A COMPRESSIVE SENSING FRAMEWORK FOR MULTIRATE SIGNAL ESTIMATION

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ABSTRACT

This paper develops a novel compressive sensing setting for the multirate signal estimation problem. The multirate signal estimation task consists of estimating the values for a source signal when observed through several measurement channels sampled at different sampling rates. We demonstrate that this formulation can be recast in a compressive sensing setup. Reformulating the multirate signal estimation problem in a compressive sensing framework, enables us to infuse the sparse signal estimation and reconstruction methodologies into this multirate setting in a novel manner. We show that for sparse signals sampled through a multirate multichannel system, the compressive sensing signal reconstruction paradigm fits in effectively. Simulations are provided demonstrating that compressive sensing based signal reconstruction for multirate signal estimation is a viable and effective alternative.

Index Terms— compressive sensing, multirate systems, signal estimation

1. INTRODUCTION

We consider a signal sensing scheme where the underlying signal is observed through a bank of measurement channels working at differing sampling rates. This type of a signal acquisition mechanism is quite general, and methods for the estimation of the underlying signal from these acquired measurements have been proposed [1–4]. Here, we consider the case where the underlying signal to be observed through this kind of a mechanism is compressible in some transform domain. The maturing field of compressive sensing is based on the premise that when linear projections onto random vectors are utilized to acquire a compressible signal, it is possible to reconstruct the signal from a number of measurements far fewer than its dimensionality. In this work we show the that the multichannel multirate signal acquisition mechanism can actually be thought of as a compressive sensing type data sensing method. We present numerical results which confirm that when the signal to be observed through the multichannel multirate system is compressible in the DCT domain, compressive sensing based reconstruction from the measurements works effectively.

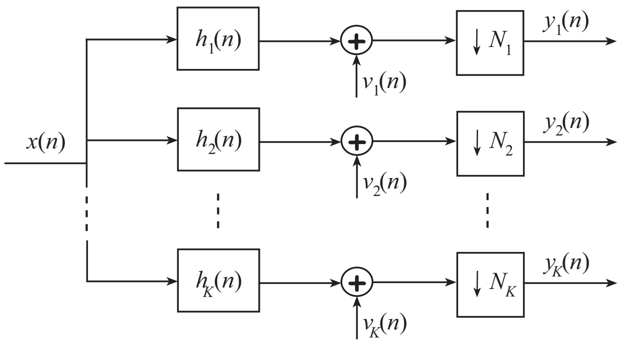

Fig. 1. Multirate multichannel signal observation mechanism.

2. MULTIRATE SIGNAL ESTIMATION PROBLEM FORMULATION

We assume a signal acquisition setting where a directly unobservable message signal $x(n)$ is observed through a bank of K sensors working at individual sampling rates. Each sensor bank consists of an FIR filter followed by a downsampler with downsampling ratio, N_k. Each distinct filter models the bandwidth limitations, transmission channel impairments or other nonidealities for the corresponding sensor bank. The differing downsampling ratios model differences in the sampling ratios of sensors in a multisensor structure. The setup as presented in Fig.1 has been considered in a number of papers.

In [1] the authors consider the problem of estimating the power spectral density for the random signal $x(n)$ observed as in Fig. 1. It is suggested to solve this ill-posed inverse problem using the principle of maximum entropy. The maximum entropy power spectrum estimate is utilized to further estimate the signal $x(n)$ in a least-squares formulation. In [2] an optimal linear filtering approach was developed for the stochastic multirate signal estimation problem for any number of observation signals with arbitrary sampling rates. In [3], the optimum linear filtering approach was applied to the high resolution signal and image reconstruction task. [4] proposes an iterative maximum entropy algorithm for the estimation of the power spectral density of the underlying stochastic signal $x(n)$, when observed through a sampling mechanism as depicted in Fig. 1.

3. COMPRESSIVE SENSING PRIOR ART

Signal processing based on sparse representations has been a subject of active research. A novel signal sensing and reconstruction paradigm based on sparse representation has been developed under the title of "compressive sensing" (or alternately "compressive sampling") [5]. For a discrete signal $\boldsymbol{x} \in \mathbb{R}^n$, the compressive sensing (CS) data acquisition step is realized by projecting the signal onto a set of sensing vectors $\{\boldsymbol{\phi}_j\}_{j=1}^{m}$. The observed values are $y_j = \langle \boldsymbol{x}, \boldsymbol{\phi}_j \rangle$, $j = 1 \dots m$. Usually $m \ll n$, hence this step brings about the much deliberated joint data acquisition and compression property of compressive sensing. The sensing vectors can get arranged as the rows of a single sensing matrix $\boldsymbol{\Phi}_{m \times n}$. The data acquisition step can be summarized in the form of the underdetermined equation

