LECTURE NOTES - IV

« FLUID MECHANICS »

Prof. Dr. Atıl BULU

Istanbul Technical University
College of Civil Engineering
Civil Engineering Department
Hydraulics Division

CHAPTER 4

BASIC EQUATIONS FOR ONE-DIMENSIONAL FLOW

4.1. EULER'S EQUATION OF MOTION

Consider a streamline and select a small cylindrical fluid system for analysis as shown in Fig. 4.1.

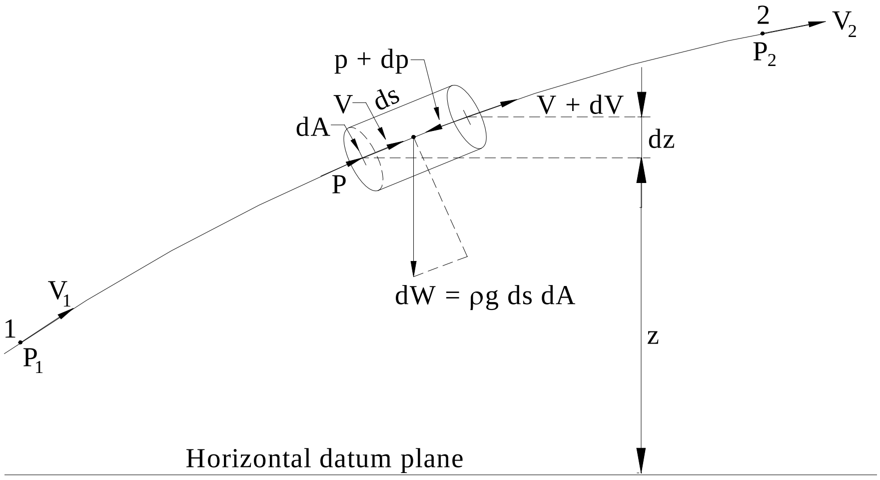

Fig. 4.1

The forces tending to accelerate the cylindrical fluid system are: forces on the ends of the system,

$$pdA - (p + dp)dA = -dpdA$$

and the component of weight in the direction of motion,

$$-\rho gdsdA\frac{dz}{ds} = -\rho gdAdz$$

The differential mass being accelerated by the action of these differential forces is,

$$dm = \rho dsdA$$

Applying Newton's second law dF = dm×a along the streamline and using the one-dimensional expression for acceleration gives

$$-dpdA - \rho gdAdz = (\rho dsdA)V\frac{dV}{ds}$$

Dividing by ρdA produces the one dimensional Euler equation,

$$\frac{dp}{\rho} + VdV + gdz = 0$$

This equation is divided by g and written

$$d\left(\frac{p}{\gamma} + \frac{V^2}{2g} + z\right) = 0$$

4.2. BERNOULLI'S EQUATION

The one-dimensional Euler equation can be easily integrated between any points (because γ and g are both constants) to obtain

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

As points 1 and 2 are any two arbitrary points on the streamline, the quantity

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = H = \text{Constant} \qquad (4.1)$$

Applies to all points on the streamline and thus provides a useful relationship between pressure p, the magnitude V of the velocity, and the height z above datum. Equ. (4.1) is known as the *Bernoulli equation* and the *Bernoulli constant H* is also termed the *total head*.

Examination of the Bernoulli terms of Equ. (4.1) reveals that p/γ and z are respectively, the pressure (either gage or absolute) and potential heads and may be visualized as vertical distances. The sum of velocity head $V^2/2g$ and pressure head p/γ could be measured by placing a tiny open tube in the flow with its open end upstream. Thus Bernoulli equation may be visualized for liquids as in Fig. 4.2, the sum of the terms (total head) being the constant distance between the horizontal datum plane and the *total headline* or *energy line (E.L.)*. The *piezometric head line* or *hydraulic grade line (H.G.L.)* drawn through the tops of the piezometer columns gives a picture of the pressure variation in the flow; evidently

1) Its distance from the stream tube is a direct measure of the static pressure in the flow,
2) Its distance below the energy line is proportional to the square of the velocity.

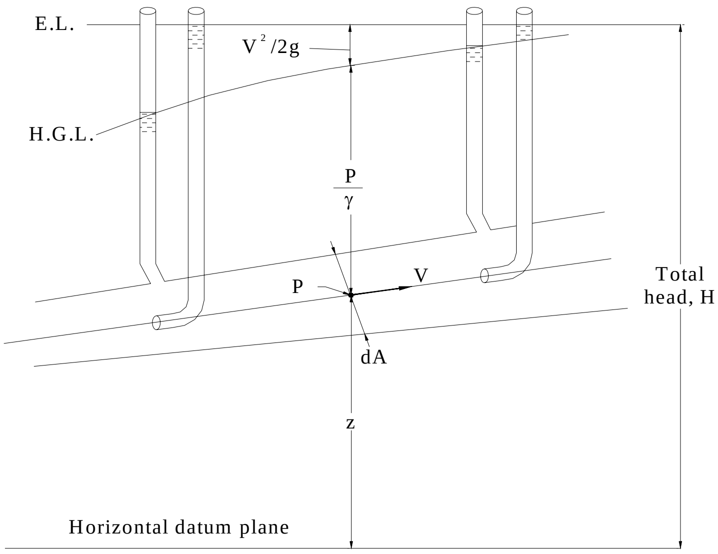

Fig. 4.2

4.3. MECHANICAL ENERGY OF A FLOWING FLUID

An element of fluid, as shown in Fig. 4.3, will posses potential energy due to its height z above datum and kinetic energy due to its velocity V, in the same way as any other object.

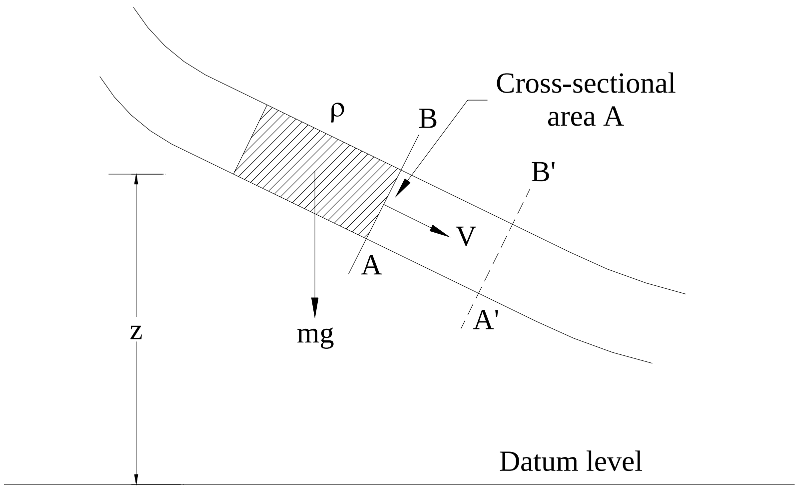

Fig. 4.3

For an element of weight mg,

Potential energy = mgz

Potential energy per unit weight = z (4.2)

Kinetic energy = $\frac{1}{2}mV^2$

Kinetic energy per unit weight = $\frac{V^2}{2g}$ (4.3)

A steadily flowing stream of fluid can also do work because of its pressure. At any given cross-section, the pressure generates a force and, as the fluid flows, this cross-section will move forward and so work will be done. If the pressure at section AB is p and the area of the cross-section is A,

Force exerted on AB = pA

After a weight mg of fluid has flowed along the stream tube, section AB will have moved to A'B':

Volume passing AB = $\frac{mg}{\rho g} = \frac{m}{\rho}$

Therefore,

Distance AA' = $\frac{m}{\rho A}$

Work done = Force×Distance AA'

$$= pA \times \frac{m}{\rho A}$$

Work done per unit weight = $\frac{p}{\rho g} = \frac{p}{\gamma}$ (4.4)

The term p/γ is known as the flow work or *pressure energy*. Note that term pressure energy refers to the energy of a fluid when flowing under pressure. The concept of pressure energy is sometimes found difficult to understand. In solid body mechanics, a body is free to change its velocity without restriction and potential energy can be freely converted to kinetic energy as its level falls. The velocity of a stream of fluid which has a steady volume rate of flow (discharge) depends on the cross-sectional area of the stream. Thus, if the fluid flows in a uniform pipe, its velocity cannot change and so the conversion of potential energy to kinetic energy cannot take place as the fluid loses elevation. The surplus energy appears in the form of an increase in pressure. As a result, pressure energy can be regarded as potential energy in transit.

Comparing the results obtained in Eqs. (4.2), (4.3) and (4.4) with Equ. (4.1), it can be seen that the three terms Bernoulli's equation are the pressure energy per unit weight, the kinetic energy per unit weight, and the potential energy per unit weight; the constant H is the total energy per unit weight. Thus, Bernoulli's equation states that, for steady flow of a frictionless fluid along a streamline, the total energy per unit weight remains constant from point to point although its division between the three forms of energy may vary:

Pressure energy per unit weight + Kinetic energy per unit weight + Potential energy per unit weight = Total energy per unit weight = Constant

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = H \qquad (4.5)$$

Each of these terms has the dimensions of a length, or head, and they are often referred to as the *pressure head* p/γ, the *velocity head* $V^2/2g$, the *potential head* z and the *total head* H. Between any two points, suffixes 1 and 2, on a streamline, Equ. (4.5) gives

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2 \qquad (4.6)$$

or

Total energy per unit weight at 1 = Total energy per unit weight at 2

In formulating Equ. (4.6), it has been assumed that no energy has been supplied to or taken from the fluid between points 1 and 2. Energy could have been supplied by introducing a pump; equally, energy could have been lost by doing work against friction in a machine such as a turbine. Bernoulli's equation can be expanded to include these conditions, giving

Total energy per unit weight at 1	=	Total energy per unit weight at 2	+	Loss Per unit weight	+	Work done per unit weight	-	Energy supplied per unit weight

EXAMPLE 4.1: A fire engine develops a head of 50 m, i.e., it increases the energy per unit weight of the water passing through it by 50 m. The pump draws water from a sump at A through a 150 mm diameter pipe in which there is a loss of energy per unit weight due to friction $h_1 = 5V_1^2/2g$ varying with the mean velocity V_1 in the pipe, and discharges it through a 75 mm nozzle at C, 30 m above the pump, at the end of a 100 mm diameter delivery pipe in which there is a loss of energy per unit weight $h_2 = 12V_2^2/2g$. Calculate,

a) The velocity of the jet issuing from the nozzle at C,
b) The pressure in the suction pipe at the inlet to the pump at B.

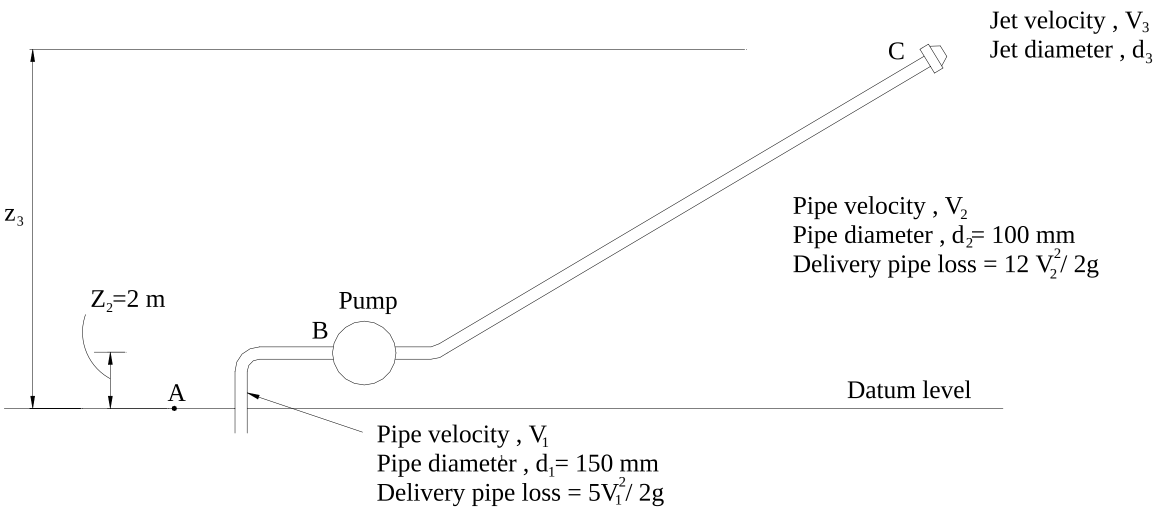

Fig.4.4

SOLUTION:

a) We can apply Bernoulli's equation in the form of Equ. (4.6) between two points, one of which will be C, since we wish to determine the jet velocity V_3, and the other point at which conditions are known, such as a point A on the free surface of the sump where the pressure will be atmospheric, so that $p_A = 0$, the velocity V_A will be zero if the sump is large, and A can be taken as datum level so that $z_A = 0$. Then,

Total energy per unit weight at A	=	Total energy per unit weight at C	+	Loss in inlet pipe	-	Energy per unit weight supplied by pump	+	Loss in discharge pipe	(I)

Total energy per unit weight at A $= \dfrac{p_A}{\gamma} + \dfrac{V_A^2}{2g} + z_A = 0$

Total energy per unit weight at C $= \dfrac{p_C}{\gamma} + \dfrac{V_3^2}{2g} + z_3$

p_C = Atmospheric pressure = 0

$z_3 = 30 + 2 = 32m$

Therefore,

Total energy per unit weight at C $= 0 + \dfrac{V_3^2}{2g} + 32 = \dfrac{V_3^2}{2g} + 32$

Loss in inlet pipe, $h_1 = 5\dfrac{V_1^2}{2g}$

Energy per unit weight supplied by pump = 50 m

Loss in delivery pipe, $h_2 = 12\dfrac{V_2^2}{2g}$

Substituting in (I),

$$0 = \frac{V_3^2}{2g} + 32 + 5\frac{V_1^2}{2g} - 50 + 12\frac{V_2^2}{2g}$$

$$V_3^2 + 5V_1^2 + 12V_2^2 = 2 \times g \times 18 \qquad \text{(II)}$$

From the continuity of flow equation,

$$\frac{\pi d_1^2}{4}V_1 = \frac{\pi d_2^2}{4}V_2 = \frac{\pi d_3^2}{4}V_3$$

Therefore,

$$V_1 = \left(\frac{d_3}{d_1}\right)^2 V_3 = \left(\frac{75}{150}\right)^2 V_3 = \frac{1}{4}V_3$$

$$V_2 = \left(\frac{d_3}{d_2}\right)^2 V_3 = \left(\frac{75}{100}\right)^2 V_3 = \frac{9}{16}V_3$$

Substituting in Equ. (II),

$$V_3^2\left[1+5\left(\frac{1}{4}\right)^2+12\left(\frac{9}{16}\right)^2\right] = 2\times g\times 1$$

$$5.109V_3^2 = 2\times g\times 18$$

$$V_3 = 8.31 m/\sec$$

b) If p_B is the pressure in the suction pipe at the pump inlet, applying Bernoulli's equation to A and B,

Total energy per unit weight at A	=	Total energy per unit weight at B	+	Loss in inlet pipe

$$0 = \frac{p_B}{\gamma} + \frac{V_1^2}{2g} + z_2 + 5\frac{V_1^2}{2g}$$

$$\frac{p_B}{\gamma} = -z_2 - 6\frac{V_1^2}{2g}$$

$$z_2 = 2m \quad , V_1 = \frac{1}{4}V_3 = \frac{8.31}{4} = 2.08 m/\sec$$

$$\frac{p_B}{\gamma} = -2 - 6\frac{2.08^2}{2g} = -3.32m$$

$$p_B = -3.32\times 1 = -3.32 t/m^2$$ (below atmospheric pressure)

4.4. THE WORK-ENERGY EQUATION

The application of work-energy principles to fluid results in a powerful relationship between fluid properties, work done, and energy transported. The Bernoulli equation is then seen to be equivalent to the mechanical work-energy equation for ideal fluid flow.

