LECTURE NOTES - VII

« FLUID MECHANICS »

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CHAPTER 7

TWO-DIMENSIONAL FLOW OF THE REAL FLUIDS

7.1. INTRODUCTION

Two-dimensional flow problems may easily be solved by potential flow approach as was explained in Chapter 6.

In order to use the ideal fluid assumption for the flow of real fluids, shearing stress that occurs during the fluid motion should be so small to affect the motion. Since shearing stress may be calculated by Newton's viscosity law by $\tau = \mu du/dy$, two conditions should be supplied to have small shearing stresses as;

a) *The viscosity of the fluid must be small*: the fluids as water, air, and etc can supply this condition. This assumption is not valid for oils.
b) *Velocity gradient must be small*: This assumption cannot be easily supplied because the velocity of the layer adjacent to the surface is zero. In visualizing the flow over a boundary surface it is well to imagine a very thin layer of fluid adhering to the surface with a continuous increase of velocity of the fluid. This layer is called as *viscous sublayer*.

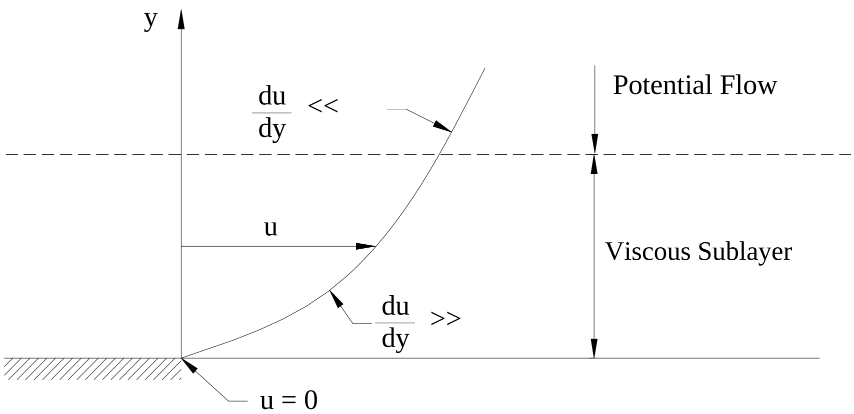

Fig. 7.1

Flow field may be examined by dividing to two zones.

a) *Viscous sublayer zone*: In this layer, velocity gradient is high and the flow is under the affect of shearing stress. Flow motion in this zone must be examined as real fluid flow.
b) *Potential flow zone*: The flow motion in this zone may be examined as ideal fluid flow (potential flow) since velocity gradient is small in this zone.

7.2. BASIC EQUATIONS

Continuity equation for two-dimensional real fluids is the same obtained for two-dimensional ideal fluid. (Equ. 6.3)

Head (energy) loss h_L must be taken under consideration in the application of energy equation. Hence, the Bernoulli equation (Equ. 6.6) may be written as,

$$\frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2 + h_L \tag{7.1}$$

On the same streamline between points 1 and 2 in a flow field.

Forces arising from the shearing stress must be added to the Euler's equations (Eqs. 6.4 and 6.5) obtained for two-dimensional ideal fluids.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{7.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g + \upsilon\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{7.3}$$

These equations are called *Navier-Stokes* equations. The last terms in the parentheses on the right side of the equations are the result of the viscosity effect of the real fluids. If $\upsilon \to 0$, the Navier-Stokes equations take the form of Euler equations. (Eqs. 6.4 and 6.5)

7.3.TWO-DIMENSIONAL LAMINAR FLOW BETWEEN TWO PARALLEL FLAT PLANES

Continuity equation for two-dimensional flow,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Since the flow is uniform, $\partial u/\partial x = 0$, therefore $\partial v/\partial y = 0$. (Fig. 7.2). Using the boundary conditions, for y = 0, u = 0 and $v \cong 0$.

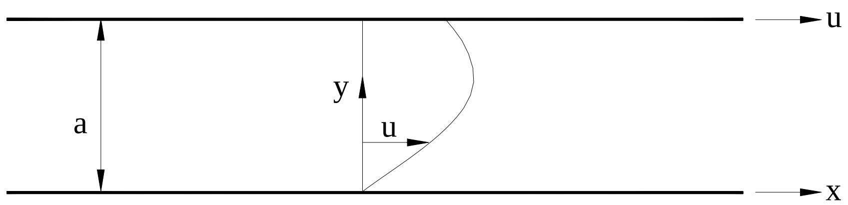

Fig. 7.2

Writing the Navier-Stokes equation for the x-axis,

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\vartheta\frac{\partial^2 u}{\partial x^2}+\vartheta\frac{\partial u^2}{\partial y^2}$$

yields

$$\vartheta\frac{d^2u}{dy^2}=\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Since υ = μ/ρ, by taking integration

$$u=\frac{1}{\mu}\frac{\partial p}{\partial x}\frac{y^2}{2}+C_1 y+C_2$$

C_1 and C_2 integration constants may be found by using boundary conditions. For y = 0, u = 0, and y = a, u = U.