$$\boldsymbol{y} = \boldsymbol{\Phi} \boldsymbol{x} \tag{1}$$

where $\boldsymbol{y} \in \mathbb{R}^m$ denotes the observation vector. The reconstruction part of the compressive sensing paradigm handles the ill-posed inverse problem forming an estimate $\widehat{\boldsymbol{x}}$ based on the observation vector $\boldsymbol{y}$. The gist of compressive sensing paradigm is by realizing that under the assumption of a sparsity prior for $\boldsymbol{x}$, the reconstruction step can be reformatted as an optimization problem. The assumption is that the signal $\boldsymbol{x}$ has a sparse (or more generally compressible) representation in a transform domain expressed by some basis matrix $\boldsymbol{\Psi}$. That is, $\boldsymbol{x} = \boldsymbol{\Psi\alpha}$, where $\boldsymbol{\alpha}$ is an S-sparse vector. By S-sparsity it is meant that $\|\boldsymbol{\alpha}\|_0 \leqslant S$. Here, $\|(\cdot)\|_0 = \#supp(\cdot)$ is the ℓ_0 pseudo norm. After forcing the sparsity constraint, the compressive sensing reconstruction procedure boils down to finding

$$\widehat{\boldsymbol{\alpha}} = \text{argmin } \|\boldsymbol{\alpha}\|_0 \text{ subject to } \|\boldsymbol{\Phi\Psi\alpha} - \boldsymbol{y}\|_2 \leqslant \epsilon \tag{2}$$

The computational complexity for the solution of this constrained minimization is known to be NP-hard. Compressive sensing idea gets attractive when this prohibitive optimization based reconstruction procedure gets replaced with a much lesser demanding ℓ_1-norm based optimization.

$$\widehat{\boldsymbol{\alpha}} = \text{argmin } \|\boldsymbol{\alpha}\|_1 \text{ subject to } \|\boldsymbol{\Phi\Psi\alpha} - \boldsymbol{y}\|_2 \leqslant \epsilon \tag{3}$$

This recovery procedure works such that, among all $\boldsymbol{\alpha}$ consistent with the observation, the one with the smallest absolute sum of the coefficients is chosen. For the ℓ_0-norm however, the number of nonzero coefficients is forced to be minimal. It has been established that under certain conditions the ℓ_1-norm based optimization outputs the exact ℓ_0-norm solutions with very high probability [6, 7].

The ℓ_1-norm based optimization criterion leads to well studied solution algorithms such as basis pursuit. The sparse solutions can also get recovered via greedy algorithms, of which matching pursuit and its numerous variations are the best known. In this paper, in the recovery step we stick with optimization based methods. In particular we utilize the ℓ_1-Magic toolbox developed by Candès and Romberg, [8].

4. MULTIRATE OBSERVATIONS MEET COMPRESSIVE SENSING

In this section it is shown that the multirate data acquisition scheme as in Fig.1 can be recast as a CS type data acquisition protocol. The filtering and downsampling steps can be conjoined in a single linear projection operator. The unification of the random filtering and downsampling steps in a single channel as a CS operator was previously considered in [9]. Here we go further and translate a multirate and *multichannel* data collection procedure into the CS paradigm.

4.1. The CS projection matrix for multirate measurements

We assume FIR filters with impulse responses $\boldsymbol{h}_i$ of length H in the individual channels of Fig.1. Hence, the observation vectors can be written as

$$\begin{aligned} \boldsymbol{y}_i &= \boldsymbol{D}_{\downarrow}^{i} \left(\boldsymbol{h}_i * \boldsymbol{x}\right) \\ &= \boldsymbol{D}_{\downarrow}^{i} \boldsymbol{H}_i \boldsymbol{x} \\ &= \boldsymbol{\Phi}_i \boldsymbol{x} \end{aligned} \tag{4}$$

Here $\boldsymbol{D}_{\downarrow}^{i}$ is the downsampling matrix with the downsampling ratio N_i. $\boldsymbol{H}_i$ is the convolution matrix corresponding to the FIR filter with the impulse response $\boldsymbol{h}_i$ in the i^{th} channel. $\boldsymbol{\Phi}_i = \boldsymbol{D}_{\downarrow}^{i} \boldsymbol{H}_i$ is the observation matrix for the i^{th} channel. Each one of these matrices is a band matrix with a quasi-Toeplitz structure. Each row includes as the nonzero part the impulse response $\boldsymbol{h}_i$, and each row is a shifted version of the row above by N_i.

