LECTURE NOTES - VI

« FLUID MECHANICS »

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CHAPTER 6

TWO-DIMENSIONAL IDEAL FLOW

6.1 INTRODUCTION

An ideal fluid is purely hypothetical fluid, which is assumed to have no viscosity and no compressibility, also, in the case of liquids, no surface tension and vaporization. The study of flow of such a fluid stems from the eighteenth century hydrodynamics developed by mathematicians, who, by making the above assumptions regarding the fluid, aimed at establishing mathematical models for fluid flows. Although the assumptions of ideal flow appear to be far obtained, the introduction of the boundary layer concept by Prandtl in 1904 enabled the distinction to be made between two regimes of flow: that adjacent to the solid boundary, in which viscosity effects are predominant and, therefore, the ideal flow treatment would be erroneous, and that outside the boundary layer, in which viscosity has negligible effect so that idealized flow conditions may be applied.

The ideal flow theory may also be extended to situations in which fluid viscosity is very small and velocities are high, since they correspond to very high values of Reynolds number, at which flows are independent of viscosity. Thus, it is possible to see ideal flow as that corresponding to an infinitely large Reynolds number and zero viscosity.

6.2. CONTINUITY EQUATION

The control volume ABCDEFGH in Fig. 6.1 is taken in the form of a small prism with sides dx, dy and dz in the x, y and z directions, respectively.

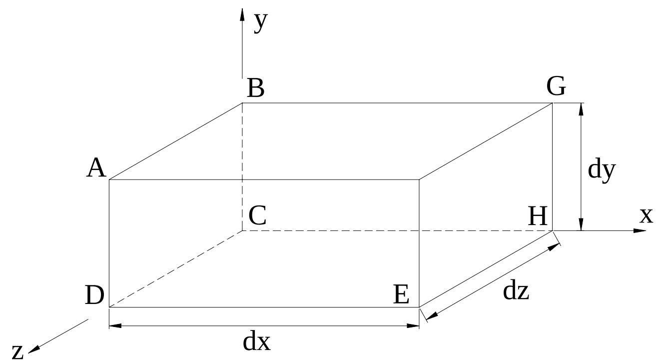


Fig. 6.1

The mean values of the component velocities in these directions are u, v, and w. Considering flow in the x direction,

Mass inflow through ABCD in unit time $= \rho u dy dz$

In the general case, both specific mass ρ and velocity u will change in the x direction and so,

Mass outflow through EFGH in unit time $= \left[\rho u + \frac{\partial(\rho u)}{\partial x} dx \right] dydz$

Thus,

Net outflow in unit time in x direction $= \frac{\partial(\rho u)}{\partial x} dxdydz$

Similarly,

Net outflow in unit time in y direction $= \frac{\partial(\rho v)}{\partial y} dxdydz$

Net outflow in unit time in z direction $= \frac{\partial(\rho w)}{\partial z} dxdydz$

Therefore,

Total net outflow in unit time $= \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] dxdydz$

Also, since $\partial\rho/\partial t$ is the change in specific mass per unit time,

Change of mass in control volume in unit time $= -\frac{\partial \rho}{\partial t} dxdydz$

(the negative sign indicating that a net outflow has been assumed). Then,

Total net outflow in unit time = Change of mass in control volume in unit time

$$\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] dxdydz = -\frac{\partial \rho}{\partial t} dxdydz$$

or

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = -\frac{\partial \rho}{\partial t} \tag{6.1}$$

Equ. (6.1) holds for every point in a fluid flow whether steady or unsteady, compressible or incompressible. However, for incompressible flow, the specific mass ρ is constant and the equation simplifies to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{6.2}$$

For two-dimensional incompressible flow this will simplify still further to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6.3}$$

EXAMPLE 6.1: The velocity distribution for the flow of an incompressible fluid is given by u = 3-x, v = 4+2y, w = 2-z. Show that this satisfies the requirements of the continuity equation.

SOLUTION: For three-dimensional flow of an incompressible fluid, the continuity equation simplifies to Equ. (6.2);

$$\frac{\partial u}{\partial x} = -1, \frac{\partial v}{\partial y} = 2, \frac{\partial w}{\partial z} = -1$$

and, hence,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -1 + 2 - 1 = 0$$

Which satisfies the requirement for continuity.

6.3. EULER'S EQUATIONS

Euler's equations for a vertical two-dimensional flow field may be derived by applying Newton's second law to a basic differential system of fluid of dimension dx by dz (Fig. 6.2).

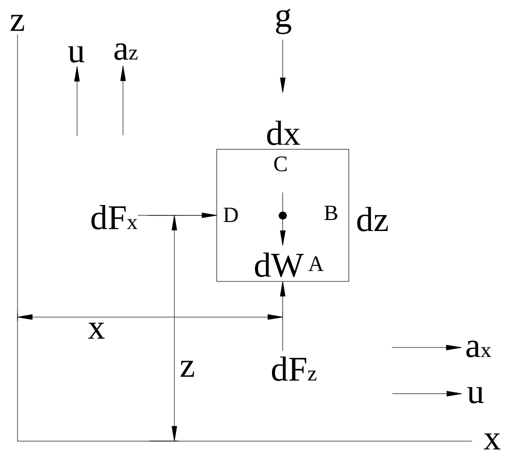


Fig. 6.2

The forces dF_x and dF_z on such an elemental system are,

$$dF_x = -\frac{\partial p}{\partial x} dxdz$$

$$dF_z = -\frac{\partial p}{\partial z} dxdz - \rho g dxdz$$

The accelerations of the system have been derived for steady flow (Equ. 3.5) as,

$$a_x = u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}$$
$$a_z = u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}$$

