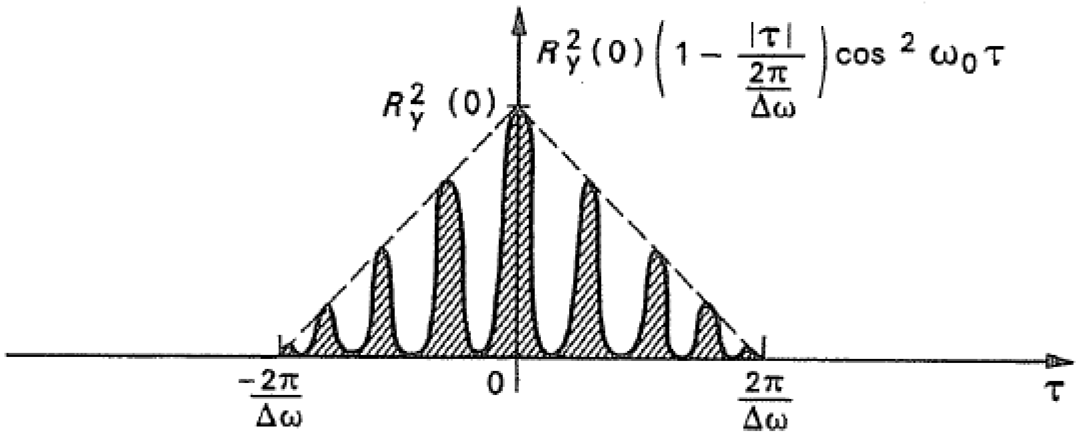
in which case we can make a further simplification of the integrand to obtain

$$\sigma^2 \simeq \frac{2}{T}\int_{-2\pi/\Delta\omega}^{2\pi/\Delta\omega} R_y^2(0)\left\{1 - \frac{|\tau|}{(2\pi/\Delta\omega)}\right\}\cos^2\omega_0\tau\,\mathrm{d}\tau.$$



Variance of the measurement

- The integral is represented by the shaded area shown in the figure. Provided that the bandwidth $\Delta\omega$ is small compared with the center frequency of the filter ω_0, then there will be many cycles of $\cos^2\omega_0\tau$ inside the triangular envelope of the figure, and in this case, the shaded area is equal to half the total area enclosed by the dotted triangle in figure.



- Hence the integral is given approximately by: $R_y^2(0)\cdot\left(\frac{2\pi}{\Delta\omega}\right)\cdot\frac{1}{2}$

  and so we arrive to the approximate result: $\sigma^2 \simeq \frac{2\pi}{T\Delta\omega}R_y^2(0)$

Variance of the measurement

- Putting the symbol B to denote the bandwidth in Hz, so that

$$B = \frac{\Delta\omega}{2\pi} \qquad \sigma^2 \simeq \frac{1}{BT} R_y^2(0)$$

- This is the simplified version of

$$\sigma^2 = \left\{ \frac{1}{T^2} \int_0^T \mathrm{d}t_1 \int_0^T \mathrm{d}t_2 \, E[y^2(t_1) y^2(t_2)] \right\} - (E[y^2])^2$$

we have been looking for. It gives the variance of the output of a spectrum analyzer. Since the correct output should be from

$$\sigma^2 = E[z^2] - (E[z])^2$$

$$E[z] = E[y^2] = R_y(0) = m \qquad \text{(say)}$$

where m denotes the mean output of the analyzer.

Variance of the measurement

- The equation $\sigma^2 \simeq \frac{1}{BT} R_y^2(0)$ may finally be written as: $\frac{\sigma}{m} \simeq \frac{1}{\sqrt{BT}}$ and it is clear that if the standard deviation of the measurement is to be small, we must have: $BT \gg 1$
  which confirms the approximation $T \gg \frac{2\pi}{\Delta\omega}$ in evaluating the integrals.
- We are now in a position to appreciate the basic dilemma which arises when measuring spectral density. For high resolution, the filter bandwidth of the analyzer, B, must be small, while for good statistical reliability B must be large compared with 1/T. This can only be achieved if the averaging time T is long, which means that the sample function x(t) must last for a long time and that the experimenter must be prepared to wait a sufficiently long time to achieve an accurate result.

Example

Example

Determine the standard deviation of the measurement of spectral density by an analyser with a $\frac{1}{3}$-octave bandwidth and (a) a 1 s averaging time and (b) a 10 s averaging time when the centre frequency is (i) 10 Hz, (ii) 100 Hz and (iii) 1000 Hz.