The observations from the different channels come together to form the single big observation vector $\boldsymbol{y}$.

$$\begin{aligned} \boldsymbol{y} &= \left[\boldsymbol{y}_1^T \dots \boldsymbol{y}_k^T\right]^T \\ &= \boldsymbol{\Phi}_{\text{MR}} \boldsymbol{x} \end{aligned} \tag{5}$$

The overall CS projection matrix for the multirate, multichannel signal observation setting is denoted by $\boldsymbol{\Phi}_{\text{MR}}$. $\boldsymbol{\Phi}_{\text{MR}}$ is generated by concatenating all the observation matrices $\boldsymbol{\Phi}_i$ corresponding to the individual channels together.

$$\boldsymbol{\Phi}_{\text{MR}} = \begin{bmatrix} \boldsymbol{\Phi}_1 \\ \vdots \\ \boldsymbol{\Phi}_K \end{bmatrix} \tag{6}$$

Assuming that the observed signal $\boldsymbol{x}$ has a sparse representation $\boldsymbol{x} = \boldsymbol{\Psi\alpha}$ in some basis $\boldsymbol{\Psi}$, the CS multirate data acquisition can be written as

$$\boldsymbol{y} = \boldsymbol{\Theta}_{\text{MR}} \boldsymbol{\alpha} \tag{7}$$

Here, $\boldsymbol{\Theta}_{\text{MR}} = \boldsymbol{\Phi}_{\text{MR}} \boldsymbol{\Psi}$ is the sensing matrix starting from the sparse domain.

4.2. Properties of the CS projection matrix

The conditions for robust signal reconstruction in CS have been examined in the literature. In general the maximum compression ratio between the observed signal dimension and the number of CS measurements guaranteeing successful reconstruction depends on: 1) the sparseness of the observed signal, and 2) the incoherence between the sparsity transform $\mathbf{\Psi}$ and the sensing operator $\mathbf{\Phi}$. Results for CS reconstruction robustness have been reported for randomized sensing matrices [10]. In a general form, for a sensing matrix $\mathbf{\Phi}$ generated with entries from a random distribution, ℓ_1 minimization leads to correct reconstruction with overwhelming probability when the number of observations satisfies

$$m \geqslant const \cdot \mu^2(\mathbf{\Phi}, \mathbf{\Psi}) \cdot S \cdot \log n \qquad (8)$$

Here, $\mu(\mathbf{\Phi}, \mathbf{\Psi})$ denotes the coherence between the matrices and gives a measure for the similarity between them. Random observation matrices are very likely to have small coherence with any fixed sparsity transform. Hence, they lead to universal CS samplers fit for signals sparse in any arbitrary base. For a coherence value $\mu(\mathbf{\Phi}, \mathbf{\Psi})$ close to one, $S \log n$ observations will suffice for successful reconstruction. Another approach to quantify the robustness of the CS reconstruction or the suitability of a certain observation matrix is through the definition of so called "*Restricted Isometry Properties*" (RIP) [11]. Observation matrices generated through random distributions or randomized selections can be proven to satisfy certain RIP and are provably robust for CS. However, testing the RIP for a general (non-random) fixed matrix is a combinatorial problem of exponential complexity. There are current attempts at finding subexponential-time algorithms for testing the RIP property (or the suitability as a CS observation matrix) of a given generic matrix [12]. However, testing the RIP property for the sensing matrix $\mathbf{\Theta}_{\mathrm{MR}}$ occurring in the multirate signal observation setup is beyond the scope of this paper. Instead, we try to establish experimental verification for the performance of this matrix via numerical simulations. Results suggest that $\mathbf{\Phi}_{\mathrm{MR}}$ works well as a CS sensing matrix when the sparsity transform is the discrete cosine transform.

5. NUMERICAL RESULTS

Numerical results for CS based reconstruction for multirate signal observation problem are presented. The experiments study the probability of exact reconstruction for the novel CS based approach to signal reconstruction from multichannel multirate observations. In the reconstruction from the CS measurements step, we utilize the ℓ_1-Magic toolbox as developed by Candès and Romberg, [8]. This toolbox offers efficient convex optimization algorithm realizations for the basis pursuit problem. In the experiments, signal length is fixed at $n = 128$ and the sparsity is fixed at $S = 10$. For each point in the plots, 1000 trials are realized. In each realization, H filter impulse response tabs for each of the distinct K channels are generated randomly from an $\mathcal{N}(0, 1)$ distribution. The sparse signal $\boldsymbol{x}$ to be sensed for each realization, has a sparse representation $\boldsymbol{\alpha}$ such that the S nonzero coefficients of $\boldsymbol{\alpha}$ occur in random positions and take their values from an $\mathcal{N}(0, 1)$ distribution.