Applying Newton's second law by equating the differential forces to the products of the mass of the system and respective accelerations gives,

$$-\frac{\partial p}{\partial x}dxdz = \rho dxdz\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right)$$
$$-\frac{\partial p}{\partial z}dxdz - \rho g dxdz = \rho dxdz\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right)$$

and by cancellation of dxdz and slight arrangement, the *Euler equations* of two-dimensional flow in a vertical plane are

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \qquad (6.4)$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + g \qquad (6.5)$$

Accompanied by the equation continuity,

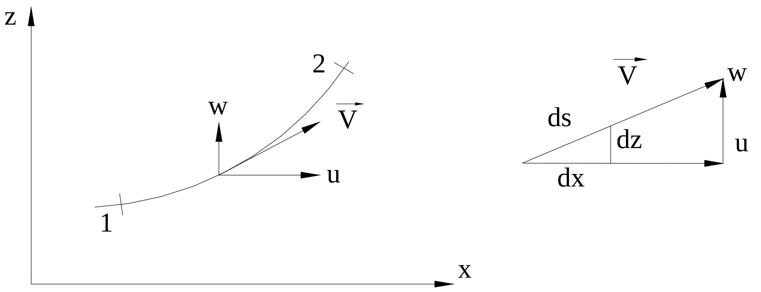
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad (6.3)$$

The Euler equations form a set of three simultaneous partial differential equations that are basic to the solution of two-dimensional flow field problems; complete solution of these equations yields p, u and w as functions of x and z, allowing prediction of pressure and velocity at any point in the flow field.

6.4. BERNOULLI'S EQUATION

Bernoulli's equation may be derived by integrating the Euler equations for a constant specific weight flow. Multiplying Equ. (6.4) by dx and Equ. (6.5) by dz and integrating from 1 to 2 on a streamline give

$$\int_1^2 u\frac{\partial u}{\partial x}dx + \int_1^2 w\frac{\partial u}{\partial z}dx = -\frac{1}{\rho}\int_1^2 \frac{\partial p}{\partial x}dx$$
$$\int_1^2 u\frac{\partial w}{\partial x}dz + \int_1^2 w\frac{\partial w}{\partial z}dz = -\frac{1}{\rho}\int_1^2 \frac{\partial p}{\partial z}dz - g\int_1^2 dz$$



However, along a streamline in any steady flow dz/dx=w/u and therefore udz = wdx. If we collect the both equations,

$$\int_1^2 \left(u\frac{\partial u}{\partial x} + w\frac{\partial w}{\partial x} \right) dx + \int_1^2 \left(u\frac{\partial u}{\partial z} + w\frac{\partial w}{\partial z} \right) dz = -\frac{1}{\rho}\int_1^2 \left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial z}dz \right) - g\int_1^2 dz$$

Since $u\frac{\partial u}{\partial x} = \frac{\partial \left(u^2/2\right)}{\partial x}$, arranging the equation yields,

$$\int_1^2 \left[\frac{\partial \left(u^2/2\right)}{\partial x}dx + \frac{\partial \left(u^2/2\right)}{\partial z}dz \right] + \int_1^2 \left[\frac{\partial \left(w^2/2\right)}{\partial x}dx + \frac{\partial \left(w^2/2\right)}{\partial z}dz \right] = -\frac{1}{\rho}\int_1^2 \left[\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial z}dz \right] - g\int_1^2 dz$$

Since the terms in each bracket is a total differential, by integrating gives

$$\frac{u_2^2}{2} - \frac{u_1^2}{2} + \frac{w_2^2}{2} - \frac{w_1^2}{2} = -\frac{1}{\rho}\left(p_2 - p_1\right) - g\left(z_2 - z_1\right)$$

By remembering that $V^2 = u^2 + w^2$, the equation takes the form of

$$\frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2 \qquad (6.6)$$

This equation is the well-known Bernoulli equation and valid on the streamline between points 1 and 2 in a flow field.

6.5. ROTATIONAL AND IRROTATIONAL FLOW

Considerations of ideal flow lead to yet another flow classification, namely the distinction between rotational and irrotational flow.

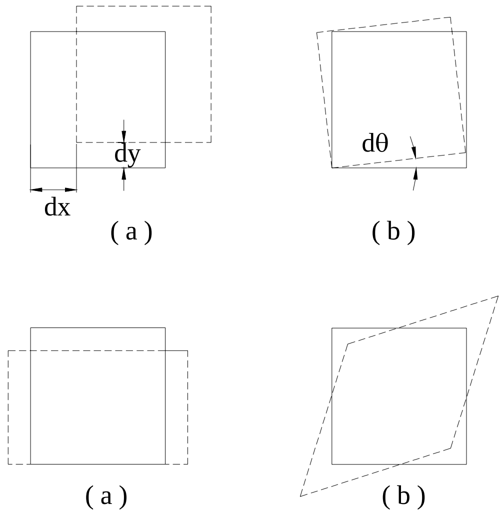


Fig. 6.2 and 6.3

Basically, there are two types of motion: translation and rotation. The two may exist independently or simultaneously, in which case they may be considered as one superimposed on the other. If a solid body is represented by square, then pure translation or pure rotation may be represented as shown in Fig. 6.2 (a) and (b), respectively.

If we now consider the square to represent a fluid element, it may be subjected to deformation. This can be either linear or angular, as shown in Fig. 6.3 (a) and (b), respectively.