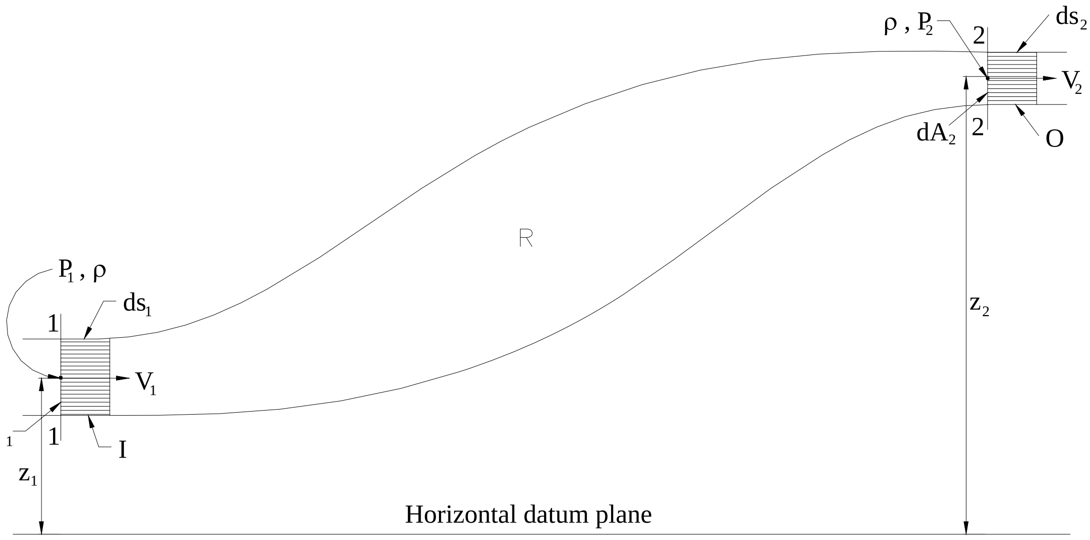

Fig. 4.5

Consider the differential stream tube section shown in Fig. 4.5 and the fluid system that occupies zones I and R of the control volume 1221 at time t and zones R and O at time t+dt. For steady flow the continuity Equ. (3.8) gives

$$dA_1V_1 = dA_2V_2 \qquad \text{or} \qquad dA_1ds_1 = dA_2ds_2$$

From dynamics, the mechanical work-energy relation (which is only an integrated form of Newton's second law) states that the work dW (expressed as a force acting over a distance) done on a system produces an equivalent change in the sum of the kinetic (KE) and potential (PE) energies of the system, that is, in time dt

$$dW = d(KE+PE) = (KE+PE)_{t+dt} - (KE+PE)_t$$

Now

$$(KE+PE)_t = (KE+PE)_R + (KE+PE)_I$$

$$(KE+PE)_t = (KE+PE)_R + \frac{1}{2}(\rho dA_1ds_1)V_1^2 + \gamma(dA_1ds_1)z_1$$

$$(KE+PE)_{t+dt} = (KE+PE)_R + (KE+PE)_O$$

$$(KE+PE)_{t+dt} = (KE+PE)_R + \frac{1}{2}(\rho dA_2ds_2)V_2^2 + \gamma(dA_2ds_2)z_2$$

Because kinetic energy of translation is $mV^2/2$ and potential energy is equivalent to the work of raising the weight of fluid in a zone to a height z above the datum.

The external work done on the system is all accomplished on cross sections 11 and 22 because there is no motion perpendicular to the stream tube so the lateral pressure forces can do no work. Also because all internal forces appear in equal and opposite pairs, there is no network done internally. The work done by the fluid entering I on the system in time dt is the *flow work.*

$$\left(p_1 dA_1\right)ds_1$$

As the system does work on the fluid in time O in time dt, the work done on the system is

$$-\left(p_2 dA_2\right)ds_2$$

In sum then

$$\left(p_1 dA_1\right)ds_1 - \left(p_2 dA_2\right)ds_2 = \frac{1}{2}\left(\rho dA_2 ds_2\right)V_2^2 + \gamma\left(dA_2 ds_2\right)z_2 - \frac{1}{2}\left(\rho dA_1 ds_1\right)V_1^2 - \gamma\left(dA_1 ds_1\right)z_1$$

Dividing by $dA_1 ds_1 = dA_2 ds_2$ produces

$$p_1 - p_2 = \frac{1}{2}\rho V_2^2 + \gamma z_2 - \frac{1}{2}\rho V_1^2 - \gamma z_1$$

When rearranged, this is recognized as Bernoulli's equation,

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 \qquad (4.7)$$

Which can be interpreted now as a mechanical energy equation. Terms such as p_1/γ, $V_1^2/2g$, z have the units of meters which represent energy per unit weight of fluid.

4.5. KINETIC ENERGY CORRECTION FACTOR

The derivation of Bernoulli's equation has been carried out for a stream tube assuming a uniform velocity across the inlet and outlet sections. In a real fluid flowing in a pipe or over a solid surface, the velocity will be zero at the solid boundary and will increase as the distance from the boundary increases. The kinetic energy per unit weight of the fluid will increase in a similar manner. If the cross-section of the flow is assumed to be composed of a series of small elements of area dA and the velocity normal to each element is u, the total kinetic energy passing through the whole cross-section can be found by determining the kinetic energy passing through an element in unit time and then summing by integrating over the whole area of the section,

$$\text{Kinetic energy} = \int_A \frac{u^2}{2} dm = \int_A \frac{u^2}{2}\rho u dt dA = \frac{\rho dt}{2}\int_A u^3 dA$$

Which can be readily integrated if the exact velocity is known. It is, however, more convenient to express the kinetic energy in terms of average velocity V at the section and a *kinetic energy correction factor* α such that

$$\text{K.E.} = \alpha \frac{V^2}{2} m = \alpha \frac{\rho dt}{2} A V^3$$

In which m = ρAVdt is the total mass of the fluid flowing across the cross-section during dt. By comparing the two expressions for kinetic energy, it is obvious that

$$\alpha = \frac{1}{AV^3} \int_A u^3 dA \qquad (4.8)$$

Mathematically, the cube of the average is less than the average of cubes, that is,

$$V^3 < \frac{1}{A} \int_A u^3 dA$$

The numerical value of α will always be greater than 1. Then Equ. (4.6) takes the form of Equ. (4.9) by taking kinetic energy correction factor, α, as

$$\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 \qquad (4.9)$$

The factor α depends on the shape of the cross-section and the velocity distribution. In most engineering problems of turbulent flow in circular pipes, α has a numerical value ranging from 1.01 to 1.10. Here α can be assumed to be unity without introducing any serious error.

The energy equation (Equ.4.9) for the flow system becomes

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 \qquad (4.7)$$

Which is identical to the energy equation for fluid flow along a streamline.

4.6. APPLICATIONS OF BERNOULLI'S EQUATION

Although there is always some friction loss in the flow of real fluids, in many engineering problems the assumption of frictionless flow may yield satisfactory results.

An important feature of Bernoulli's equation is in its graphical representation of the three terms, p/γ, z, and $V^2/2g$, at each section of the flow system. In Fig. 4.6 there is shown a typical example of steady flow of an *ideal* fluid from a large reservoir through a system of pipes varying in size and terminating in a nozzle.

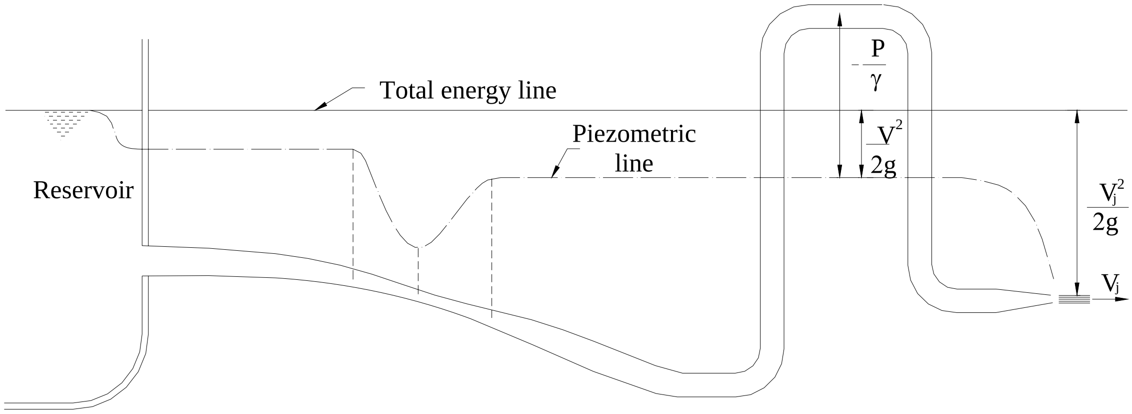

Fig. 4.6

Since the cross-sectional area of the reservoir is very large and there the velocity is zero, both the total energy line and piezometric line coincide with the free surface in the reservoir. The total energy line of the whole flow system must be horizontal if the flow is assumed to be frictionless (ideal). The flow in the pipes at various sections follows the continuity principle, that is, Q = VA. The volume rate of flow (discharge) Q can be determined by writing Bernoulli's equation to relate flow conditions at the free surface of the reservoir and the jet at the nozzle outlet. Thus the velocity head $V^2/2g$ at any section is determined and, finally the piezometric line is sketched in at a distance of $V^2/2g$ below the total energy line. The distance between the piezometric line and the line of the pipe is the pressure head p/γ at any section. If the piezometric line stays above the centerline of the pipe, the pressure head is positive. Otherwise, negative pressure prevails throughout the region where the piezometric line falls below the centerline of the pipe. Care must be taken in dealing with the regions of negative pressure because of the adverse effect of *cavitation*, a phenomenon closely associated with the regions where the local pressure drops to the *vapor pressure* of the liquid flowing in the system. At the region of low pressure liquid tends to vaporize and water bubbles start to form. These vapor bubbles are carried downstream and subsequently collapse in the zones of higher pressure. The repeated collapsing of vapor bubbles produces extremely high hydrodynamic pressures upon the solid boundaries, frequently causing severe physical damages to the boundary material. In order to avoid cavitation, it is essential to eliminate the zone of low pressure where vaporization may take place. The position of the piezometric line yields visual information on the pressure conditions of the whole flow system.

EXAMPLE 4.2: The 10 cm (diameter) siphon shown in Fig. 4.7 is filled with water and discharging freely into the atmosphere through a 5 cm (diameter) nozzle. Assume frictionless flow and find, a) the discharge in cubic meter per second and, b) the pressure at C, D, E, and F.

SOLUTION:

a) Since the flow is assumed to be frictionless, Bernoulli's equation (Equ. 4.6) can be applied to points A and G with elevation datum at G and zero gage pressure as pressure datum:

$$\frac{p_A}{\gamma}+z_A+\frac{V_A^2}{2g}=\frac{p_G}{\gamma}+z_G+\frac{V_G^2}{2g}$$

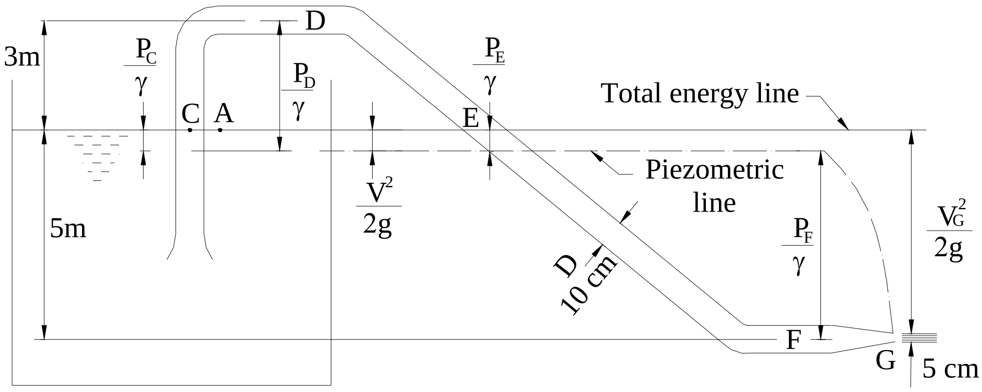

Fig. 4.7

Point A is at the reservoir surface where atmospheric pressure prevails and velocity is negligible. Therefore,

$$0+5+0=0+0+\frac{V_G^2}{2g}$$

$$V_G=\sqrt{2g\times 5}=9.90\,m/\sec$$

$$Q=A_GV_G=\frac{\pi}{4}\times 0.05^2\times 9.90=0.0194\,m^3/\sec$$

b) From the continuity equation the velocity in the 10 cm diameter siphon is,

$$V=\frac{Q}{A}=\frac{4Q}{\pi D^2}=\frac{4\times 0.0194}{\pi\times 0.10^2}=2.48\,m/\sec$$

By using the same elevation and pressure data as in (a) and writing Bernoulli's equation to relate flow conditions at A and C,

$$0+5+0=\frac{p_C}{\gamma}+5+\frac{2.48^2}{2g}$$

from which

$$\frac{p_C}{\gamma}=-0.31m \text{ of water}$$

$$p_C=-0.31t/m^2$$

The minus sign indicates vacuum.