We must first calculate the bandwidth in the three cases. Let f_1 be the lower cut-off frequency. Then for a $\frac{1}{3}$-octave filter the upper cut-off frequency is

$$\sqrt[3]{2}\, f_1 = 1{\cdot}26\, f_1.$$

Also, if the centre frequency is f_0

$$\frac{f_0}{f_1} = \frac{1{\cdot}26\, f_1}{f_0}$$

so that

$$f_0 = 1{\cdot}12\, f_1$$

and the bandwidth is

$$0{\cdot}26\, f_1 = 0{\cdot}23\, f_0.$$

Hence the filter bandwidths are (i) 2·3 Hz, (ii) 23 Hz and (iii) 230 Hz approximately.

The ratio of standard deviation to mean output, σ/m, can now be tabulated by putting numbers into (9.22).

Example

	(a) Averaging time $T = 1$ s	(b) Averaging time $T = 10$ s
Case (i) $f_0 = 10$ Hz	$\frac{\sigma}{m} = 0{\cdot}66$	$\frac{\sigma}{m} = 0{\cdot}21$
Case (ii) $f_0 = 100$ Hz	$\frac{\sigma}{m} = 0{\cdot}21$	$\frac{\sigma}{m} = 0{\cdot}07$
Case (iii) $f_0 = 1000$ Hz	$\frac{\sigma}{m} = 0{\cdot}07$	$\frac{\sigma}{m} = 0{\cdot}02$

This means that, when an experimenter makes a spot measurement of spectral density, his result is subject to an error on account of the fluctuations of the output meter. The standard deviation of the measurement (and therefore of the error) is expressed above as a ratio of the mean output level m.

Before leaving this subject it should be mentioned that the output of a spectral analyser will also be susceptible to a steady state or *bias error* when the filter bandwidth covers a range of frequencies in which the spectral density is changing rapidly with frequency. The mean output of the analyser is, from (7.18),

$$E[y^2] = 2\int_{\omega_0-\Delta\omega/2}^{\omega_0+\Delta\omega/2} H_0^2 S_x(\omega)\,\mathrm{d}\omega$$

and our assumption in (9.3) that this may be written

$$E[y^2] \simeq 2H_0^2 S_x(\omega_0)\,\Delta\omega$$

is of course not accurate if $S_x(\omega)$ is changing fast with frequency in the band $\Delta\omega$. The only practical solution to this problem is to achieve greater resolution by employing an analyser with narrower band filters (provided that it is possible to use a long enough averaging time to keep the variance of the measurement acceptable).

Analysis of finite length records

- We have seen how the standard deviation of a single measurement of spectral density is affected by the bandwidth B and averaging time T of an analogue spectrum analyser and, to a good approximation, the ratio of the standard deviation σ to its mean value m is independent of the variance of the input signal and depends only on B and T, according to the fundamental result

$$\frac{\sigma}{m} \simeq \frac{1}{\sqrt{BT}}$$

- If the instrument's averaging time is T, its output at any instant is based only on the input values for the immediately preceding time interval T,

$$z(t) = \frac{1}{T}\int_0^T y^2(t)\,\mathrm{d}t.$$

- The basic problem is that it is just not possible to calculate the precise estimate of spectral density when only a limited length of data is available for analysis.

Analysis of finite length records

- Now, with an analogue spectrum analyzer, operating on a continuous random process, we can go on taking instantaneous measurements of spectral density and by watching the movement of the instrument’s output meter, quickly appreciate the variability of the output.

- Usually, an analogue instrument has two or more alternative averaging times and several different bandwidths, and by altering the settings of the instrument, it is soon possible to judge how the instrument has to be adjusted to give reasonable accuracy with an acceptable bandwidth, and therefore adequate resolution of close spectral peaks.

- However, analogue spectrum analysis takes time, since many separate readings have to be taken to cover a wide range of frequencies; also the maximum resolution obtainable from analogue filters is limited.