The rotational movement can be specified in mathematical terms. Fig.6.4 shows the rotation of a rectangular fluid element in a two-dimensional flow.

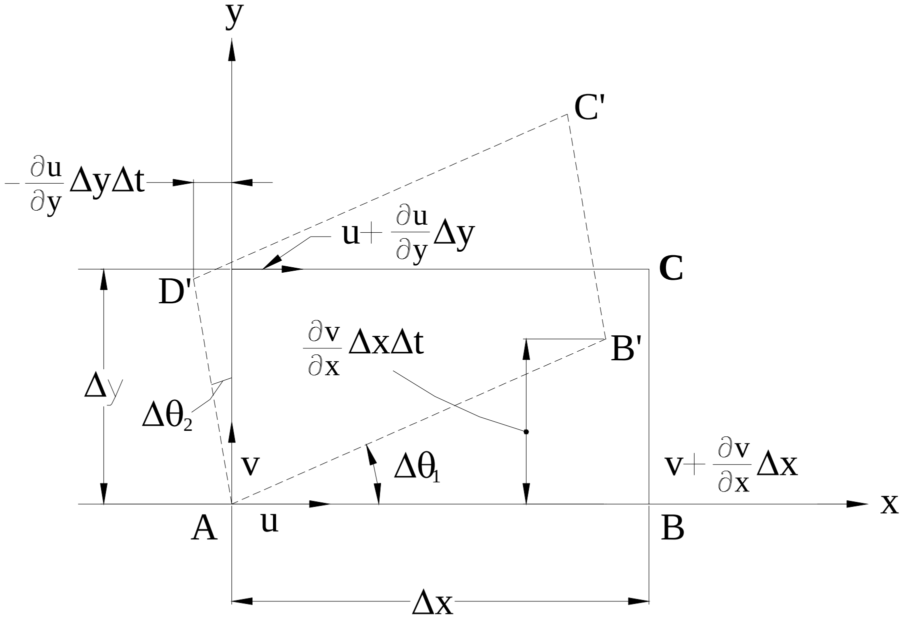


Fig. 6.4

During the time interval Δt the element ABCD has moved relative to A to a new position, which is indicated by the dotted lines. The angular velocity (w_{AB}) of line AB is,

$$w_{AB} = \lim_{\Delta t \to 0} \frac{\Delta \theta_1}{\Delta t} = \lim_{\Delta t \to 0} \frac{(\partial v / \partial x)\Delta x \Delta t}{\Delta x \Delta t} = \frac{\partial v}{\partial x}$$

Similarly, the angular velocity (w_{AD}) of line AD is

$$w_{AD} = \lim_{\Delta t \to 0} \frac{\Delta \theta_2}{\Delta t} = -\frac{\partial u}{\partial y}$$

The average of the angular velocities of these two line elements is defined as the *rotation* w of the fluid element ABCD. Therefore,

$$w = \frac{1}{2}\left(w_{AB} + w_{AD}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{6.7}$$

The condition of irrotationality for a two-dimensional flow is satisfied when the rotation w is everywhere zero, so that

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \text{or} \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \tag{6.8}$$

For a three-dimensional flow, the condition of irrotationality requires that the rotation about each of three axes, which are parallel to x, y and z-axes must be zero. Therefore, the following three equations must be satisfied:

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \tag{6.9}$$

EXAMPLE 6.2: The velocity components in a two-dimensional velocity field for an incompressible fluid are expressed as

$$u = \frac{y^3}{3} + 2x - x^2 y$$

$$v = xy^2 - 2y - \frac{x^3}{3}$$

Show that these functions represent a possible case of an irrotational flow.

SOLUTION: The functions given satisfy the continuity equation (Equ. 6.3), for their partial derivatives are

$$\frac{\partial u}{\partial x} = 2 - 2xy \qquad \text{and} \qquad \frac{\partial v}{\partial y} = 2xy - 2$$

so that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 - 2xy + 2xy - 2 = 0$$

Therefore they represent a possible case of fluid flow. The rotation w of any fluid element in the flow field is,

$$w = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x}\left(xy^2 - 2y - \frac{x^3}{3}\right) - \frac{\partial}{\partial y}\left(\frac{y^3}{3} + 2x - x^2 y\right)\right]$$

$$= \frac{1}{2}\left[\left(y^2 - x^2\right) - \left(y^2 - x^2\right)\right] = 0$$

6.6. CIRCULATION AND VORTICITY

Consider a fluid element ABCD in rotational motion. Let the velocity components along the sides of the element be as shown in Fig. 6.5.

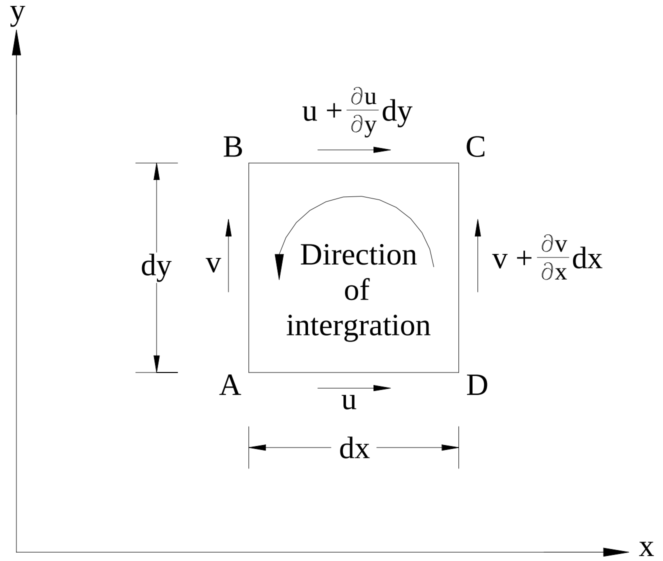


Fig. 6.5

Since the element is rotating, being part of rotational flow, there must be a resultant peripheral velocity. However, since the center of rotation is not known, it is more convenient to relate rotation to the sum of products of velocity and distance around the contour of the element. Such a sum is the line integral of velocity around the element and it is called *circulation*, denoted by Γ. Thus,