Similarly, between A and D

$$0+5+0=\frac{p_D}{\gamma}+8+\frac{2.48^2}{2g}$$

$$\frac{p_D}{\gamma}=-3.31m \text{ of water}$$

$$p_D=-3.31t/m^2$$

Between A and E,

$$0+5+0=\frac{p_E}{\gamma}+5+\frac{2.48^2}{2g}$$

$$\frac{p_E}{\gamma}=-0.31m \text{ of water}$$
$$p_E=-0.31t/m^2$$

Between A and F,

$$0+5+0=\frac{p_F}{\gamma}+0+\frac{2.48^2}{2g}$$

$$\frac{p_F}{\gamma}=4.69m \text{ of water}$$
$$p_F=4.69t/m^2$$

The total energy line and piezometric line shown in Fig. 4.7 are self-explanatory.

4.6.1. Torricelli's Theorem

The classical *Torricelli's theorem,* which was formulated through experimentation, states that the velocity of liquid flowing out of an *orifice* is proportional to the square root of the height of liquid above the center of the orifice. This statement can readily be proved by applying Bernoulli's equation.

The reservoir in Fig. 4.8 is filled with a liquid of specific weight γ to a height h above the center of a round orifice O in its side. It is assumed, 1) That both the free surface of the liquid in the reservoir and the liquid jet are exposed to the atmospheric pressure, that is, $p_A = p_0$; 2) That the liquid surface in the reservoir remains constant; and 3) That the surface area of the reservoir is large compared with the cross-sectional area of the orifice. Thus the velocity head at A may be neglected, that is, $V_A^2/2g \cong 0$. Therefore, if the elevation at the center of the orifice is taken as a datum, Bernoulli's equation for flow conditions between A and O becomes,

$$\frac{p_A}{\gamma}+h+0=\frac{p_O}{\gamma}+0+\frac{V_o^2}{2g}$$

or, solving for V_o,

$$V_O=\sqrt{2gh} \qquad (4.10)$$

Which is known as Torricelli's theorem. It is worth noting here that the *velocity of efflux* is identical to the velocity attained in free fall.

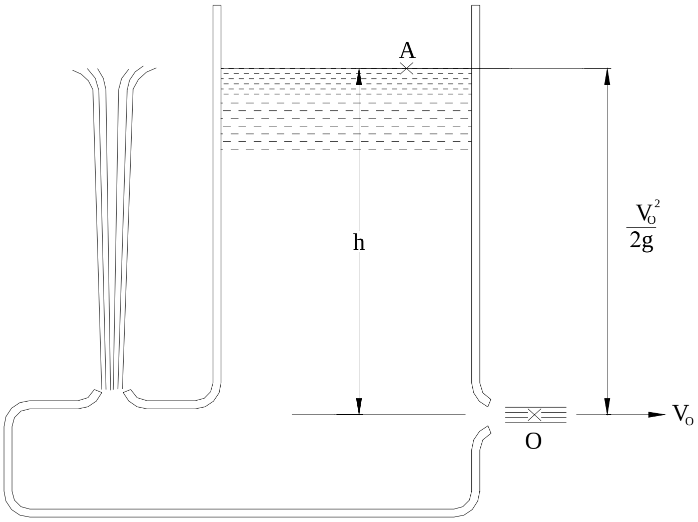

Fig. 4.8

EXAMPLE 4.3: The pressurized tank shown has a circular cross-section 2 m in diameter. Oil is drained through a nozzle 0.08m in diameter in the side of the tank. Assuming that the air pressure is maintained constant, how long does it take to lower the oil surface in the tank by 2 m? The specific weight of the oil in the tank is 0.75 t/m^3 and that of mercury is 13.6 t/m^3.

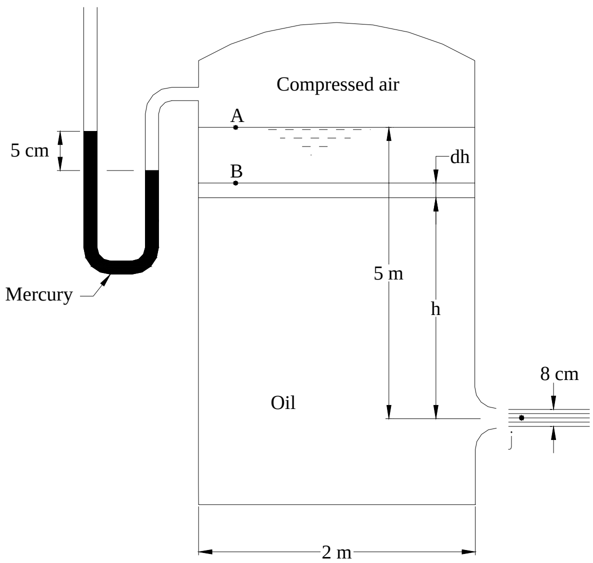

Fig. 4.9

SOLUTION: The pressure head in the tank above the oil surface is maintained at a constant value of,

$$\frac{p_A}{\gamma_{oil}} = 0.05 \times \frac{13.6}{0.75} = 0.91m \text{ of oil}$$

Since the oil surface drops constantly, the discharge out of the nozzle will vary with time. By neglecting the friction loss, Bernoulli's equation (Equ. 4.6) is written for flow conditions between a point B in the oil when the oil surface is at a height h above the center of the nozzle and the point j in the jet.

$$\frac{p_B}{\gamma_{oil}} + z_B + \frac{V_B^2}{2g} = \frac{p_j}{\gamma_{oil}} + z_j + \frac{V_j^2}{2g}$$

The cross-sectional area of the tank is very much larger than the jet area; V_B is approximately zero for all practical purposes. The jet issuing out of the nozzle is subject to the atmospheric pressure. By choosing the center of the nozzle as the elevation datum and the local atmospheric pressure as the pressure datum,

$$0.91 + h + 0 = 0 + 0 + \frac{V_j^2}{2g}$$

$$V_j = \sqrt{2g(0.91 + h)}$$

Which is the instantaneous velocity of the jet when the oil surface is at h above the center of the nozzle.

From the continuity equation the total discharge out of the nozzle during a small interval dt must be equal to the reduction in the volume of oil in the tank. Thus,

$$A_j V_j dt = A_t dh$$

or

$$\frac{\pi}{4} \times 0.08^2 \sqrt{2g(h + 0.91)}dt = -\frac{\pi}{4} \times 2^2 \times dh$$

from which

$$\int_0^t dt = -\frac{625}{\sqrt{2g}} \int_0^3 (0.91 + h)^{-1/2} dh$$

By integrating

$$t = -\frac{625}{\sqrt{2g}} \left[2(0.91 + h)^{1/2}\right]_5^3 = 128 \sec$$

4.6.2. Free Liquid Jet

A free liquid jet is actually a streamline along which the pressure is atmospheric. Accordingly, Bernoulli's equation may be applied to the whole *trajectory* of the jet if the air resistance is neglected. With the pressure head p/γ equal to zero, the sum of velocity head and elevation head above any arbitrarily chosen datum must remain constant for all points along the trajectory (Fig. 4.10).

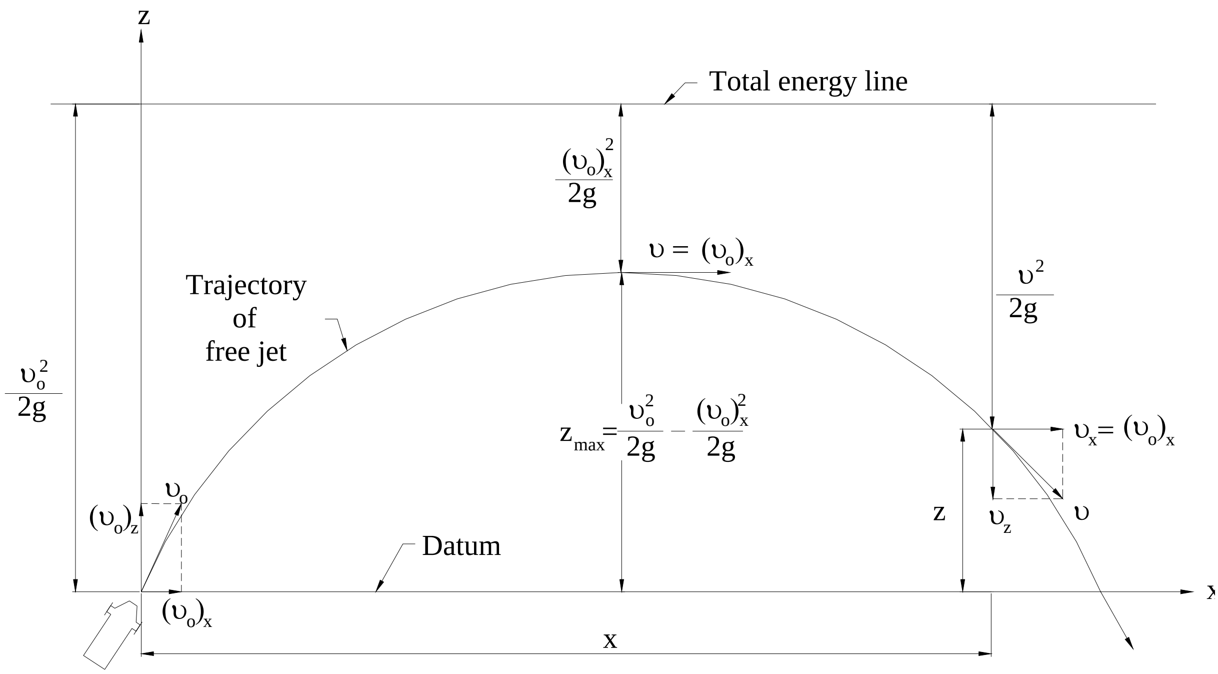

Fig. 4.10

Therefore,

$$z + \frac{V^2}{2g} = \text{Constant}$$

The total energy line is shown to be horizontal and at a distance of $V^2/2g$ above trajectory. The velocity at any point of the jet may be determined from its components V_x and V_y. Thus V equals $(V_x^2 + V_y^2)^{1/2}$. Here basic equations of projectile motion in physics are used to determine the velocity components at any point along the trajectory:

$$V_x = (V_0)_x$$
$$V_z = (V_o)_z - gt$$

The coordinates of the trajectory are expressed as follows:

$$x = (V_0)_x t$$
$$z = (V_0)_z t - \frac{1}{2} g t^2$$

Where t is the elapsed after the liquid jet leaves the nozzle.

EXAMPLE 4.4: Water is discharged from a 5 cm (diameter) nozzle, which is inclined at a 30^0 angle above the horizontal. If the jet strikes the ground at a horizontal distance of 5 m and a vertical distance of 1 m from the nozzle as shown in Fig. 4.11, what is the discharge in cubic meter per second?

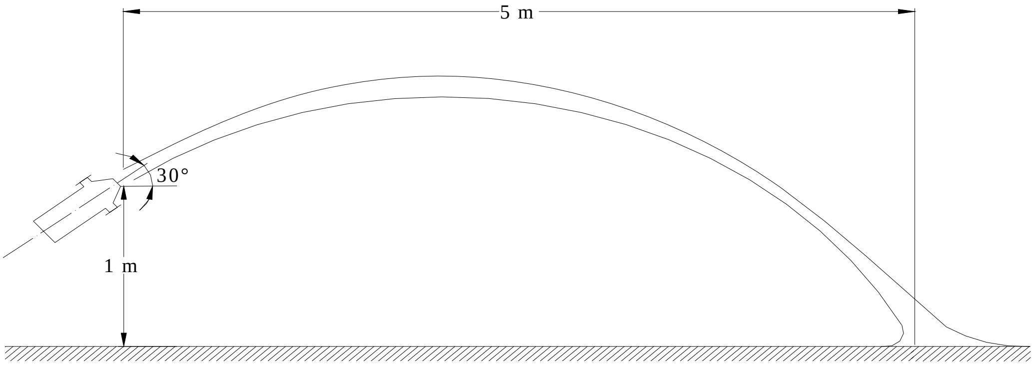

Fig. 4.11

SOLUTION:

$$(V_O)_x = V_j Cos30^0 = 0.866V_j$$
$$(V_O)_z = V_j Sin30^0 = 0.5V_j$$

Therefore the two coordinate equations for the trajectory are

$$x = (V_j Cos30^0)t$$

$$z = (V_j Sin30^0)t - \frac{1}{2}gt^2$$

By eliminating t and solving for V_j from these two equations,

$$V_j = \frac{x}{Cos30^0}\left[\frac{g}{2(xTan30^0 - z)}\right]^{1/2}$$
$$= \frac{5}{0.866}\left[\frac{9.81}{2(5\times 0.577 - 1)}\right]^{1/2}$$

$$V_j = 9.31 m/\text{sec}$$

Hence

$$Q = A_j V_j = \frac{\pi}{4}\times 0.05^2 \times 9.31 = 0.0183 m^3/\text{sec}$$

4.6.3. Venturimeter

If the flow constriction in a pipe, as shown in Fig. 4.12, is well streamlined, the loss of energy is practically equal to zero. The difference in velocity heads, $\Delta(V^2/2g)$, at two sections across the constriction results in a change in potential heads, $\Delta h_{pz} = \Delta(p/\gamma) + \Delta z$. Such a device is used for metering the quantity of flow Q in the pipe system by measuring the difference in potential head.