The velocity distribution equation over the y-axis may be found as,

$$u=U\frac{y}{a}-\frac{ay}{2\mu}\frac{\partial p}{\partial x}\left(1-\frac{y}{a}\right) \tag{7.4}$$

$\frac{\partial p}{\partial x}<0$ in the flow direction.

If we write Navier-Stokes equation for the y-axis, and by using the above defined conditions,

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-g+\vartheta\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

yields

$$\frac{1}{\rho}\frac{\partial p}{\partial y}=-g \quad \text{and} \quad \frac{\partial p}{\partial y}=-\gamma$$

This is the hydrostatic pressure distribution that was obtained before.

Special cases;

a) If $\frac{\partial p}{\partial x}=0$, then

$$u=U\frac{y}{a} \tag{7.5}$$

This flow is known as *Coutte flow*. The velocity distribution is linear across the channel section. (Fig. 7.3)

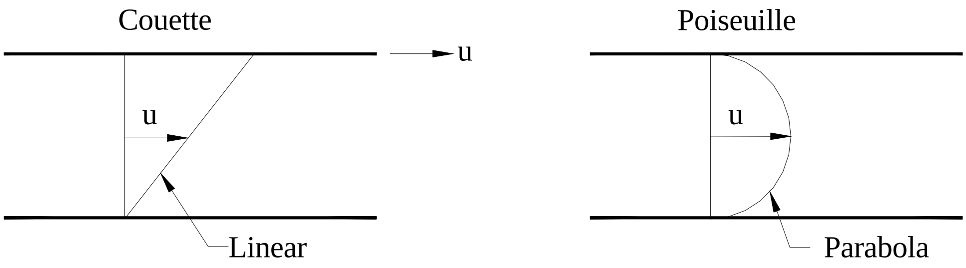

Fig. 7.3

b) If the upper plate is stationary, u = 0. The velocity distribution will take the form of

$$u = -\frac{ay}{2\mu}\frac{\partial p}{\partial x}\left(1 - \frac{y}{a}\right) \tag{7.6}$$

The velocity distribution curve of the laminar is a parabola with its vertex at the centerline of the flow channel. The pressure gradient is negative since there is a pressure drop in flow direction. The maximum velocity occurs at the center of the channel for y = a/2, that is

$$u_{\max} = -\frac{a^2}{8\mu}\frac{\partial p}{\partial x} \tag{7.7}$$

Since dq = udy, the discharge q of the laminar flow per unit width of the channel may be found by the integration of dq. Thus,

$$\begin{aligned} q &= \int_0^a \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(ay - y^2\right)dy \\ &= \frac{1}{12\mu}\left(-\frac{\partial p}{\partial x}\right)a^3 \end{aligned} \tag{7.8}$$

The average velocity V is,

$$V = \frac{q}{b} = \frac{1}{12\mu}\left(-\frac{\partial p}{\partial x}\right)a^2 \tag{7.9}$$

Which is two-thirds of the maximum velocity, u_{max}.

The pressure drop ($p_1 - p_2$) between any two chosen sections 1 and 2 in the direction of flow at a distance L = ($x_2 - x_1$) apart can be determined by integrating (Equ.7.9) with respect to x since the flow is steady;

$$\int_{p_1}^{p_2} -\partial p = \int_{x_1}^{x_2} \frac{12\mu V}{a^2}\partial x$$

and

$$p_1 - p_2 = \frac{12\mu V}{a^2}\left(x_2 - x_1\right) = \frac{12\mu V}{a^2}L \tag{7.10}$$

When the channel is inclined, the term ($-\partial p/\partial x$) in these equations is replaced by $-\partial(p+\gamma z)/\partial x$, ant the term p_1-p_2 on the left-hand side of Equ. (7.10) then becomes $(\gamma z_1+p_1)-(\gamma z_2+p_2)$.

7.4. LAMINAR FLOW IN CIRCULAR PIPES: HAGEN-POISEUILLE THEORY

The derivation of the Hagen-Poiseuille equation for laminar flow in straight, circular pipes is based on the following two assumptions;

a) The viscous property of fluid follows Newton's law of viscosity, that is, $\tau=\mu(du/dy)$,
b) There is no relative motion between fluid particles and solid boundaries, that is, no slip of fluid particles at the solid boundary.