$$\Gamma = \oint \vec{V} \cdot d\vec{s} \tag{6.10}$$

Circulation is, by convention, regarded as positive for anticlockwise direction of integration. Thus, for the element ABCD, from side AD

$$\Gamma_{ABCD} = udx + \left(v + \frac{\partial v}{\partial x} dx \right) dy - \left(u + \frac{\partial u}{\partial y} dy \right) dx - vdy$$

$$= \frac{\partial v}{\partial x} dxdy - \frac{\partial u}{\partial y} dydx$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dydx$$

Since

$$\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \zeta$$

For the two-dimensional flow in the x-y Ë–aWe, is the *vorticity* of the element about the z-axis, ζ_z. ThÔ product dxdy is the area of the element dA.

Th)s

$$\Gamma_{ABCD} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy = \zeta_z dA$$

It is seen, therefore, that the circulation[XrWund X contour is equal to the sum of the vorticities within%the area of contour. This is known as Stokes' thXo   em and may be stated mathematically, for a general case of any contoué›C   (Fig_6.6) as

$$\Gamma_C = \oint V_s \cdot ds = \int_A \zeta \cdot dA \qquad (6.11)$$

Fig. 6.6

The above considerations<indicate that, for irrotational flow, since vorticity iÏ equal to zero, Óhe circulation around a closed contour through which fluid és moving, must be equal to zero.

6.7. STREAM FUNCTIO…

A *stream function* ψ is a mathematical device, which describes the formŒof any particular pattern of flow. In Fig. 6.7 let P (x, y) represent a movable point in the plane of motion of a steady, two-dimensional flow, and cönsider the flow to have unit thicrness perpendicular to the xy-plane.

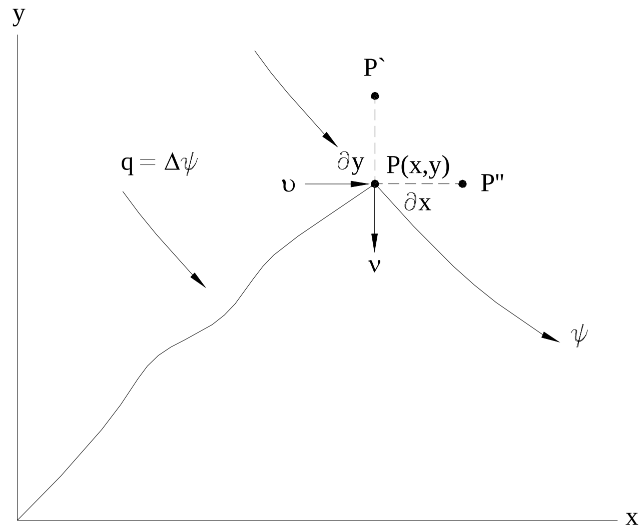


Fig. 6.7

The volume rate of flow across any line connecting OP is a function of the position of P and defined as the stream function ψ:

$$\psi = f(x, y)$$

Stream function ψ has a unit of cubic meter per second per meter thickness (normal to the xy-plane).

The two components of velocity, u and v can be expressed in terms of ψ. If the point P in Fig. 6.7 is displaced an infinitesimal distance ∂y is ∂ψ = u·∂y. Therefore,

$$u = \frac{\partial \psi}{\partial y} \qquad (6.12)$$

Similarly,

$$v = -\frac{\partial \psi}{\partial x} \qquad (6.13)$$

When these values of u and v are substituted into Eqs. (3.6), the differential equation for streamlines in two-dimensional flow becomes

$$\frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = 0$$

By definition, the left-hand side of this equation is equal to the total differential dψ when ψ = f (x, y). Thus,

$$d\psi = 0$$

and

ψ = C (constant along a streamline) (6.14)

Equ. (6.14) indicates that the general equation for the streamlines in a flow pattern is obtained when ψ is equated to a constant. Different numerical values of the constant in turn define streamlines. As an example, the stream function for a steady two-dimensional flow at 90^0 corner (shown in Fig. 6.8) takes the following form:

$$\psi = xy$$

The general equation for the streamlines of such a flow is obtained when ψ = C (constant), that is,

$$xy = C$$

Which indicates that the streamlines are a family of rectangular hyperbolas. Different numerical values of C define different streamlines as shown in Fig. 6.8. Obviously, the volume rate of flow between any two streamlines is equal to the difference in numerical values of their constants.