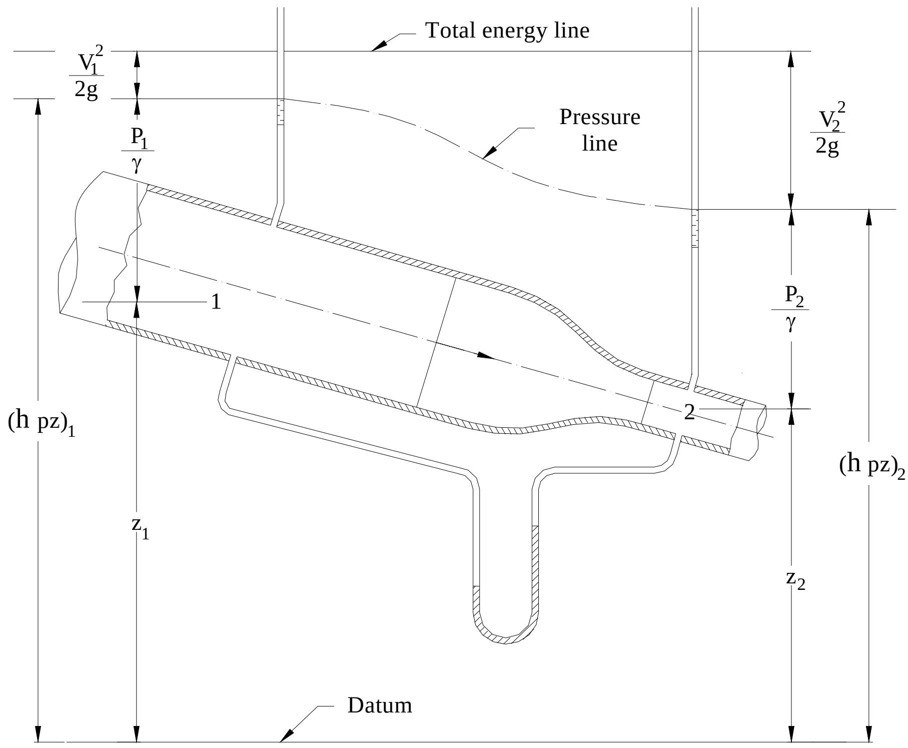

Fig. 4.12

The Bernoulli's equation for the two sections is

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}$$

and the continuity equation is

$$Q = A_1 V_1 = A_2 V_2$$

These two equations may be solved for V_1 and V_2 it the two potential heads $(h_{pz})_1=(p_1/\gamma)+z_1$ and $(h_{pz})_2=(p_2/\gamma)+z_2$ are known. Hence, Q may be calculated. The difference in potential heads may be measured either by means of two piezometric columns at two sections or by using a differential manometer connected to the two sections.

EXAMPLE 4.5: The inclined Venturimeter shown in Fig. 4.13 is installed in a 20 cm (diameter) water pipe line and has a throat diameter of 10 cm. Water flows in the upward direction. For a manometer reading of 25 cm of mercury, what is the discharge in cubic meter per second? The specific weight of mercury is 13.6 gr/cm^3.

SOLUTION: Denote h as the vertical distance between the throat and the water-mercury interface in the manometer tube and y as the vertical distance between the centers of the two sections in which manometer tappings are located. If the friction loss of flow through the Venturimeter is neglected, Bernoulli's equation can be applied to sections A and t:

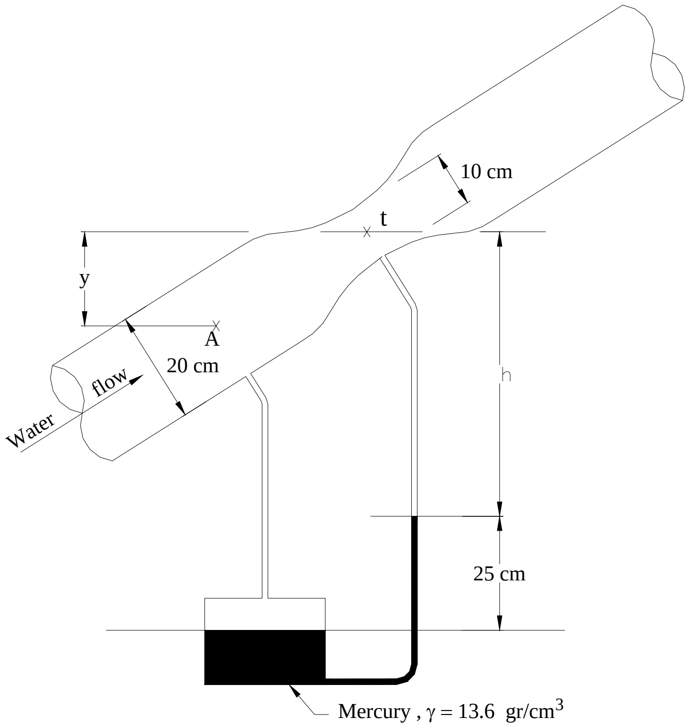

Fig. 4.13

$$\frac{p_A}{\gamma_w}+z_A+\frac{V_A^2}{2g}=\frac{P_t}{\gamma_w}+z_t+\frac{V_t^2}{2g}$$

By taking section A as the elevation datum,

$$\frac{p_A}{\gamma_w}+0+\frac{V_A^2}{2g}=\frac{P_t}{\gamma_w}+y+\frac{V_t^2}{2g} \qquad \text{(a)}$$

The difference in pressure head ($p_A/\gamma_w - p_t/\gamma_w$) is determined by writing pressure equation for the differential manometer starting at section A:

$$\frac{p_A}{\gamma_w}+(h-y+0.25)-0.25\times 13.6-h=\frac{p_t}{\gamma_w}$$

From this

$$\frac{p_A}{\gamma_w}-\frac{p_t}{\gamma_w}=y+0.25\times 13.6-0.25$$

Which now is substituted into Equ. (a) to obtain,

$$\frac{V_t^2}{2g}-\frac{V_A^2}{2g}=0.25\times 13.6-0.25=3.15m \qquad \text{(b)}$$

The continuity equation of flow is

$$A_A V_A = A_t V_t$$

$$\frac{\pi}{4} \times 0.2^2 \times V_A = \frac{\pi}{4} \times 0.1^2 \times V_t$$

$$V_A = \frac{1}{4} V_t$$

Then, by substituting $V_t/4$ for V_A in Equ. (b),

$$\frac{V_t^2}{2g}\left(1 - \frac{1}{16}\right) = 3.15$$

Hence the velocity at the throat

$$V_t = \sqrt{\frac{2g \times 3.15 \times 16}{15}} = 8.12\, m/\text{sec}$$

The discharge is

$$Q = A_t V_t = \frac{\pi}{4} \times 0.10^2 \times 8.12 = 0.064\, m^3/\text{sec}$$

4.6.4. Stagnation Tube

A stagnation tube, such as the one shown in Fig. 4.14, is simply a bent tube with its opening pointed upstream toward the approaching flow. The tip of the stagnation tube is *stagnation point*, and stagnation tube, therefore measures the *stagnation pressure* (or the total pressure), which is the sum of *static pressure* and the *dynamic pressure*.

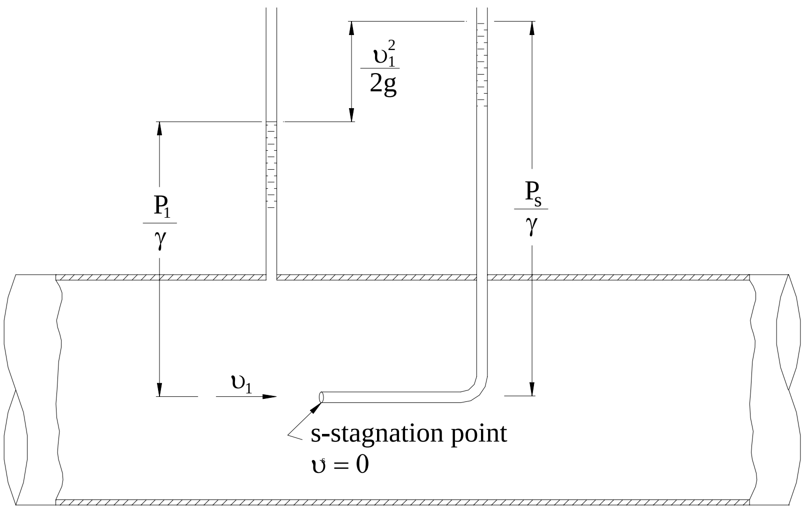

Fig. 4.14

By using the Bernoulli equation and taking the stagnation point s be section 2 as datum, the stagnation pressure p_s is determined:

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}=\frac{p_s}{\gamma}+0$$

or

$$p_s = p_1 + \frac{1}{2}\rho V_1^2 \qquad (4.11)$$

Stagnation pressure = Static pressure + Dynamic pressure

4.6.5. Pitot Tube

A typical *Pitot tube* is shown schematically in Fig. 4.15. It consists of a stagnation tube surrounded by a closed outer (static pressure) tube with annular space in between them. Small holes are drilled through the outer tube to measure the static pressure. The stagnation tube in the center measures the stagnation (total) pressure which is the sum of the static and the dynamic pressure.

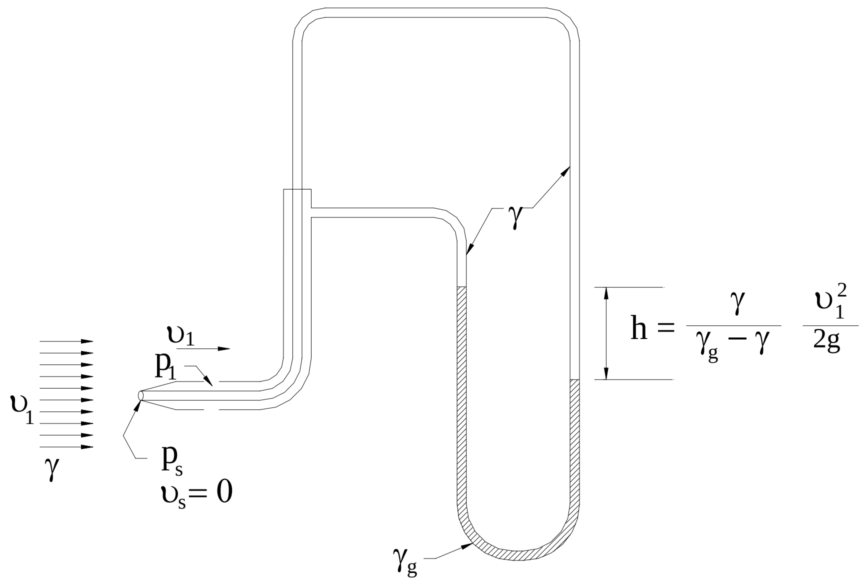

Fig. 4.15

When the two tubes are connected to a differential pressure-measuring device, the resulting difference in the pressure, p_s-p_1, is a direct measure of the velocity V_1:

$$V_1 = \sqrt{\frac{2(p_s - p_1)}{\rho}} \qquad (4.12)$$

EXAMPLE 4.6: The Pitot tube in Fig. 4.16 is carefully aligned with an air stream of specific weight 1.23 kg/m^3. If the attached differential manometer shows a reading of 150 mm of water, what is the velocity of the air stream?

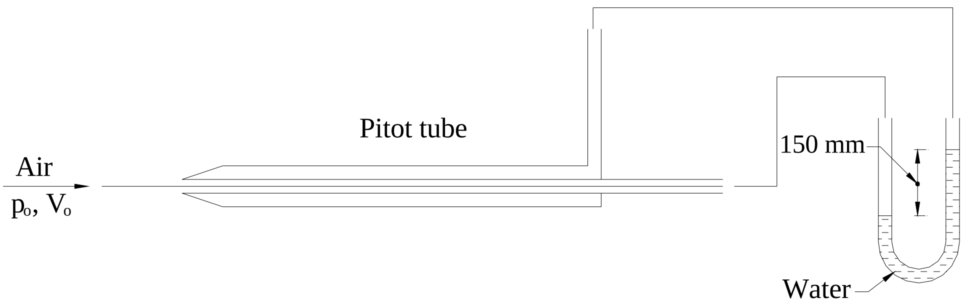

Fig. 4.16

SOLUTION: Stagnation pressure will be found at the tip of the Pitot tube. Assuming that the holes in the barrel of the static tube will collect the static pressure p_0 in the undisturbed air stream, the manometer will measure (p_s-p_0). Applying Equ. (4.11),

$$p_s = p_0 + \frac{1}{2}\rho V_0^2$$

Therefore,

$$0.15(1 - 0.00123) = \frac{1}{2} \times \frac{0.00123}{9.81} \times V_0^2$$

$$V_0 = 48.90\, m/\text{sec}$$

4.6.6. Flow Over a Weir

The Bernoulli principle may be applied to problems of open flow such as the overflow structure of Fig. 4.17. Such problems feature a moving liquid surface in contact with the atmosphere and flow pictures dominated by gravitational action. A short distance upstream from the structure, the streamlines will be straight and parallel and the velocity distribution will be uniform.

