Discrete Mathematics

Trees

H. Turgut Uyar   Ayşegül Gençata   Emre Harmancı

2007

Content

Tree

Definition

tree:

Graph G is called a tree if G is connected and contains no cycles.

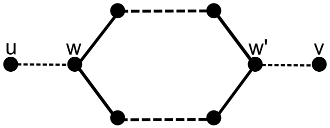
- Graph whose connected components are trees: *forest*

Tree Theorems

Theorem

There is exactly one path between a node pair in a tree.

- There is a path because the tree is connected.
- If there were more than one path:



Tree Theorems

Theorem

The following statements are equivalent for a loop-free undirected graph $G(V, E)$ and $|V| \geq 2$:

1. *G is a tree (is connected and has no cycles.)*
2. *there is exactly one path between each node pair.*
3. *G is connected, but if an edge is removed it becomes disconnected.*
4. *G does not have a cycle, but if an edge is added between any two nodes a cycle is formed.*

- proof method: $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$

Tree Theorems

Theorem

The following statements are equivalent for a loop-free undirected graph $T(V, E)$:

1. *T is a tree.*
2. *T is connected, and $|V| = |E| + 1$*
3. *T contains no cycles, and $|V| = |E| + 1$*

Tree Theorems

Theorem

For every tree $T(V, E)$, if $|V| \geq 2$, then T has at least two pendant vertices.

Proof.

- $2|E| = \sum_{v \in V} d_v$
- Assume there is one vertex of degree 1:
  $\Rightarrow 2|E| \geq 2(|V| - 1) + 1$
  $\Rightarrow 2|E| \geq 2|V| - 1$
  $\Rightarrow |E| \geq |V| - \frac{1}{2} > |V| - 1$ contradiction

$\square$

Spanning Tree

Definition

spanning tree:
A subgraph T of a graph G is called a spanning tree of G, if T is a tree and T includes all vertices of G.

Definition

minimum spanning tree:
A spanning tree for which the sum of the edge weights is minimum.

Kruskal's Algorithm

Kruskal's algorithm

1. Set the counter $i \leftarrow 1$, select $e_1 \in E$, where $wt(e_1)$ is minimum
2. For $1 \leq i \leq n-2$:
   if edges $e_1, e_2, \ldots, e_i$ have been selected, select e_{i+1} from the remaining edges in G so that:
   - $wt(e_{i+1})$ is minimum
   - The subgraph determined by $e_1, e_2, \ldots, e_i, e_{i+1}$ contains no cycles.
3. Replace $i \leftarrow i+1$
   - If $i = n-1$ the subgraph with edges $e_1, e_2, \ldots, e_{n-1}$ is an optimal spanning tree of G.
   - If $i < n-1$ return to step 2.

Rooted Tree

- The tree T is a directed tree, if all edges of T are directed.
- T is called a rooted tree if there is a unique vertex r, called the root, with indegree of 0, and for all other vertices v the indegree is 1.
  - All vertices with outdegree 0 are called leaf.
  - All other vertices are called branch node or internal node.

Node Level

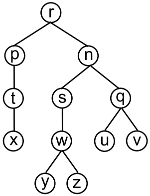
Definition

level:
distance from root

- parent: preceeding node
- child: succeeding node

Example

Example



- root: r
- leaves: $x\ y\ z\ u\ v$
- internal nodes: $p\ n\ t\ s\ q\ w$
- parent of y : w
  children of w : y and z

Binary Rooted Trees

- *binary rooted tree*: Every node has at most 2 children
- *complete* binary rooted tree: Every node has 0 or 2 children

Operation tree

- Mathematical operations can be represented by trees.
- root and internal nodes contain operators
- the leaves contain variables/constants

Traversal of Operation Tree

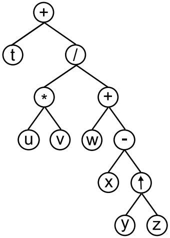
1. inorder: left sub-tree, root, right sub-tree
2. preorder: root, left sub-tree, right sub-tree
3. postorder:left sub-tree, right sub-tree, root

- Changing operator precedence requires parantheses in inorder notation.

Example

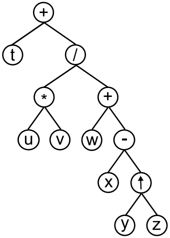
Example

$$t + \frac{u \cdot v}{w + x - y^z}$$



Example of preorder traversal

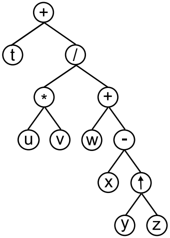
Example



$$+ \ t \ / \ * \ u \ v \ + \ w \ - \ x \ \uparrow \ y \ z$$

Example of inorder traversal

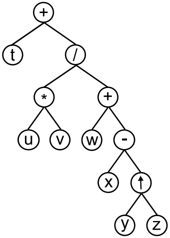
Example



$$t \; + \; u \; * \; v \; / \; w \; + \; x \; - \; y \; \uparrow \; z$$

Example of postorder traversal

Example



$t\ u\ v\ *\ w\ x\ y\ z\ \uparrow\ -\ +\ /\ +$

m-ary Tree

Definition

m-ary tree:

The outdegree of all vertices, except leaves, is m.

m-ary Trees

Theorem

In an m-ary tree:

- *number of nodes n*
- *number of leaves l*
- *number of internal nodes i*

- $n = m \cdot i + 1$
- $l = n - i = m \cdot i + 1 - i = \ (m-1) \cdot i + 1$
- $i = \frac{l-1}{m-1}$

Example

Example

How many matches are played in a tennis tournament of 27 players?

- a leaf for each player: $l = 27$
- an internal node for each match: $m = 2$
- number of matches: $i = \frac{l-1}{m-1} = \frac{27-1}{2-1} = 26$