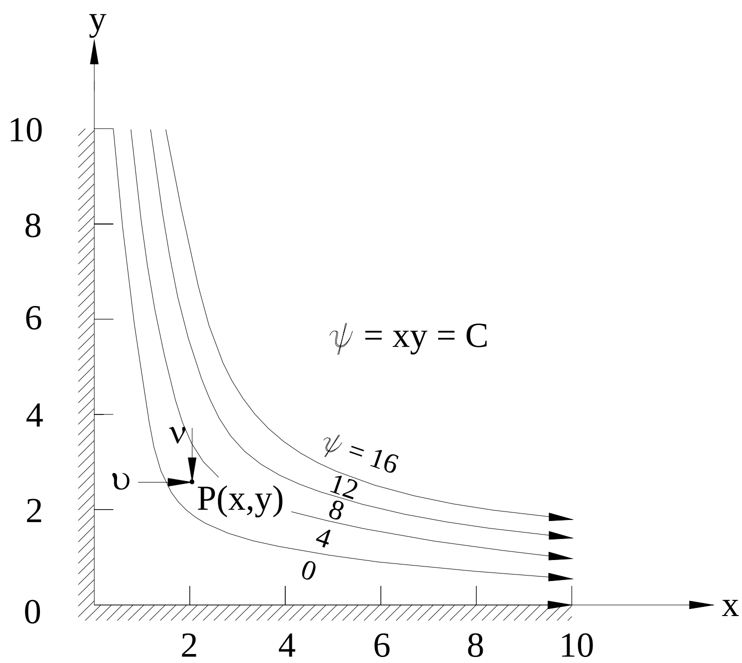


Fig. 6.8

EXAMPLE 6.3: A stream function is given by

$$\psi = 3x^2 - y^3$$

Determine the magnitude of velocity components at the point (3,1).

SOLUTION: The x and y components of velocity are given by

x-component: $u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(3x^2 - y^3\right) = -3y^2$

y-component: $v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left(3x^2 - y^3\right) = -6x$

At the point (3,1)

u = -3 and v = -18

and the total velocity is the vector sum of the two components.

$$\vec{V} = -3\vec{i} - 18\vec{j}$$

Note that ∂u/∂x=0 and ∂v/∂y=0, so that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Therefore the given stream function satisfies the continuity equation.

The equation for vorticity,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{6.14}$$

may also be expressed in terms of ψ by substituting Eqs. (6.12) and (6.13) into Equ. (6.14)

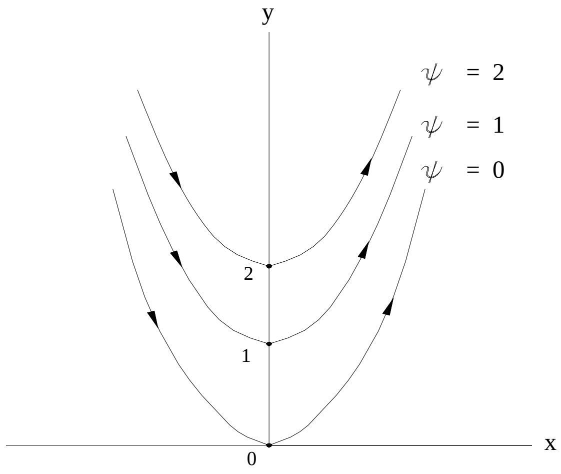
$$\zeta = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}$$

However, for irrotational flows, ζ = 0, and the classic Laplace equation,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0$$

results. This means that the stream functions of all irrotational flows must satisfy the Laplace equation and that such flows may be identified in this manner; conversely, flows whose ψ does not satisfy the Laplace equation are rotational ones. Since both rotational and irrotational flow fields are physically possible, the satisfaction of the Laplace equation is no criterion of the physical existence of a flow field.

EXAMPLE 6.4: A flow field is described by the equation $\psi = y - x^2$. Sketch the streamlines ψ = 0, ψ=1, and ψ = 2. Derive an expression for the velocity V at any point in the flow field. Calculate the vorticity.



SOLUTION: From the equation for ψ, the flow field is a family of parabolas symmetrical about the y-axis with the streamline ψ = 0 passing through the origin of coordinates.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(y - x^2\right) = 1$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left(y - x^2\right) = 2x$$

Which allows the directional arrows to be placed on streamlines as shown.
The magnitude V of the velocity may be calculated from

$$V = \sqrt{u^2 + v^2} = \sqrt{1 + 4x^2}$$

and the vorticity by Equ. (6.14)

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(1) = 2\sec^{-1} \quad \text{(Counter clockwise)}$$

Since $\zeta \neq 0$, this flow field is seen to be rotational one.

6.8. VELOCITY POTENTIAL FUNCTIONS

When the flow is irrotational, a mathematical function called the *velocity potential* function ϕ may also be found to exist. A velocity potential function ϕ for a steady, irrotational flow in the xy-plane is defined as a function of x and y, such that the partial derivative ϕ with respect to displacement in any chosen direction is equal to the velocity in that direction. Therefore, for the x and y directions,

$$u = \frac{\partial \phi}{\partial x} \tag{6.15}$$

$$v = \frac{\partial \phi}{\partial y} \qquad (6.16)$$

These equations indicate that the velocity potential increases in the direction of flow. When the velocity potential function ϕ is equated to a series of constants, equations for a family of equipotential lines are the result.

The continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (6.3)$$

may be written in terms of ϕ by substitution Eqs. (6.15) and (6.16) into the Equ. (6.3), to yield The Laplacian differential equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = 0 \qquad (6.17)$$

Thus all practical flows (which must conform to the continuity principle) must satisfy the Laplacian equation in terms of ϕ.

Similarly, the equation of vorticity,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad (6.14)$$

may be put in terms of ϕ to give

$$\zeta = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y}$$

from which a valuable conclusion may be drawn: Since,

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial x \partial y}$$

the vorticity must be zero for the existence of a velocity potential. From this it may be deduced that only irrotational ($\zeta = 0$) flow fields can be characterized by a velocity potential ϕ; for this reason *irrotational* flows are also known as *potential* flows.