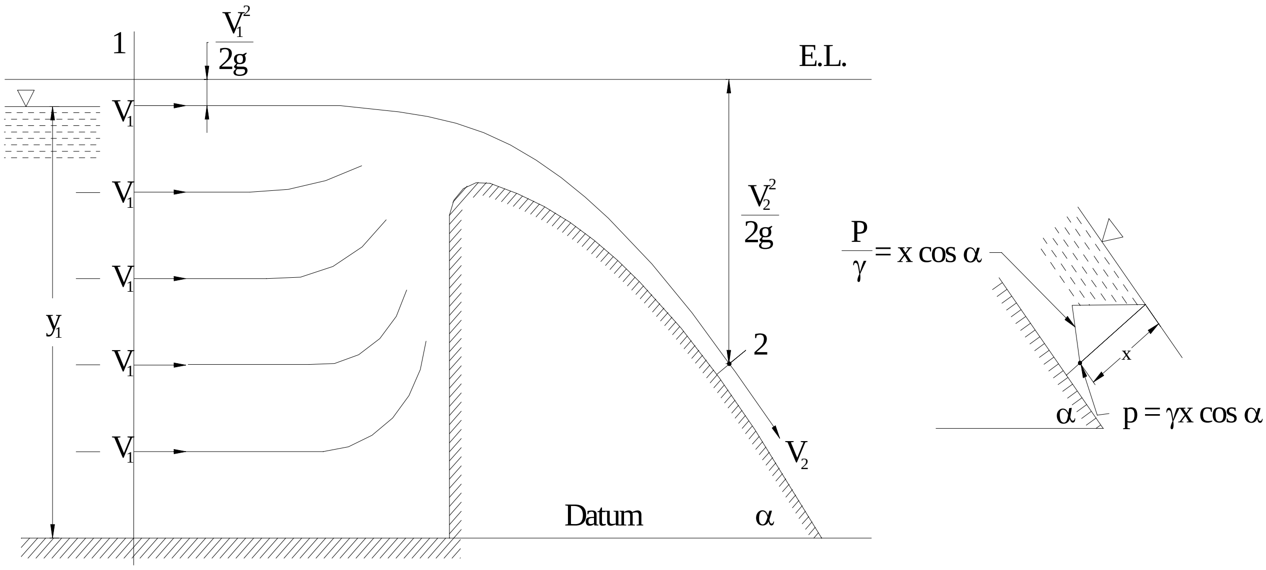

Fig. 4.17

In this region the quantity z+p/γ will be constant, the pressure distribution hydrostatic, and the hydraulic grade (piezometric) line (for all stream tube) located in the liquid surface; the energy line will be horizontal and located $V_1^2/2g$ above the liquid surface. With atmospheric pressure on the liquid surface the stream tube in the liquid surface behaves as a free jet allowing all surface velocities to be computed from the positions of liquid surface and energy line. The prediction of velocities elsewhere in the flow field where stream tubes are severely convergent or curved is outside the province of one-dimensional flow. At section 2, however (if the streamlines there are assumed straight and parallel), the pressures and velocities may be computed from the one-dimensional assumption.

EXAMPLE 4.7: Refer to Fig. 4.17. At section 2 the water surface is at elevation 30.5 m, and the 60^0 spillway surface at elevation 30 m. The velocity in the water surface V_{S2} at section 2 is 6.1 m/sec. Calculate the pressure and velocity on the spillway face at section 2. If the bottom of the approach channel is at elevation 29 m, calculate the depth and velocity in the approach channel.

SOLUTION:

Thickness of sheet of water at section 2 = $\dfrac{30.5-30}{Cos60^0}=1m$

Pressure on spillway face at section 2 = $1\times 1\times Cos60^0 0.5 t/m^2$

Elevation of energy line = $30.5+\dfrac{6.1^2}{2g}=32.40m$

$$32.4=\frac{0.5}{1}+\frac{V_{F2}^2}{2g}+3.0$$

$$V_{F2}=6.1m/\sec$$

Which is to be expected from the one-dimensional assumption. Evidently all velocities through section 2 are 6.1 m/sec, so

$q=1\times 1\times 6.1=6.1m^3/\sec$ per meter of spillway length

At section 1,

$$y_1+29.0=z_1+\frac{p_1}{\gamma}$$

and applying the Bernoulli equation,

$$y_1+\frac{V_1^2}{2g}=y_1+\left(\frac{1}{2g}\right)\times\left(\frac{6.1}{y_1}\right)^2=3.4m$$

Solving this cubic equation by trial and error, the roots are y_1 = 3.22 m, 0.85 m and -0.69 m. Obviously the second and third roots are invalid here, so depth in approach channel will be 3.22 m. The velocity V_1 may be computed from

$$\frac{V_1^2}{2g} = 3.40 - 3.22 = 0.22m$$

or from

$$V_1 = \frac{6.10}{3.22}$$

Both of which give V_1 = 1.9 m/sec.

4.1.6. The Power of a Stream of Fluid

A stream of fluid could do work as a result of its pressure p, velocity V and elevation z and that the total energy per unit weight H of the fluid is given by

$$H = \frac{p}{\gamma} + \frac{V^2}{2g} + z$$

If the weight per unit time of fluid is known, the power of the stream can be calculated, since

Power = Energy Per unit time = (Weight/Unit time)×(Energy/Unit weight)

If Q is the volume rate (discharge) of flow,

Weight per unit time $= \gamma Q = \rho g Q$

Power $= \rho g Q H = \rho g H \left(\frac{p}{\gamma} + \frac{V^2}{2g} + z \right)$

$$\text{Power} = pQ + \frac{1}{2}\rho V^2 Q + \rho g Q z \qquad (4.13)$$

EXAMPLE 4.8: Water is drawn from a reservoir, in which the water level is 240 m above datum, at the rate of 0.13 m^3/sec. The outlet of the pipeline is at datum level and is fitted with a nozzle to produce a high-speed jet to drive a turbine of the Pelton wheel type. If the velocity of the is 66 m/sec, calculate

a) The power of the jet,
b) The power supplied from the reservoir,
c) The head used to overcome losses,
d) The efficiency of the pipeline and nozzle in transmitting power.

SOLUTION:

a) The jet issuing from the nozzle will be at atmospheric pressure and at datum level so that, in Equ. (4.13), p = 0 and z = 0. Therefore,

Power of jet $= \frac{1}{2}\rho V^2 Q$

$$= \frac{1}{2} \times \frac{1000}{9.81} \times 66^2 \times 0.13 = 28862\, kgm/\sec$$
$$= 28862 \times 9.81 = 283140W = 283.14KW$$

b) At the reservoir, the pressure is atmospheric and the velocity of the free surface is zero so that, in Equ. (4.13), p = 0, V = 0. Therefore,

Power supplied from reservoir = $\rho Qgz = \gamma Qz$

$$= 1000 \times 0.13 \times 240 = 31200\, kgm/\sec$$
$$= 31200 \times 9.81 = 306072W = 306.72KW$$

c) If, H_1 = Total head at the reservoir, H_2 = Total head at the jet, h = Head lost in transmission,

Power supplied from reservoir = $\gamma Q H_1 = 31200\, kgm/\sec$

Power of issuing jet = $\gamma Q H_2 = 28862\, kgm/\sec$

Power lost in transmission = $\gamma Q h = 2338\, kgm/\sec$

Head lost in pipe = h = (Power lost)/(γQ)

$$h = \frac{2338}{1000 \times 0.13} = 17.98m$$

d) Efficiency of transmission = (Power of jet)/(Power supplied by reservoir)

Efficiency of transmission = $\frac{28862}{31200} = 0.925 = 92.5\%$

4.7. IMPULSE-MOMENTUM EQUATION: CONSERVATION OF MOMENTUM

The impulse-momentum equation for fluid flow can be derived from the well-known Newton's second law of motion. The resultant force $\vec{F}$ acting on a mass particle m is equal to the time rate of change of linear momentum of the particle:

$$\vec{F} = \frac{d\left(m\vec{V}\right)}{dt}$$

This law applies equally well to a system of mass particles. The internal forces between any two mass particles of the system exists in pairs. They are both equal and opposite of each other and, therefore, will cancel out as set forth in Newton's third law of motion. The external forces acting on mass particles of the system can then be summed up and equated to the time of change of the linear momentum of the whole system,

$$\sum \vec{F} = \frac{\sum d\left(m\vec{V}\right)}{dt} \tag{4.13}$$

If we define momentum by $\vec{I} = m\vec{V}$, then

$$\sum \vec{F} = \frac{\sum d\vec{I}}{dt} \tag{4.14}$$

Extending Newton's second law of motion to fluid flow problem, take as a free body the fluid mass included between sections 1-1 and 2-2 within a length of a flow channel (Fig. 4.18).

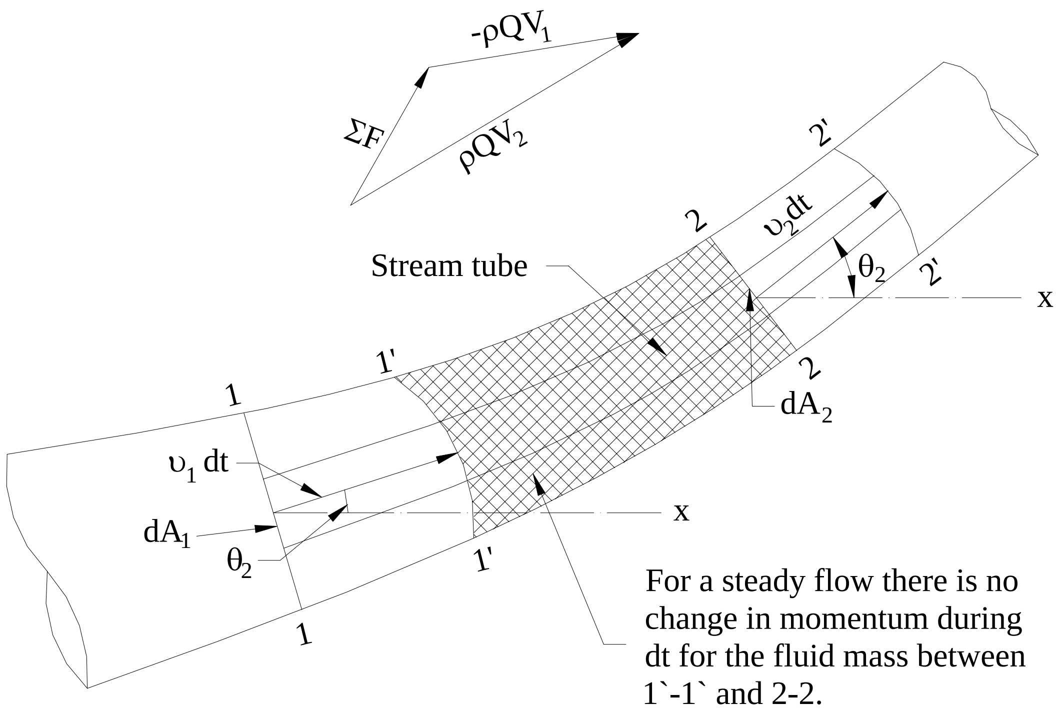

Fig. 4.18

The fluid mass of the free body 1-1 and 2-2 at time t_1 moves to a new position 1'-1' and 2'-2' at time t_2 when t_2-t_1 equals to dt. Sections 1'-1' and 2'-2' are curved because the velocities of flow at these sections are *non-uniform*. It should be noted that, for steady flow, the following continuity equation holds.

[Fluid mass within section 1-1 and 1'-1'] = [Fluid mass within section 2-2 and 2'-2']

$$M(11'1'1) = M(22'2'2)$$

The momentum of the system at time t,

$$\vec{I}_t = \vec{I}(11'1'1)_t + \vec{I}(1'221')_t$$

$$\vec{I}_t = \rho u_1 dA_1 dt \vec{u}_1 + \vec{I}(1'221')_t \qquad (4.15)$$

The momentum of the system at time t+dt,

$$\vec{I}_{t+dt} = \vec{I}(1'221')_{t+dt} + \vec{I}(22'2'2)_{t+dt}$$

$$\vec{I}_{t+dt} = \vec{I}(1'221')_{t+dt} + \rho u_2 dA_2 dt \vec{u}_2 \qquad (4.16)$$

Change of momentum at dt time,

$$d\vec{I} = \vec{I}_{t+dt} - \vec{I}_t \qquad (4.17)$$

Furthermore, when the flow is steady,

$$\vec{I}(1'221')_t = \vec{I}(1'221')_{t+dt}$$

From Eqs. (4.15) and (4.16),

$$d\vec{I} = \rho u_2 dA_2 dt \vec{u}_2 - \rho u_1 dA_1 dt \vec{u}_1$$

or

$$\frac{d\vec{I}}{dt} = \rho u_2 dA_2 \vec{u}_2 - \rho u_1 dA_1 \vec{u}_1 \qquad (4.18)$$

From Eqs. (4.14) and (4.18),

$$\sum \vec{F} = \rho u_2 dA_2 \vec{u}_2 - \rho u_1 dA_1 \vec{u}_1 \qquad (4.19)$$

This equation is known as *impulse-momentum equation*.

If, however, average velocities V_1 and V_2 at sections 1-1 and 2-2 may be determined, the impulse-momentum equation may be written,

$$\sum \vec{F} = \rho V_2 dA_2 \vec{V}_2 - \rho V_1 dA_1 \vec{V}_1 \qquad (4.20)$$

By taking integral over the areas,

$$\int \sum \vec{F} = \rho V_2 \vec{V}_2 \int_{A_2} dA_2 - \rho V_1 \vec{V}_1 \int_{A_1} dA_1$$

$$\vec{K} = \rho V_2 A_2 \vec{V}_2 - \rho V_1 A_1 \vec{V}_1 \qquad (4.21a)$$

Here, $\vec{K} = \int \sum \vec{F}$ is the total of the external forces acting on the control volume.