Fig. 7.4 illustrates the laminar motion of fluid in a horizontal circular pipe located at a sufficiently great distance from the entrance section when a steady laminar flow occurs in a straight stretch of horizontal pipe, a pressure gradient must be maintained in the direction of flow to overcome the frictional forces on the concentric cylindrical surfaces, as shown in Fig. 7.4.

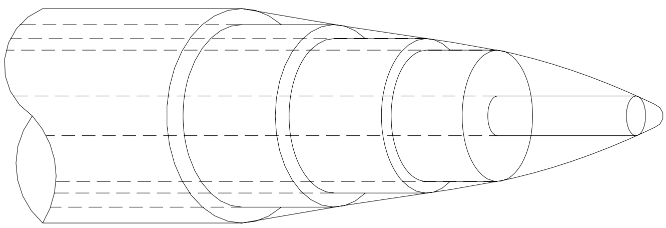

Fig. 7.4

Each concentric cylindrical layer of fluid is assumed to slide over the other in an axial direction. For practical purposes the pressure may be regarded as distributed uniformly over any chosen cross section of the pipe.

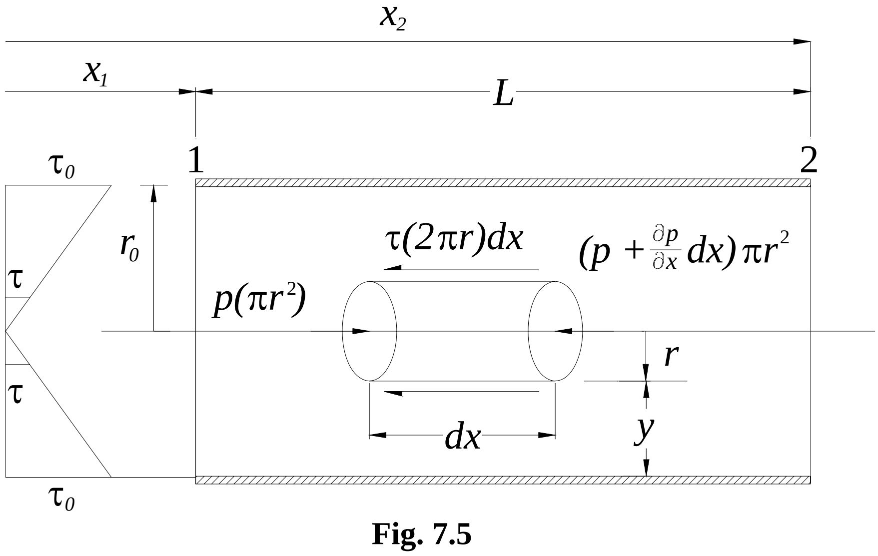

Fig. 7.5

A concentric cylinder of fluid is chosen as a free body (Fig. 7.5). Since the laminar motion of fluid is steady, the momentum equation for the flow of fluid through the chosen free body (in the absence of gravitational forces) is,

$$p\pi r^2 - \left(p + \frac{\partial p}{\partial x}dx\right)\pi r^2 - \tau(2\pi r)dx = 0$$

Which after simplifying, becomes,

$$\tau = -\frac{\partial p}{\partial x}\frac{r}{2} \tag{7.11}$$

The pressure gradient $\partial p/\partial x$ in the direction of flow depends on x only for any given case of laminar flow. The minus sign indicates a decrease of fluid pressure in the direction of flow in a horizontal pipe, since flow work must be performed on the free body to compensate for the frictional resistance to the flow. Equ. (7.11) shows that the shearing stress is zero at the center of pipe (r=0) and increase linearly with the distance r from the center, attaining its maximum value, $\tau_0=(-\partial p/\partial x)(r_0/2)$, at the pipe wall ($r=r_0$).

In accordance with the first assumption, τ equals $\mu(\partial u/\partial y)$. Since y equals r_0-r, it follows ∂y equals $-\partial r$. Newton's law of viscosity then becomes

$$\tau = -\mu\frac{\partial u}{\partial r} \tag{7.12}$$

in which the minus sign predicts mathematically that u decreases with r. By combining Eqs. (7.11) and (7.12),

$$-\mu\frac{\partial u}{\partial r} = -\frac{\partial p}{\partial x}\frac{r}{2}$$

or

$$\partial u = \frac{1}{2\mu}\frac{\partial p}{\partial x}rdr \tag{7.13}$$

Since $\partial p/\partial x$ is not a function of r, the integration of this differential equation with respect to r then yields the expression for point velocity u:

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x}r^2 + C$$

The integration constant C can be evaluated by means of the second assumption, that is, u=0 at $r=r_0$= Therefore, $C = -(\partial p/\partial x)r_0^2/4\mu$, and

$$u = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)\left(r_0^2 - r^2\right) \tag{7.14}$$

Which is an equation of parabola.