RELATION BETWEEN STREAM FUNCTION AND VELOCITY POTENTIAL

A geometric relationship between streamlines and equipotential lines may be derived from the foregoing equations and restatement of certain mathematical definitions; the latter are (with definitions of u and v inserted)

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -vdx + udy$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = udx + vdy$$

However, along a streamline ψ is constant and $d\psi = 0$, so along a streamline,

$$\frac{dy}{dx} = \frac{v}{u}$$

also along any equipotential line ϕ is constant and $d\phi = 0$, so along an equipotential line;

$$\frac{dy}{dx} = -\frac{v}{u}$$

The geometric significance of this is seen in Fig. 6.9. *The equipotential lines are normal to the streamlines*. Thus the streamlines and equipotential lines (for potential flows) form a net, called a *flow net*, of mutually perpendicular families of lines, a fact of great significance for the study of flow fields where formal mathematical expressions of ϕ and ψ are unobtainable. Another feature of the velocity potential is that the value of ϕ drops *along the direction of the flow*, that is, $\phi_3 < \phi_2 < \phi_1$.

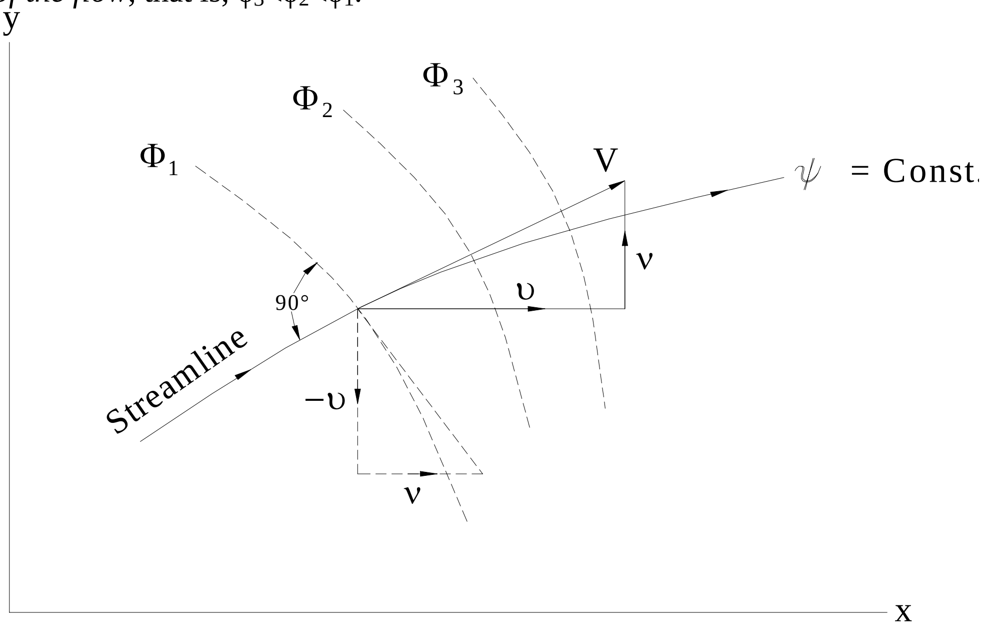


Fig. 6.9

It is important to note that the stream functions are not restricted to irrotational (potential) flows, whereas the velocity potential function exists only when the flow is irrotational because the velocity potential function always satisfies the condition of irrotationality (Equ. 6.8). The partial derivative of u in Equ. (6.15) is always equal to the partial derivative v in Equ. (6.16)

For any flow pattern the velocity potential function ϕ is related to the stream function ψ by the means of the two velocity components, u and v, at any point (x, y) in the Cartesian coordinate system in the form of the two following equations:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad (6.18)$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad (6.19)$$

EXAMPLE 6.5: A stream function in a two-dimensional flow is ψ = 2xy. Show that the flow is irrotational (potential) and determine the corresponding velocity potential function ϕ.

SOLUTION: The given stream function satisfies the condition of irrotationality, that is,

$$w = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

$$= \frac{1}{2}\left[\frac{\partial^2}{\partial x^2}(2xy) + \frac{\partial^2}{\partial y^2}(2xy)\right] = 0$$

which shows that the flow is irrotational. Therefore, a velocity potential function ϕ will exist for this flow.

By using Equ. (6.18)

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x$$

Therefore,

$$\phi = \int 2x \partial x = x^2 + f_1(y) \qquad \text{(a)}$$

From Equ. (6.19)

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(2xy) = -2y$$

From this equation,

$$\phi = \int -2y \partial y = -y^2 + f_2(x) \qquad \text{(b)}$$

The velocity potential function,

$$\phi = x^2 - y^2 + C$$

satisfies both ϕ functions in Equations a and b.

EXAMPLE 6.6: In a two-dimensional, incompressible flow the fluid velocity components are given by: u = x – 4y and v = -y - 4x. Show that the flow satisfies the continuity equation and obtain the expression for the stream function. If the flow is potential (irrotational) obtain also the expression for the velocity potential.

SOLUTION: For incompressible, two-dimensional flow, the continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

but u = x – 4y and v = -y – 4x.

$$\frac{\partial u}{\partial x}=1 \qquad \frac{\partial v}{\partial y}=-1$$

Therefore, 1 – 1 = 0 and the flow satisfies the continuity equation.