For a steady flow of an incompressible fluid, the impulse-momentum equation for fluid flow may be simplified to the following form by first applying the continuity principle, that is, $Q = A_1V_1 = A_2V_2$, to the flow system:

$$\vec{K} = \rho Q\left(\vec{V}_2 - \vec{V}_1\right) \qquad (4.21b)$$

If we take the velocity components for x and y-axes, Equ. (4.21b) takes the form of,

$$K_x = \rho Q\left(V_{2x} - V_{1x}\right)$$
$$K_y = \rho Q\left(V_{2y} - V_{1y}\right) \qquad (4.22)$$

These components can be combined to give the resultant force,

$$\vec{K} = \sqrt{K_x^2 + K_y^2} \qquad (4.23)$$

If D'Alembert's principle is applied to the flow system, the system is brought into relative equilibrium with the inclusion of inertia forces. The value of K is positive in the direction if V is assumed to be positive.

For any control volume, the total force K that acts upon it in a given direction will be made up of three component forces:

$\vec{K}_1 =$ Force exerted in the given direction on the fluid in the control volume by any solid body within the control volume or coinciding with the boundaries of the control volume.

$\vec{K}_2 =$ Force exerted in the given direction on the fluid in the control volume by body forces such as gravity. (Weight of the control volume).

$\vec{K}_3 =$ Force exerted in the given direction on the fluid in the control volume by the fluid outside the control volume. (Pressure forces, always gage pressures were taken).

$\rho Q\vec{V}_1 =$ Rate of change of momentum in the direction of $\vec{V}_1$.

$\rho Q\vec{V}_2 =$ Rate of change of momentum in the opposite direction of $\vec{V}_2$.

Thus,

$$\vec{K} = \vec{K}_1 + \vec{K}_2 + \vec{K}_3 = \rho Q\left(\vec{V}_2 - \vec{V}_1\right) \tag{4.24}$$

The force F exerted by the fluid on the solid body inside or coinciding with the control volume in the given direction will be equal and opposite to $\vec{K}_1$ so that $\vec{F} = -\vec{K}_1$. The directions of the forces acting on a control volume can be shown schematically in Fig. (4.19).

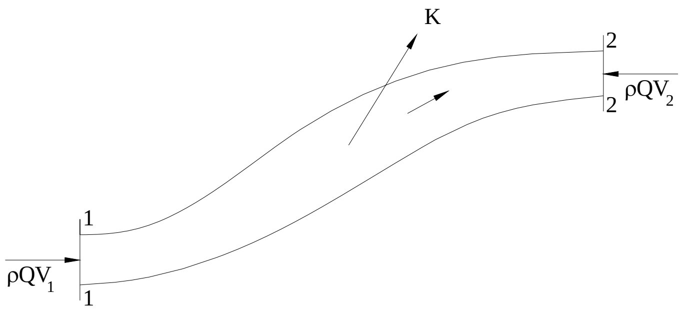

Fig. 4.19

4.7.1. Momentum Correction Factor

The momentum equation (4.24) is based on the assumption that the velocity is constant across any given cross-section. When a real fluid flows past a solid boundary, shear stresses are developed and the velocity is no longer uniform over the cross-section. In a pipe, for example, the velocity will vary from zero at the wall to a maximum at the center. Then, Equ. (4.24) takes the form of,

$$\vec{K} = \rho Q\left(\beta_2\vec{V}_2 - \beta_1\vec{V}_1\right)$$

Dimensionless *momentum correction factor* β accounts for the non-uniform distribution of velocity across the flow section. Obviously,

$$\beta = \frac{1}{AV^2}\int_A v^2 dA \tag{4.26}$$

and the numerical value of β is always greater one. In problems of turbulent flow in pipes, β is approximately equal to one.

Thus, Equ. (4.25) takes the form of Equ. (4.24) for the most practical problems of turbulent flow.

$$\vec{K} = \rho Q\left(\vec{V}_2 - \vec{V}_1\right) \tag{4.24}$$

4.7.2. Application of the Impulse-Momentum Equation

The impulse-momentum equation, together with the energy equation, and the continuity equation, furnishes the basic mathematical relationships for solving various engineering problems in fluid mechanics. In contrast to the energy equation, which is a *scalar* equation, each term in the impulse-momentum equation represents a *vector* quantity. The energy equation describes the conservation of energy and the average changes in the energies of flow along a flow passage, whereas the impulse-momentum equation relates the over-all forces on the boundaries of a chosen region in a flow channel without regard to the internal flow phenomenon. In many instances, however, both the impulse-momentum equation and the energy equation are complementary to each other.

Since the impulse-momentum equation relates the resultant external forces on a chosen free body of fluid in a flow channel to the change of momentum flux at the two end sections, it is especially valuable in solving those problems in fluid mechanics in which detailed information on the flow process may be either lacking or rather difficult to evaluate. In general, the impulse-momentum equation is used to solve the following two types of flow problems.

1) To determine the resultant forces exerted on the boundaries of a flow passage by a stream of flow as the flow *changes its direction or its magnitude of velocity or both*. Problems of this type include pipe bends and reducers, stationary and moving vanes, and jet propulsion. In such cases, although the fluid pressures on the boundaries may be determined by means of the energy equation, the resultant forces on the boundaries must be determined by integrating the pressure forces.
2) To determine the flow characteristics of non-uniform flows in which *an abrupt change of flow section occurs*. Problems of this type, such as a sudden enlargement in a pipe system or a hydraulic jump in an open-channel flow, cannot be solved by using the energy equation alone, because there is usually an unknown quantity of energy loss involved in each of these flow processes. The impulse-momentum equation must be used first to determine the flow characteristics. Then the energy equation may be used to evaluate the amount of energy loss in the flow process.

4.7.1.1. Force Exerted by a Flowing Fluid on a Contracting Pipe on a Horizontal Plane

The change of momentum of a fluid flowing through a pipe bend induces a force on the pipe. Consider the pipe bend shown in Fig. 4.20.

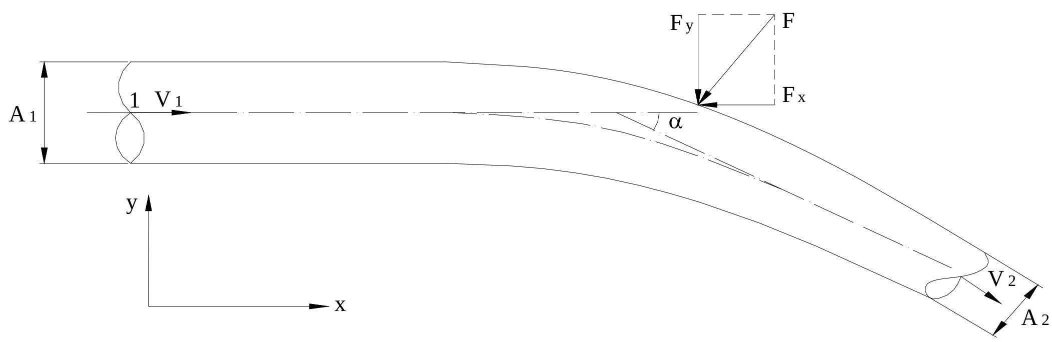

Fig. 4.20

The fluid enters the bend with velocity V_1 through area A_1 and leaves with a velocity V_2 through area A_2 after having been turned through an angle α. Let F be the force required to hold the pipe in equilibrium against the pressure of the fluid. F_x and F_y are the component forces in the negative x and y directions, respectively. Equ. (4.24) can now be employed to determine the magnitude of the force F. Along the x-direction we have,

$$\rho Q\left(V_2 Cos\alpha - V_1\right) = -F_x + p_1 A_1 - p_2 A_2 Cos\alpha \qquad \text{(a)}$$

and in the y-direction we have,

$$\rho Q\left(-V_2 Sin\alpha + 0\right) = -F_y + p_2 A_2 Sin\alpha \qquad \text{(b)}$$

Then the total force is given by,

$$F = \sqrt{F_x^2 + F_y^2} = \begin{cases} \rho^2 Q^2\left(V_1^2 + V_2^2 - 2V_1V_2 Cos\alpha\right) + p_1^2 A_1^2 + p_2^2 A_2^2 - \\ 2p_1 p_2 A_1 A_2 Cos\alpha + 2\rho Q\left[p_1 V_1 A_1 + p_2 V_2 A_2 - \left(p_2 V_1 A_2 + p_1 V_2 A_1\right)Cos\alpha\right] \end{cases}^{1/2} \qquad \text{(c)}$$

But, from the continuity equation the velocities are related by,

$$V_2 = \frac{A_1}{A_2} V_1$$

Thus, Equ. (c) becomes,

$$F = \left\{ \begin{aligned} &\rho^2 Q^2 V^2\left(1 - 2\frac{A_1}{A_2}Cos\alpha + \frac{A_1^2}{A_2^2}\right) + 2\rho Q^2\left[p_1 + p_2 - \left(p_1\frac{A_1}{A_2} + p_2\frac{A_2}{A_1}\right)Cos\alpha\right] \\ &+ p_1^2 A_1^2\left[1 + \left(\frac{p_2}{p_1}\right)^2\left(\frac{A_2}{A_1}\right)^2 - 2\left(\frac{p_2}{p_1}\right)\left(\frac{A_2}{A_1}\right)Cos\alpha\right] \end{aligned} \right\}^{1/2} \qquad \text{(d)}$$

For equal areas ($A_1 = A_2$), Equ. (d) reduces to,

$$F = \left\{2\rho Q^2\left(1 - Cos\alpha\right)\left(\rho V_1^2 + p_1 + p_2\right) + p_1^2 A_1^2\left[1 + \left(\frac{p_2}{p_1}\right)^2 - 2\left(\frac{p_2}{p_1}\right)Cos\alpha\right]\right\}^{1/2}$$

and, if the bend is 90^0, the force for the constant area bend becomes,

$$F = \left\{2\rho Q^2\left(\rho V_1^2 + p_1 + p_2 +\right) + p_1^2 A_1^2\left[1 - \left(\frac{p_1}{p_2}\right)^2\right]\right\}^{1/2}$$

EXAMPLE 4.8: When 300 lt/sec of water flow through this vertical 300 mm by 200 mm pipe bend, the pressure at the entrance is 7 t/m^2. Calculate the force by the fluid on the bend if the volume of the bend is 0.085 m^3.

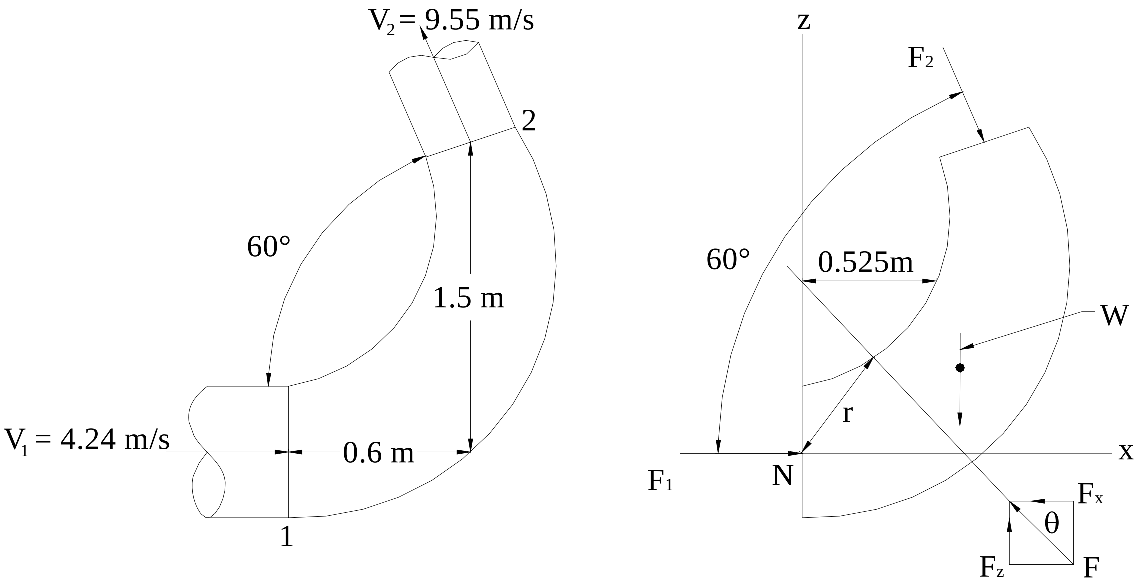

Fig. 4.21

SOLUTION: From the continuity principle,

$$Q = V_1 A_1 = V_2 A_2$$

$$V_1 = \frac{4Q}{\pi D_1^2} = \frac{4 \times 0.3}{\pi \times 0.3^2} = 4.24\, m/\sec$$

$$V_2 = \frac{4Q}{\pi D_2^2} = \frac{4 \times 0.3}{\pi \times 0.2^2} = 9.55\, m/\sec$$

and from the Bernoulli equation,

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}$$

$$7 + 0 + \frac{4.24^2}{19.62} = \frac{p_2}{\gamma} + 1.5 + \frac{9.55^2}{19.62}$$

$$\frac{p_2}{\gamma} = 1.77m \quad , \; p_2 = 1.77\, t/m^2$$

Now, for the free-body diagram, the pressure forces F_1 and F_2 may be computed,

$$F_1 = \frac{\pi}{4} \times 0.3^2 \times 7 = 0.495 ton$$

$$F_2 = \frac{\pi}{4} \times 0.2^2 \times 1.77 = 0.056 ton$$

With this and velocity diagram,

$$\sum x = 0$$

$$F_x = F_1 + \rho Q V_1 + F_2 Cos60^0 + \rho Q V_2 Cos60^2$$

$$F_x = 0.495 + \frac{1}{9.81} \times 0.30 \times 4.24 + \left(0.056 + \frac{1}{9.81} \times 0.3 \times 9.55 \right) Cos60^0$$

$$F_x = 0.799 ton$$

$$\sum z = 0$$

$$F_z = \left(F_2 + \rho Q V_2 \right) Sin60^0 + \gamma \times 0.085$$

$$F_z = \left(0.056 + \frac{1}{9.81} \times 0.30 \times 9.55 \right) Sin60^0 + 1 \times 0.085$$

$$F_z = 0.386 ton$$

$$F = \sqrt{F_x^2 + F_z^2} = \sqrt{0.779^2 + 0.386^2} = 0.869 ton$$

$$Tan\theta = \frac{F_z}{F_x} = \frac{0.386}{0.799} = 0.483 \quad , \ \theta = 25.8^0$$

The plus signs confirm the direction assumptions for F_x and F_z. Therefore the force on the bend is 0.87 ton downward to the right at 25.8^0 with horizontal.