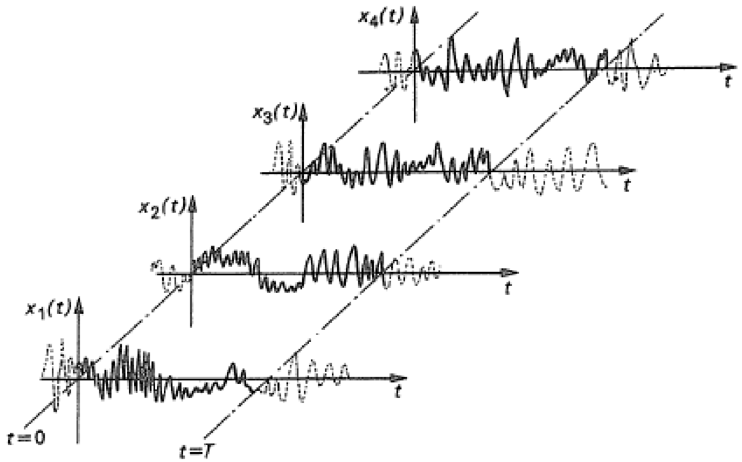
- Digital data analysis metods are therefore extensively used and virtually all random data analysis except for so-called “quick-look” spectrum analysis is now carried out digitally.

Analysis of finite length records

- The sample function x(t) is first digitized by an analogue-to-digital converter and then a digital computer is used to make calculations on the digitized data.
- Since there is a limit to the number of data points that can be fed into a computer, there is a limit to the length of the sample function that can be analyzed.
- This restricted length of sample causes the same loss of precision as that occurring in an analogue instrument with a finite averaging time.
- It turns out that accuracy still depends on record length T and bandwidth B, except that the bandwidth is no longer that of an analogue filter, but has to be interpreted in a different way.
- We shall now consider the general problem of analysing a record of finite length and introduce the concept of a spectral window which is used to define the equivalent bandwidth B_e of a digital calculation.

Analysis of finite length records

- Suppose that {x(t)} is a stationary random process consisting of an ensemble of sample functions. Since the process is stationary, each sample function theoretically goes forever. However, suppose that records are only available for the period t=0 to t=T, as shown in the figure. In this case, the autocorrelation function $R_x(\tau) = E[x(t)x(t+\tau)]$ can only be determined for $|\tau| \leq$ T. We can not therefore calculate the corresponding spectral density from $S_x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau)\,\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\tau$ since we do not know $R_x(\tau)$ for $|\tau| > T$.



Finite length records of duration T from the stationary random process $\{x(t)\}$

Analysis of finite length records

- The best we can do is to approximate $S_x(\omega)$ by truncating the integral to give the approximation $$S_x(\omega) \simeq \frac{1}{2\pi}\int_{-T}^{T} R_x(\tau)\,e^{-i\omega\tau}\,d\tau.$$
- Although as already mentioned, the calculation procedure carried out in a computer may not actually involve finding the autocorrelation function $R_x(\tau)$, nevertheless the basic difficulty which arises is the inherent loss of accuracy resulting from an approximation equivalent to the above equation.
- To illustrate how the approximate spectrum $S_x(\omega) \simeq \frac{1}{2\pi}\int_{-T}^{T} R_x(\tau)\,e^{-i\omega\tau}\,d\tau.$ is likely to differ from its true value calculated from

$$S_x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau)\,e^{-i\omega\tau}\,d\tau$$

suppose that $x(t) = a\sin(\omega_0 t + \phi)$

where the amplitude a and the frequency ω_o are constant, and the phase angle is constant for each sample function, but varies randomly from

Analysis of finite length records

sample to sample with all values between 0 and 2π being equally likely. In this case,

$$R_x(\tau) = E[x(t)x(t+\tau)] = \int_0^{2\pi} a^2 \sin(\omega_0 t + \phi)\sin(\omega_0 t + \omega_0 \tau + \phi)p(\phi)\,d\phi$$

which putting $p(\phi)=1/2\pi$, gives:

$$R_x(\tau) = \frac{a^2}{2}\cos\omega_0\tau.$$

- Now suppose that the records x(t) are only defined for t=0 to t=T, so that we only know $R_x(\tau)$ for $|\tau| \leqslant T$, and consider calculating the approximation for $R_x(\tau)$ in

$$S_x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau)\,e^{-i\omega\tau}\,d\tau$$

gives:

$$S_x(\omega) \simeq \frac{1}{2\pi}\int_{-T}^{T} \frac{a^2}{2}\cos\omega_0\tau\, e^{-i\omega\tau}\,d\tau$$
$$= \frac{1}{2\pi}\int_0^T a^2 \cos\omega_0\tau\cos\omega\tau\,d\tau$$

since $\cos\omega_0 t$ is an even function of τ, this may in turn be written as:

$$S_x(\omega) \simeq \frac{a^2}{4\pi}\int_0^T \{\cos(\omega-\omega_0)\tau + \cos(\omega+\omega_0)\tau\}\,d\tau$$