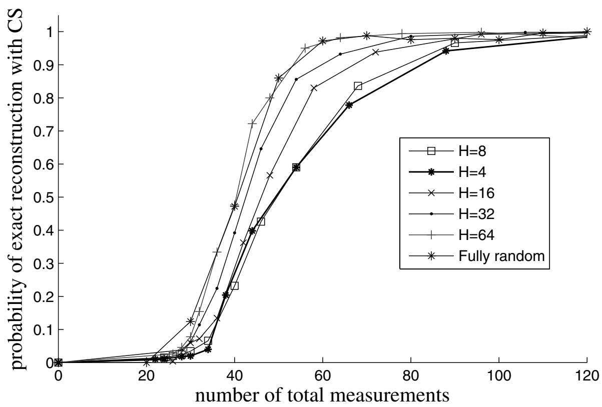

Fig. 2. Probability of exact reconstruction versus the length of the total observation vector $\boldsymbol{y}$ for $N_1 = N_2$ with differing filter lengths.

In this work we present results for signals sparse in the Discrete Cosine Transform (DCT) domain. We consider in the experiments the scenario with two and three multirate sampling channels. We present results for the case, where the the subsampling rates in the different channels are equivalent. In Fig. 2, we consider the case for two sensing channels with $N_1 = N_2$. The subsampling rates in the two channels are equal, and both channels contribute to the $\boldsymbol{y}$ vector in equal amounts. The filter lengths are chosen as $H = 4, 8, 16, 32, 64$. Secondly we consider the case with $N_1 = N_2 = N_3$ in Fig.3. In these figures each curve corresponds to a different filter tab length H. Each point on the curves gives the probability of exact reconstruction for the corresponding number of total measurements. The horizontal axis denotes the total number of observed (sensed) values, that is the size of the vector $\boldsymbol{y}$ (5), which unites the observations from the different channels. We observe that longer filter lengths translate into better reconstruction performance. Sensing matrices $\mathbf{\Phi}_{\mathrm{MR}}$ constructed using longer filter impulse responses have better coherence properties than those constructed using shorter filter impulse responses. In these figures we also present results for a fully random i.i.d sensing matrix with entries chosen from a normal distribution. We realized CS measurements and reconstruction using this fully random matrix for comparison purposes. The results for the fully random matrix, two channel multirate measurements with $N_1 = N_2$ and $H = 64$, and three channel multirate measurements with $N_1 = N_2 = N_3$ and $H = 64$ are packed together in Fig.4. These curves suggest that for sufficiently long filter length H, the multichannel multirate sampling schemes in the CS setting work with a performance comparable to fully random CS sensing matrices. The multirate multichannel data acquisition system presents a viable sensing mechanism for CS.

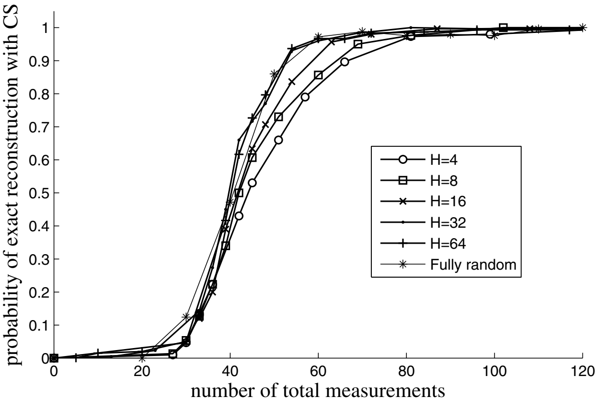

Fig. 3. Probability of exact reconstruction versus the length of the total observation vector $\boldsymbol{y}$ for $N_1 = N_2 = N_3$ with differing filter lengths.

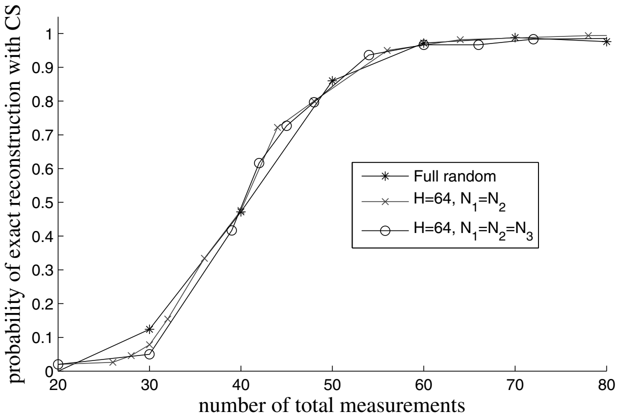

Fig. 4. Probability of exact reconstruction for fully random sensing matrix, $N_1 = N_2$ multirate system with $H = 64$ and $N_1 = N_2 = N_3$ multirate system with $H = 64$.

6. CONCLUSIONS

In this work we have established that multichannel multirate signal acquisition is a viable CS sensing mechanism for compressible signals. Numerical results suggest the suitability of this type of data acquisition for compressible signals sparse in the DCT domain. There is a plethora of subjects remaining for future work. Work on signals sparse in different transform domains is one subject to consider. Establishing RIP results for the CS matrices occurring in this acquisition setup and evaluating the effect of unequal subsampling rates in the different channels on the reconstruction performance are some of the other possible subjects for future studies.

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