To obtain the stream function, using Eqs. (6.12) and (6.13)

$$u=\frac{\partial \psi}{\partial y}=x-4y \qquad \text{(a)}$$

$$v=-\frac{\partial \psi}{\partial x}=y+4x \qquad \text{(b)}$$

Therefore, from (a),

$$\psi=\int (x-4y)\partial y+f(x)+C$$

$$=xy-2y^2+f(x)+C$$

But, if $\psi_0 = 0$ at x = 0 and y = 0, which means that the reference streamline passes through the origin, then C = 0 and

$$\psi=xy-2y^2+f(x) \qquad \text{(c)}$$

To determine f (x), differentiate partially the above expression with respect to x and equate to –v, equation (b):

$$\frac{\partial \psi}{\partial x}=y+\frac{\partial}{\partial x}f(x)=y+4x$$

$$f(x)=\int 4x\partial x=2x^2$$

Substitute into (c)

$$\psi = 2x^2 + xy - 2y^2$$

To check whether the flow is potential, there are two possible approaches:

(a) Since

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

but

$$v = -(4x + y) \qquad \text{and} \qquad u = x - 4y$$

Therefore,

$$\frac{\partial v}{\partial x} = -4 \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -4$$

so that

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -4 + 4 = 0$$

and flow is potential.

(a) Laplace's equation must be satisfied,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0$$

$$\psi = 2x^2 + xy - 2y^2$$

Therefore,

$$\frac{\partial \psi}{\partial x} = 4x + y \quad \text{and} \quad \frac{\partial \psi}{\partial y} = x - 4y$$

$$\frac{\partial^2 \psi}{\partial x^2} = 4 \quad \text{and} \quad \frac{\partial^2 \psi}{\partial y^2} = -4$$

Therefore 4 – 4 = 0 and flow is potential.

Now, to obtain the velocity potential,

$$\frac{\partial \phi}{\partial x} = u = x - 4y$$

$$\phi = \int (x - 4y)\partial x + f(y) + G$$

But $\phi_0 = 0$ at x = 0 and y = 0, so that G = 0. Therefore,

$$\phi = \frac{x^2}{2} - 4yx + f(y)$$

Differentiating with respect to y and equating to v,

$$\frac{\partial \phi}{\partial y} = -4x + \frac{d}{dy} f(y) = -4x - y$$

$$\frac{d}{dy} f(y) = -y \qquad \text{and} \qquad f(y) = -\frac{y^2}{2}$$

so that

$$\phi = \frac{x^2}{2} - 4yx - \frac{y^2}{2}$$

6.10. THE FLOW NET

In any two-dimensional steady flow problem, the mathematical solution is to determine the velocity field of flow expressed by the following two velocity components:

$$u = f_1(x, y)$$
$$v = f_2(x, y)$$

However, if the flow is irrotational, the problem can also be solved graphically by means of a *flow net* such as the one shown in Fig.6.10. This is a network of mutually perpendicular streamlines and equipotential lines. The streamlines, which show the direction of flow at any point, are so spaced that there is an equal rate of flow Δq discharging through each stream tube. The discharge Δq is equal to the change in ψ from one streamline to the next. The equipotential lines are then drawn everywhere normal to the streamlines. The spacings of equipotential lines are selected in such a way that the change in velocity potential from one equipotential line to the next is constant. Furthermore, that is, $\Delta\psi = \Delta\phi$. As a result they form approximate squares (Fig. 6.10)

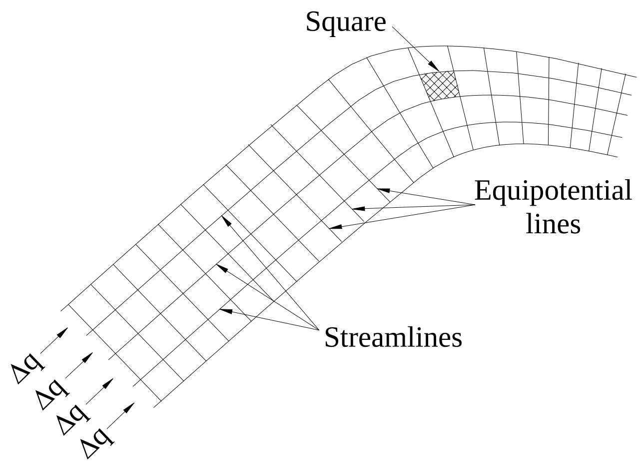


Fig.6.10

From the continuity relationship, the distances between both sets of lines must therefore be inversely proportional to the local velocities. Thus the following relation is a key to the proper construction of any flow net.

$$\frac{v_1}{v_2} = \frac{\Delta n_2}{\Delta n_1} = \frac{\Delta s_2}{\Delta s_1}$$

Where Δn and Δs are respectively the distance between streamlines and between equipotential lines.

Since there is only one possible pattern of flow for a given set of boundary conditions, a flow net, if properly constructed, represents a unique mathematical solution for a steady, irrotational flow. Whenever the flow net is used, the hydrodynamic condition of irrotationality (Equ. 6.8) must be satisfied.

The flow net must be used with caution. The validity of the interpretation depends on the extent to which the assumption of ideal (nonviscous) fluid is justified. Fortunately, such fluids as water and air have rather small viscosity so that, under favorable conditions of flow, the condition of irrotationality can be approximately attained. In practice, flow nets can be constructed for both the flow within solid boundaries (Fig.6.11) and flow around a solid body (Fig. 6.12).