Analysis of finite length records

- Which can be easily integrated to give: $S_x(\omega) \simeq \frac{a^2}{4\pi}\left\{\frac{\sin(\omega - \omega_0)T}{\omega - \omega_0} + \frac{\sin(\omega + \omega_0)T}{\omega + \omega_0}\right\}.$

- Notice that the terms on the r.h.s. of the above equation have a form similar to those in equation

$$R_x(\tau) = \frac{4S_o}{\tau}\cos\frac{\omega_2\tau}{2}\sin\frac{\omega_2\tau}{2}$$

which has been shown to become a delta function, and so the approximation for $S_x(\omega)$ given by

$$S_x(\omega) \simeq \frac{a^2}{4\pi}\left\{\frac{\sin(\omega - \omega_0)T}{\omega - \omega_0} + \frac{\sin(\omega + \omega_0)T}{\omega + \omega_0}\right\}.$$

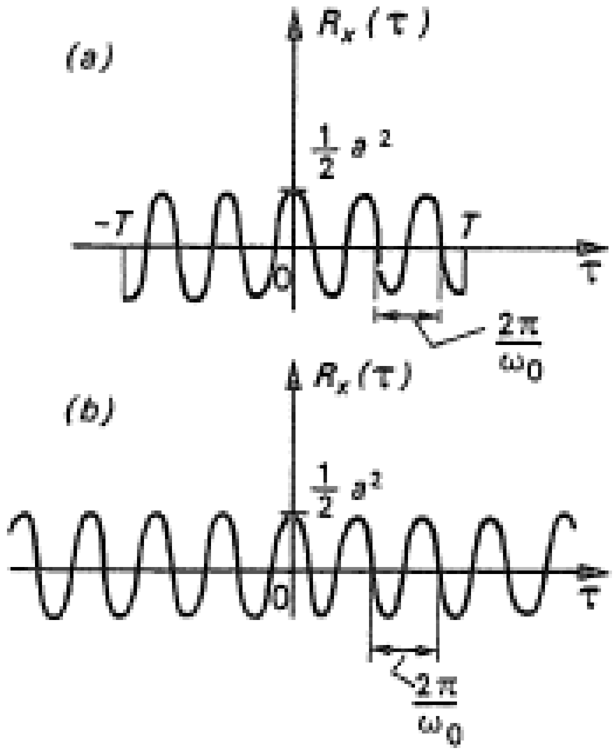
tends to two delta functions in the limit when the length of the record, T, becomes infinite.

Analysis of finite length records

- In the Figure a, the approximate result from the equation

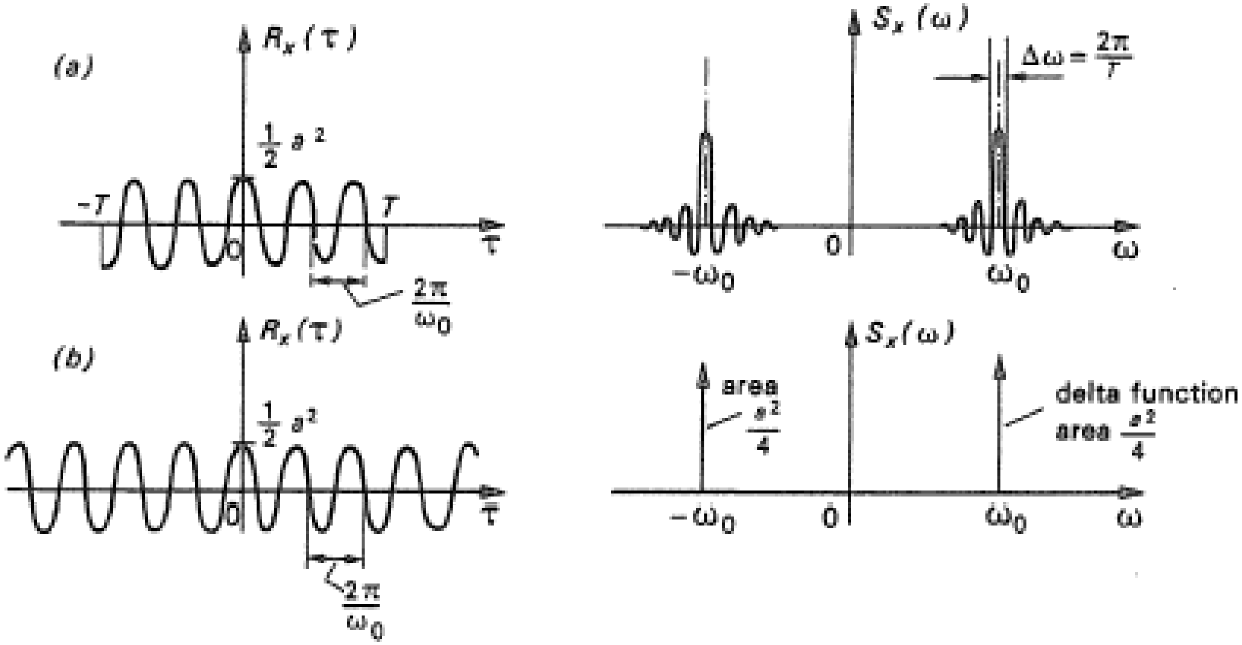
$$S_x(\omega) \simeq \frac{a^2}{4\pi}\left\{\frac{\sin(\omega-\omega_0)T}{\omega-\omega_0}+\frac{\sin(\omega+\omega_0)T}{\omega+\omega_0}\right\}.$$