The point velocity varies parabolically along a diameter, and the velocity distribution is a paraboloid of revolution for laminar flow in a straight circular pipe (Fig. 7.6).

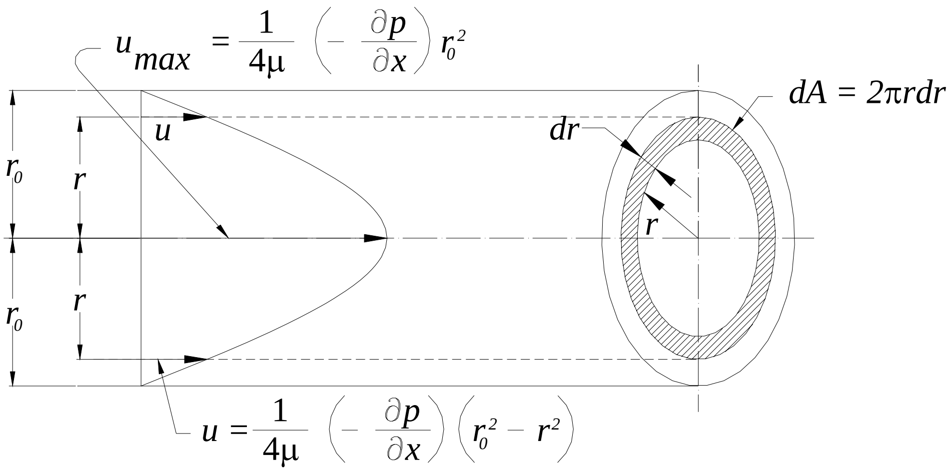

Fig. 7.6

The maximum point velocity u_{max} occurs at the center of the pipe and has the magnitude of

$$u_{\max} = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)r_0^2 \tag{7.15}$$

Thus, from Eqs. (7.14) and (7.15), the point velocity can also be expressed of the maximum point velocity as

$$u = u_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2\right] \tag{7.16}$$

The volumetric rate of flow Q through any cross section of radius r_0 is obtained by the integration of

$$dq = udA = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)\left(r_0^2 - r^2\right)(2\pi r)dr$$

as shown in Fig. 7.6. Hence,

$$\begin{aligned} Q &= \frac{\pi}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\int_0^{r_0}\left(r_0^2 - r^2\right)dr \\ &= \frac{\pi}{8\mu}\left(-\frac{\partial p}{\partial x}\right)r_0^4 \end{aligned} \tag{7.17}$$

and the average velocity V is Q/A, or

$$V = \frac{Q}{\pi r_0^2} = \frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right)r_0^2 \tag{7.18}$$

comparison of Eqs. (7.15) and (7.18) reveals that

$$V = \frac{u_{\max}}{2} \qquad (7.19)$$

Equ. (7.18) may be arranged in the following form

$$-\partial p = \frac{8\mu V}{r_0^2}\partial x$$

and then integrated with respect to x for any straight stretch of pipe between x_1 and x_2; $L=x_2-x_1$ (Fig. 7.5). Hence,

$$-\int_{p_1}^{p_2}\partial p = \frac{8\mu V}{r_0^2}\int_{x_1}^{x_2}\partial x$$

and, since D equals $2r_0$,

$$p_1 - p_2 = \frac{8\mu VL}{r_0^2} = \frac{32\mu VL}{D^2} \qquad (7.20)$$

This is usually referred to as the Hagen-Poiseuille equation.

EXAMPLE 7.1: A straight stretch of horizontal pipe having a diameter of 5 cm is used in the laboratory to measure the viscosity of crude oil (γ = 0.93 t/m^3). During a test run a pressure difference of 1.75 t/m^2 is obtained from two pressure gages, which are located 6 m apart on the pipe. Oil is allowed to discharge into a weighing tank, and a total of 550 kg of oil is collected for a duration of 3 min. Determine the viscosity of the oil.

SOLUTION: The discharge of oil flow in the pipe is

$$Q = \frac{0.550}{0.93\times 3\times 60} = 0.0033\,m^3/\sec$$

The average velocity is then

$$V = \frac{Q}{A} = \frac{4\times 0.0033}{\pi\times 0.05^2} = 1.69\,m/\sec$$

From Equ. (7.20)

$$\mu = \frac{(p_1 - p_2)D^2}{32VL}$$

$$\mu = \frac{1.75\times 0.05^2}{32\times 1.69\times 6} = 1.35\times 10^{-5}\,t\sec/m^2$$