Now, assuming that the bend is such shape that the centroid of the fluid therein is 0.525 m to the right of section 1 and that F_1 and F_2 act at the centroids of sections 1 and 2, respectively. We take moments about the center of section 1 to find the location of F,

$$-r \times 0.87 + 0.525 \times 0.085 + 1.5 \times 0.056 \times Cos60^0 + 0.6 \times 0.056 \times Sin60^0 =$$

$$-1.5 \times 9.55 \times Cos60^0 \times 0.30 \times \frac{1}{9.81} - 0.6 \times 9.55 \times Sin60^0 \times 0.30 \times \frac{1}{9.81}$$

$$r = 0.56 m$$

4.7.1.2. Reaction of a Jet

In Chapter (4.6.1) Torricelli's theorem for the efflux velocity from an orifice in a large tank was derived. The momentum theorem can now be applied to this example to determine the propulsive force created by the orifice flow. In Fig. 4.22 is shown a large tank with its surface open to the atmosphere and with an orifice area A_2.

We assume that $A_2 << A_1$. Then the velocity of the jet by Equ. (4.10),

$$V_2 = \sqrt{2gh}$$

and

$$Q = A_2 V_2$$

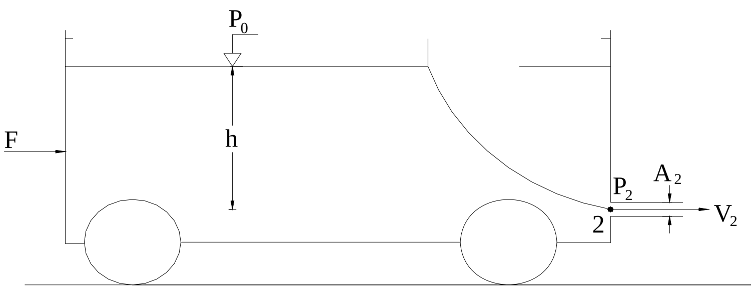

Fig. 4.22

Let F represent the force necessary to hold the tank in equilibrium. Then Equ. (4.22) becomes,

$$F = \rho Q(V_2 - 0)$$

Therefore, the propulsive force or thrust induced by the jet is given by,

$$T = -F = -\rho Q V_2$$

EXAMPLE 4.9: A jet of water of diameter d = 50 mm issues from a hole in the vertical side of an open tank which is kept filled with water to a height of 1.5 m above the center of the hole (Fig. 4.23). Calculate the reaction of the jet on the tank and its contents, a) When it is stationary, b) When it is moving with a velocity u = 1.2 m/sec in the opposite direction to the jet relative to the tank remains unchanged. In the latter case, what would be the work done per second?

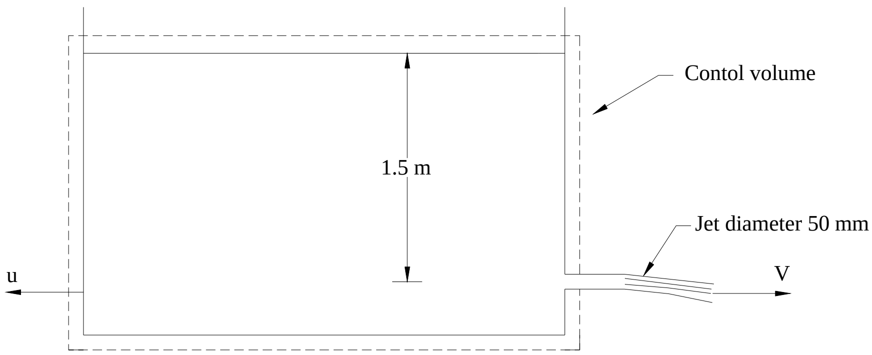

Fig. 4.23

SOLUTION: Take the control volume shown in Fig. 4.23. In Equ. (4.24), the direction under consideration will be that of the issuing jet, which will be considered as positive in the direction of the jet; therefore, $K_2 = 0$, and, if the jet is assumed to be at the same pressure as the outside of the tank, $K_3 = 0$. Force exerted by fluid system in the direction of motion,

$$R = -K_1 = -\rho Q(V_2 - V_1) \qquad \text{(a)}$$

The velocity of the jet may be found by applying Torricelli's equation (Equ. 4.10),

$$V_2 = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 1.5} = 5.42\, m/\text{sec}$$

Mass discharge per unit time,

$$M = \rho Q = \rho \frac{\pi d^2}{4} V_2 = \frac{1000}{9.81} \times \frac{\pi \times 0.05^2}{4} \times 5.42$$
$$M = 1.085\, kg \sec/m$$

a) If the tank is stationary,

V_2 = 5.42 m/sec

V_1 = Velocity at the free surface = 0

Substituting in Equ. (a), reaction of jet on the tank,

$$R = \rho Q V_2 = M V_2 = 1.085 \times 5.42 = 5.88 kg$$

b) If the tank is moving with a velocity u in the opposite direction to the of the jet, the effect is to superimpose a velocity of (–u) on the whole system:

$$V_2' = V_2 - u \qquad , V_1 = -u$$

Thus,

$$V_2 - V_1 = V_2$$

Thus, the reaction of the jet R remains unaltered at 5.88 kg.

Work done per second = Reaction × Velocity of the tank

$$W = R \times u = 5.88 \times 1.2 = 7.056\, kgm/\sec = 69.22 Watt$$

4.7.1.3.Pressure Exerted on a Plate by a Free Jet

Consider a jet of fluid directed at the inclined plate shown in Fig. 4.24. Suppose we are interested in equilibrium against the pressure of the jet.

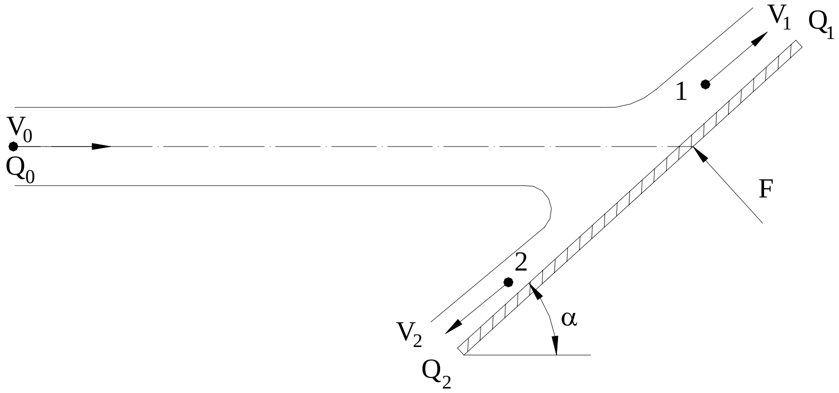

Fig. 4.24

For a free jet the static pressure is the same for all points in the jet. Thus the velocities can be related by,

$$V_0 = V_1 = V_2 \qquad \text{(a)}$$

In addition, for an inviscid fluid there can be no shearing force parallel to the plate surface; thus, the reaction force is normal to the plate surface. From the momentum theorem this force must be equal to the rate of momentum change normal to the plate surface. For this case,

$$F = \rho V_0 Q_0 Sin\alpha \qquad \text{(b)}$$

where $Q_0 = V_0 A$ and A is the cross-sectional area of the jet. It is interesting to note that the division of flow along the plate is uneven. The magnitudes of the flow rates along the plate can be determined by a consideration of the momentum theorem parallel to the plate. In this case,

$$(\rho Q_1 V_1 - \rho Q_2 V_2) - \rho Q_0 V_0 Cos\alpha = 0 \qquad \text{(c)}$$

Also, the continuity equation stipulates that,

$$Q_0 = Q_1 + Q_2 \qquad \text{(d)}$$

Combination of Eqs. (a), (c), and (d) leads to the following results for discharges:

$$Q_1 = \frac{Q_0}{2}(1 + Cos\alpha) \qquad \text{(e)}$$

$$Q_2 = \frac{Q_0}{2}(1 - Cos\alpha) \qquad \text{(f)}$$

EXAMPLE 4.10:

Calculate the force exerted by a jet of water 20 mm in diameter which strikes a flat plate at an angle of 30^0 to the normal of the plate with a velocity of 10 m/sec if, a) The plate is stationary, b) The plate is moving in the direction of the jet with a velocity of 2 m/sec.

SOLUTION:

a) The angle α shown in Fig. 4.24 is 90^0-30^0=60^0; hence the normal force [Equ.(b)] is,

$$F = \frac{1000}{9.81} \times \frac{\pi \times 0.02^2}{4} \times 10 \times 10 \times Sin60^0 = 2..77kg$$

b) The change of velocity on impact in this case is,

$$V_0 - V_p = 8\,m/\sec$$

The normal force is,

$$F = \frac{1000}{9.81} \times \frac{\pi \times 0.02^2}{4} \times 8 \times 8 \times Sin60^0 = 1.77kg$$

4.7.1.4. Stationary and Moving Vanes: The Impulse Turbine

Fig. 4.25 shows a free jet, which is deflected by a stationary curved vane through an angle α. Here the jet is assumed to impinge on the vane tangentially. Hence, there is no loss of energy because of impact. Since the friction loss of the flow passing along the smooth surface of the stationary vane is almost equal to zero, the magnitude of the jet velocity remains unchanged as it flows along the vane if the small difference in elevation between the two ends of the vane is neglected as was shown in [Equ. (c). Pressure exerted on a plate by a free jet].

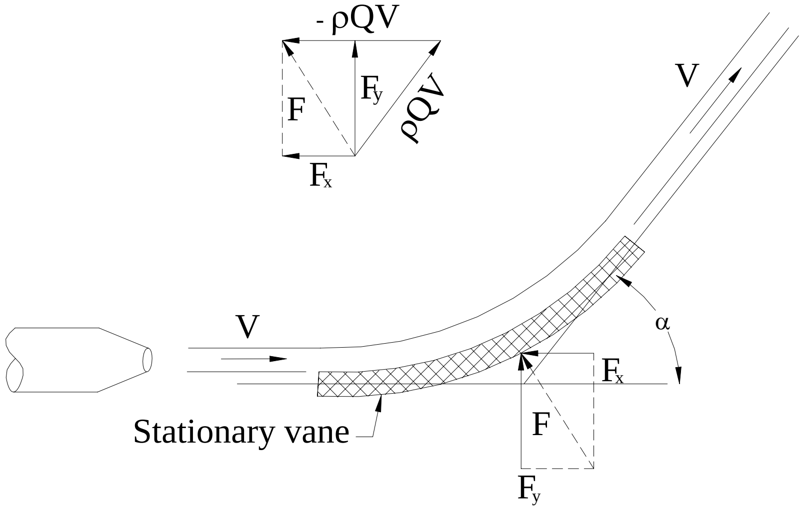

Fig. 4.25

Therefore, the two components F_x and F_y of the F exerted by the stationary vane on the jet of fluid may be determined from the following impulse-momentum equations:

$$-F_x = \rho QVCos\alpha - \rho QV$$

$$F_y = \rho QVSin\alpha - 0$$

The force components, which exerts on the vane are equal and opposite to F_x and F_y shown in Fig. 4.25.

Next, consider the moving vane in Fig. 4.26, which is moving with a velocity u in the same direction as the approaching jet. The free jet of velocity V hits the moving vane tangentially. This type of problem may be analyzed by applying the principle of relative motion to the whole system.

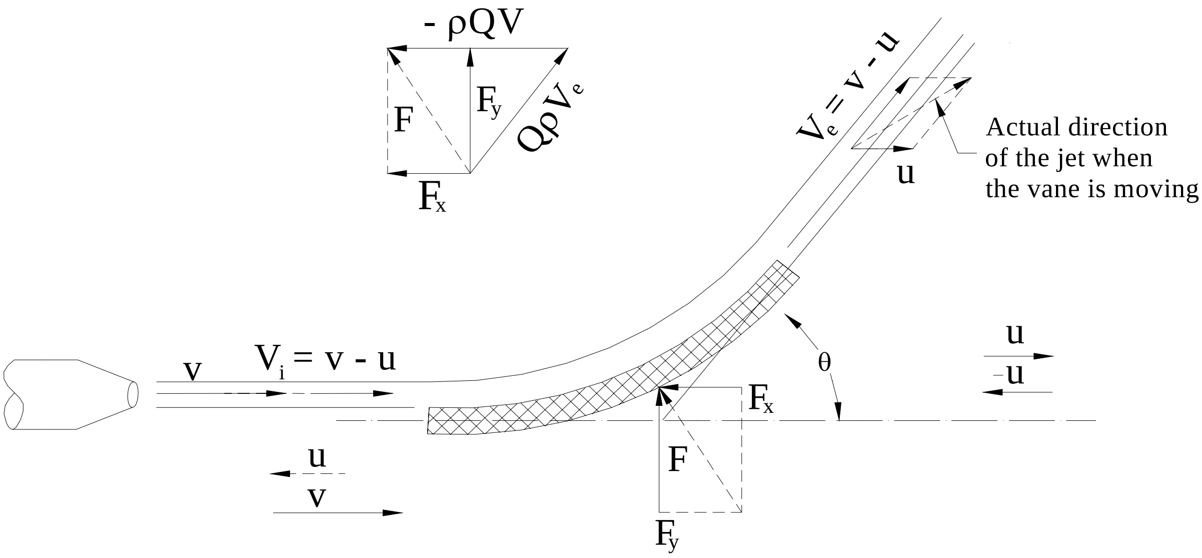

Fig. 4.26

This is done by bringing the moving vane in a stationary state before the entrance V and the exit V_e must be relative velocities of the jet at these two sections with respect to vane. The entrance velocity of the jet relative to the vane is V_i = V-u, and the magnitude of this relative velocity remains the same along the curved surface of the vane if the friction loss is assumed to be zero. Thus, V_e = V-u, and the direction of the exit velocity relative to the vane is shown in Fig. 4.26. Therefore, the force components F_x and F_y exerted by the moving vane on the jet are determined by applying the impulse-momentum equations to the flow system:

$$-F_x = Q\rho(V-u)Cos\alpha - Q\left(\frac{V_i}{V}\right)\rho(V-u)$$

$$F_x = \rho Q(V-u)(1-Cos\alpha)$$

$$F_y = \rho Q(V-u)Sin\alpha - 0$$

Again, the force components which jet exerts on the moving vane must be equal and opposite to F_x and F_y shown in Fig. 4.26.