  for T finite is shown compared with figure (b) , the exact result when $T \rightarrow \infty$. The conclusion we can draw from this is that the result of analysing only a finite length of record is to smear out a sharp spectral line over a band of frequencies of width $\Delta\omega=2\pi/T$ approximately. In order to resolve two nearby spectral peaks, the length of record T, must be long enough for their frequency difference to be large compared with 1/T Hz.



Analysis of finite length records

- However accurate the analogue to digital conversion and however large the computer, close spectral peaks can only be distinguished if the record length T is long enough.



Fourier transform of a finite length of a cosine wave compared with the transform of an infinite length

Analysis of finite length records

A further limitation on the accuracy of numerical calculations can be seen by considering the fluctuations of $S(\omega)$ on either side of $\omega = \omega_0$ in Fig. 9.4(*a*). These fluctuations remain as the length of the record increases although their frequency increases as shown in Fig. 9.5(*a*) (which is drawn for positive frequencies only). One way of removing them, or at least of reducing their

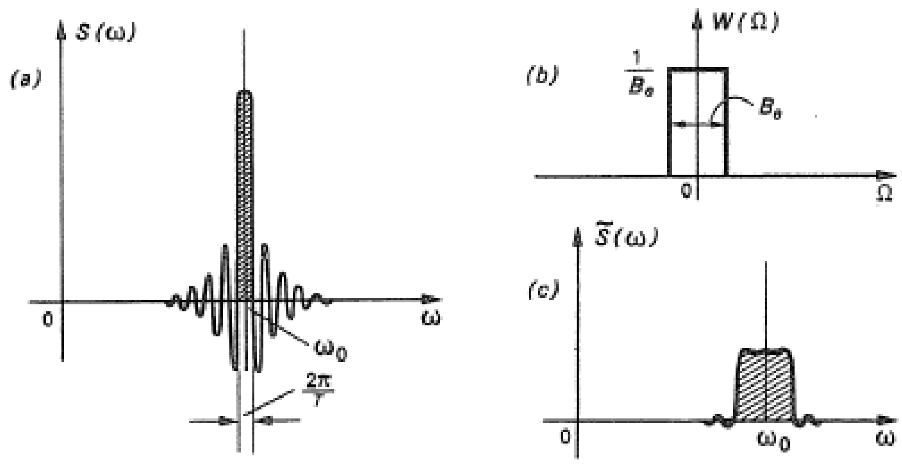


Fig. 9.5 Smoothing with a rectangular spectral window

magnitude, is to smooth the spectrum so that instead of plotting $S(\omega)$ we plot a *smoothed spectrum* $\tilde{S}(\omega)$ given by

$$\tilde{S}(\omega) = \int_{-\infty}^{\infty} W(\Omega - \omega) S(\Omega)\, d\Omega \tag{9.26}$$

where Ω is a dummy frequency variable and $W(\Omega)$ is a weighting function which satisfies