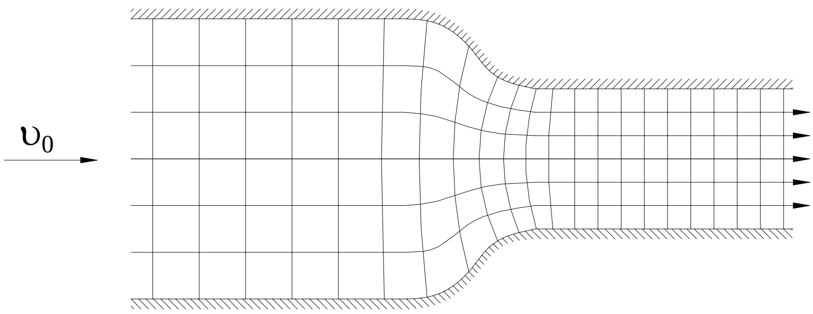


Fig. 6.11

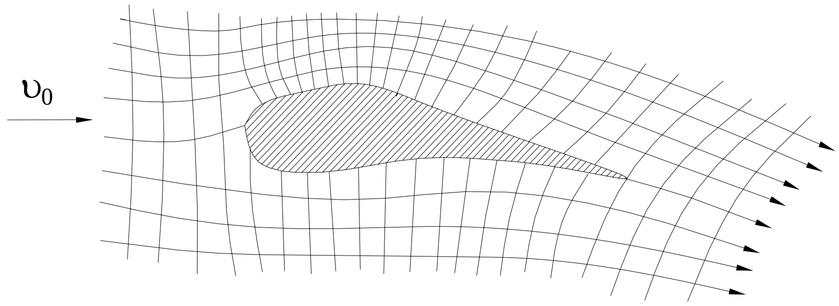


Fig. 6.12

In either flow the boundary surfaces also represent streamlines. Other streamlines are then sketched in by eye. Next, the equipotential lines are drawn everywhere normal to the streamlines. The accuracy of the flow net depends on the criterion that both sets of lines must form approximate squares. Usually a few trials will be required before a satisfactory flow net is produced.

After a correct flow net is obtained, the velocity at any point in the entire field of flow can be determined by measuring the distance between the streamlines (or the equipotential lines), provided the magnitude of velocity at a reference section, such as the velocity of flow v_0 in the straight reach of the channel in Fig. 6.11, or the velocity of approach v_0 in Fig. 6.12, is known. It is seen from both that the magnitude of local velocities depends on the configuration of the boundary surface. Both flow nets give an accurate picture of velocity distribution in the entire field of flow, except for those regions in the vicinity of solid boundaries where the effect of fluid viscosity becomes appreciable.

6.11. GROUND WATER FLOW

The flow net and flow field superposition techniques may also be applied to the flow of real fluids under some restrictions, which are frequently encountered in engineering practice. Consider the one-dimensional flow of an incompressible real fluid in a stream tube. The Bernoulli equation written in differential form is

$$d\left(\frac{p}{\gamma}+\frac{V^2}{2g}+z\right)=-dh_L$$

Suppose now that V is small (so that $dV^2/2g$ may be neglected) and the head loss dh_L given by

$$dh_L=\frac{1}{K}Vdl \qquad (6.20)$$

in which dl is the differential length along the stream tube and K is a constant. The Bernoulli equation above then reduces to

$$d\left(\frac{p}{\gamma}+z\right)=-\frac{1}{K}Vdl$$

$$V=-\frac{d}{dl}K\left(\frac{p}{\gamma}+z\right)$$

and, if this may be extended to the two-dimensional case,

$$u=-\frac{\partial}{\partial x}K\left(\frac{p}{\gamma}+z\right)=\frac{\partial\phi}{\partial x} \qquad (6.21)$$

$$v=-\frac{\partial}{\partial x}K\left(\frac{p}{\gamma}+z\right)=\frac{\partial\phi}{\partial y} \qquad (6.22)$$

and $K\left(\frac{p}{\gamma}+z\right)$ is seen to be the velocity potential of such flow field.

The conditions of the foregoing hypothetical problem are satisfied when fluid flows in a laminar condition through a homogenous porous medium. The media interest are those having a set of interconnected pores that will pass a significant volume of fluid, for example, sand, and the certain rock formations. The head-loss law (Equ. 6.20) is usually written as

$$V=K\frac{dh_L}{dl}=-K\frac{dh}{dl}$$

(where $h=p/\gamma+z$) and is an experimental relation called Darcy's law; K is known as the *coefficient of permeability*, has the dimensions of velocity, and ranges in value from 3×10^{-11} m/sec for clay to 0.3 m/sec for gravel.

A Reynolds number is defined for porous media flow as Re = Vd/ν, where V is the apparent velocity or specific discharge (Q/A) and d is a characteristic length of the medium, for example, the effective or median grain size in sand. When Re<1 the flow is laminar and Darcy's linear law is valid. If Re>1 it is likely that the flow is turbulent, that $V^2/2g$ is not negligible, and Equ. 6.20 is not valid. Note that V is not the actual velocity in the pores, but is the velocity obtained by measuring the discharge Q through an area A. The average velocity in the pores is V_p = V/n where n is the porosity of the medium;

n=(Volume of voids) / (Volume of solids plus voids)

Even though the actual fluid flow in the porous medium is viscous dominated and rotational, the "apparent flow" represented by V and the velocities u and v (Equations 6.21 and 6.22) is irrotational. Both the flow net and superposition of flow field concepts can be used. The flow net is very useful in obtaining engineering information for the "seepage flow" of water through or under structures, to wells and under drains, or for the flow of petroleum through the porous materials of subsurface "reservoirs". Flow field superposition is most useful in defining the flow pattern in ground water aquifers under the action of recharge and withdrawals wells.