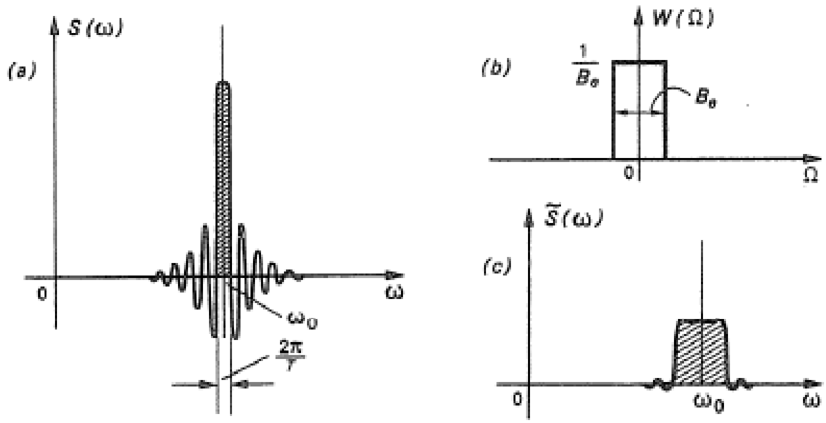
$$\int_{-\infty}^{\infty} W(\Omega)\, d\Omega = 1. \tag{9.27}$$

Analysis of finite length records

- Suppose that W(Ω) is the rectangular function shown in figure b. In this case, the smoothed form of the spectrum defined by

$$S_x(\omega) \simeq \frac{a^2}{4\pi}\left\{\frac{\sin(\omega-\omega_0)T}{\omega-\omega_0}+\frac{\sin(\omega+\omega_0)T}{\omega+\omega_0}\right\}.$$

is sketched in Figure c. Since positive and negative half cycles of S(ω) tend to cancel each other out, the smoothed spectrum has the approximate form shown and on account of $\int_{-\infty}^{\infty} W(\Omega)\,\mathrm{d}\Omega = 1.$ the shaded areas in Figure a and Figure c are approximately equal.



Smoothing with a rectangular spectral window

Analysis of finite length records

- The function W(Ω) is called a spectral window function and the shape of a graph of W(Ω) against Ω is said to be the "shape of the window". Many different shapes have been suggested, but the effective width of the window is what matters because this defines the band of frequencies over which averaging occurs.

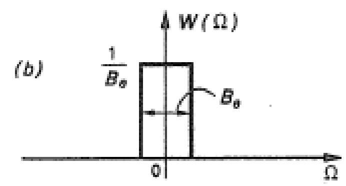
$$\tilde{S}(\omega) = \int_{-\infty}^{\infty} W(\Omega - \omega) S(\Omega) \, d\Omega$$

- For a rectangular window as shown in the figure, the bandwidth B_e is just the full width of the function. However, for other functions which may rise gradually to a peak and then fall off gradually, it is necessary to calculate an effective width and this is usually defined by the effective bandwidth B_e where,

$$B_e \int_{-\infty}^{\infty} W^2(\Omega) \, d\Omega = \left\{ \int_{-\infty}^{\infty} W(\Omega) \, d\Omega \right\}^2$$

which with $\int_{-\infty}^{+\infty} W(\Omega) d\Omega = 1$ becomes

$$B_e = \frac{1}{\int_{-\infty}^{\infty} W^2(\Omega) \, d\Omega}.$$



Analysis of finite length records

- Just as the variance of a measurement made with an analogue spectral analyser depends on the filter bandwidth and the averaging time, so the variance of any estimation of mean square spectral density depnds on (i) the effective band width of the spectral window, B_e, and (ii) the length of the record, T. It can be shown that the result

$$\frac{\sigma}{m} \cong \frac{1}{\sqrt{BT}}$$

still applies provided that the spectrum changes slowly over frequency intervals of order 1/T, i.e., provided that the record length is long enough to resolve adjacent spectral peaks. In this case, we therefore still have:

$$\frac{\sigma}{m} \simeq \frac{1}{\sqrt{(B_e T)}}$$

where σ is the standard deviation of a measurement whose mean value is m, B_e is the effective bandwidth of the spectral window and T is the record length.

Analysis of finite length records

- Furthermore, the equation

$$\frac{\sigma}{m} \simeq \frac{1}{\sqrt{(B_e T)}}$$

also applies approximately to measurements of the autocorrelation function. In this case, the effective bandwidth Be is the bandwidth of the entire input process so an individual measurement of a correlation function has greater statistical reliability than an individual measurement of spectral density from the same length of the record. However, in transforming from the time domain to the frequency domain, we are forced to employ selective filters which automatically reduce the effective bandwidth and therefore reduce the statistical reliability of the results obtained.