In mechanics the *power* developed by a working agent is defined as the rate at which work done by that agent. When a jet of fluid strikes a single moving vane of Fig. 4.26, the power developed is equal to $F_x \times u$, or

Power = $\rho Q(V-u)(1-Cos\alpha)u$

Force component F_y does not produce any power because there is no motion of the vane in the y direction.

In the engineering application of the principle of moving vane to an *impulse turbine wheel* (Fig. 4.27), a series of vanes is mounted on the periphery of a rotating wheel. The vanes are usually spaced that the entire discharge Q is deflected by vanes.

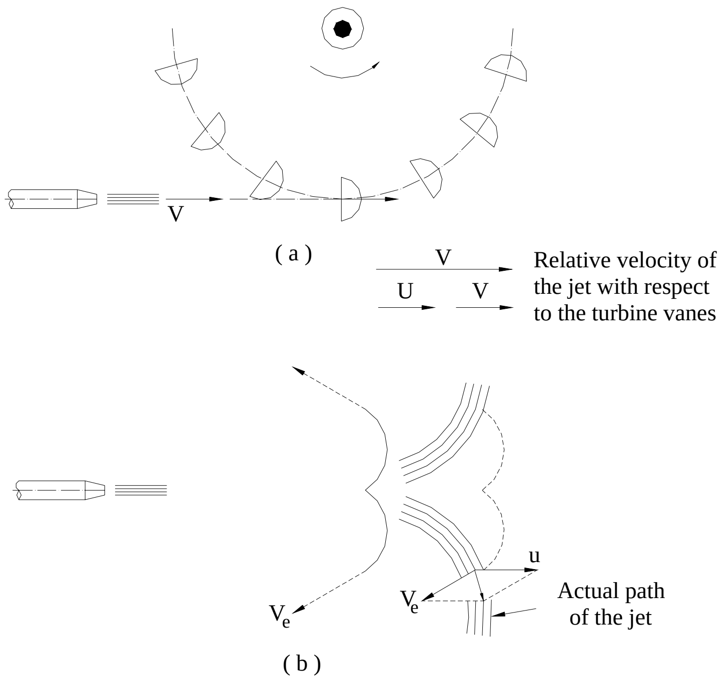

Fig. 4.27

Therefore, the total power output of a frictionless impulse turbine is,

$$P_T = \rho Q(V-u)(1-Cos\alpha)u \qquad (4.25)$$

This equation indicates that, for any free jet of discharge Q and velocity V, the power developed in an impulse turbine is seen to vary with both the deflection angle α of the vane and the velocity u at which the vanes move. Mathematically, the values of α and u to produce maximum turbine power for a given jet may be determined by taking partial derivatives $\partial P_T/\partial\alpha$ and $\partial P_T/\partial u$ and then equating them zero. Thus,

$$\frac{\partial P_T}{\partial \alpha} = \rho Q(V-u)u(1+Sin\alpha) = 0$$

and

$$\alpha = 180^0 \quad ; \ (P_T)_{\alpha=180^0} = 2\rho Qu(V-u)$$

Also,

$$\frac{\partial P_T}{\partial u} = \rho Q(1-Cos\alpha)(V-2u) = 0$$

and,

$$u = \frac{V}{2} \qquad ; \ (P_T)_{u=V/2} = \rho Q(1-Cos\alpha)\frac{V^2}{4}$$

The *maximum turbine power* is obtained when $\alpha=180^0$ and u=V/2.

Therefore,

$$(P_T)_{max} = \rho Q\frac{V^2}{2}$$

which is exactly the power in the free jet of fluid. In practice, however, the deflection angle of the vanes on an impulse wheel is found to be about 170 degrees and the periphery speed of the impulse wheel to be approximately u = 0.45V.

EXAMPLE 4.11: An impulse turbine of 1.8 m diameter is driven by a water jet of 50 mm diameter moving at 60 m/sec. Calculate the force on the blades and the power developed at 250 r/min. The blade angles are 150^0.

SOLUTION: The velocity of the impulse wheel is,

$$u = \frac{250}{60} \times 2\pi \times 0.9 = 23.6\, m/\sec$$

The flow rate is,

$$Q = \frac{\pi}{4} \times 0.05^2 \times 60 = 0.12\, m^3/\sec$$

The working component of force on the fluid is,

$$F_x = \rho Q (V - u)(1 - Cos\alpha)$$

$$F_x = \frac{1000}{9.81} \times 0.12 \times (60 - 23.6)(1 - Cos150^0)$$

$$F_x = 831 kg$$

The power developed,

$$P = F_x \times u = 831 \times 23.6 = 19612\, kgm/\sec = 192390 Watt$$

4.7.1.5. Sudden Enlargement in a Pipe System

There is a certain amount of energy lost when the fluid flows through a *sudden enlargement* in a pipe system such as that shown in Fig. 4.28.

The continuity equation for the flow is,

$$Q = A_1 V_1 = A_1 V_e = A_2 V_2$$

Since the velocity V_e of the submerged jet may be assumed to be equal to V_1, by reason of Bernoulli theorem, p_e equals to p_1. This latter condition is readily verified in the laboratory.

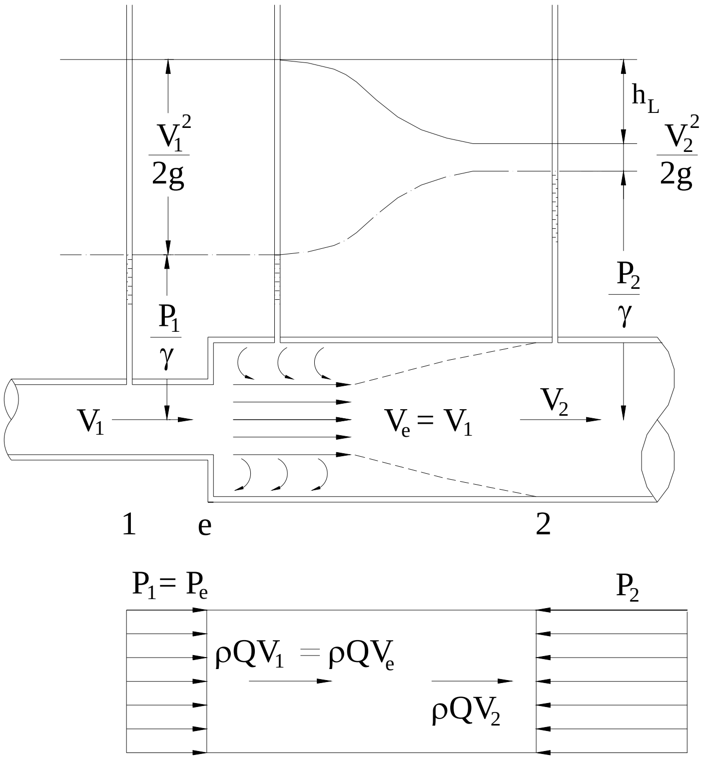

Fig. 4.28

The impulse-momentum equation for the flow of fluid in the pipe between sections e and 2 is,

$$(p_e - p_2)A_2 = \rho Q(V_2 - V_1)$$

or

$$\frac{p_1 - p_2}{\gamma} = \frac{Q}{gA_2}(V_2 - V_1) = \frac{V_1^2}{g}\left[\left(\frac{A_1}{A_2}\right)^2 - \frac{A_1}{A_2}\right]$$

The energy equation may now be written between sections 1 and 2 in the following form:

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + h_L$$

Solving these three equations simultaneously gives,

$$h_L = \frac{(V_1 - V_2)^2}{2g} = \frac{V_1^2}{2g}\left(1 - \frac{A_1}{A_2}\right)^2 \qquad (4.26)$$

This is the well-known *Borda-Carnot* equation.

4.7.1.6. Hydraulic Jump in an Open Channel Flow

A *hydraulic jump* in an open channel flow is a local phenomenon in which the surface of a rapidly flowing stream of liquid rises abruptly. This sudden rise in liquid surface is accompanied by the formation of extremely turbulent rollers on the sloping surface in the hydraulic jump as that shown in Fig. 4.29. An appreciable quantity of energy is dissipated in this process when the initial kinetic energy of flow is partly transformed into potential energy.

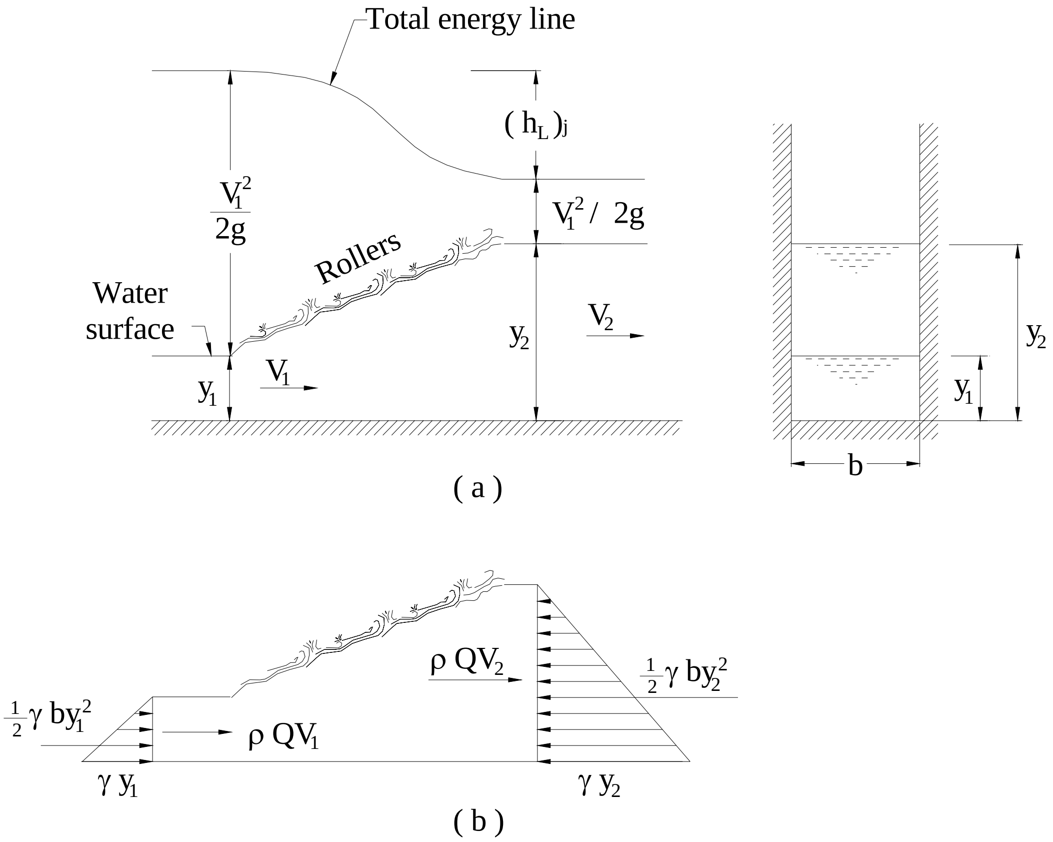

Fig. 4.29

The hydraulic jump shown in Fig. 4.29 is assumed to occur in a horizontal rectangular channel of width b (perpendicular to the plane of paper). Because the flow is guided by a solid boundary at the bottom of the channel, hydrostatic pressure distribution exists at both the upstream section 1 and downstream section 2 from the hydraulic jump. The following two assumptions are made in the mathematical analysis of a hydraulic jump: 1) The friction loss of flow at the wetted surface of the channel between sections 1 and 2 is assumed to be negligible; 2) The velocity distribution of flow at both sections is assumed to be uniform.

For steady flows the continuity equation yields,

$$Q = b\ y_1 V_1 = b y_2 V_2 \qquad \text{or} \qquad y_1 V_1 = y_2 V_2$$

and the impulse-momentum equation for the free body of the flow system (Fig.4.29b) is,

$$\frac{\gamma b y_1^2}{2} - \frac{\gamma b y_2^2}{2} = \rho Q V_2 - \rho Q V_1$$

or

$$\frac{\gamma y_1^2}{2} - \frac{\gamma y_2^2}{2} = \rho y_2 V_2^2 - \rho y_1 V_1^2$$

of the four quantities, V_1, y_1, V_2, and y_2, two must be given; the other two can be determined by the simultaneous solution of these two equations.

After the flow characteristics are all determined, the loss of energy of flow in a hydraulic jump can readily be evaluated by the application of the energy equation, that is,

$$h_L = \left(y_1 + \frac{V_1^2}{2g} \right) - \left(y_2 + \frac{V_2^2}{2g} \right) \qquad (4.27)$$

EXAMPLE 4.12: As shown in the Fig. 4.30, 10 m^3/sec of water per meter of width flows down an overflow spillway onto a horizontal floor. The velocity of flow at the toe of the spillway is 20 m/sec. Compute the downstream depth required at the end of the spillway floor to cause a hydraulic jump to form and the horsepower dissipation from the flow in the jump per meter of width.

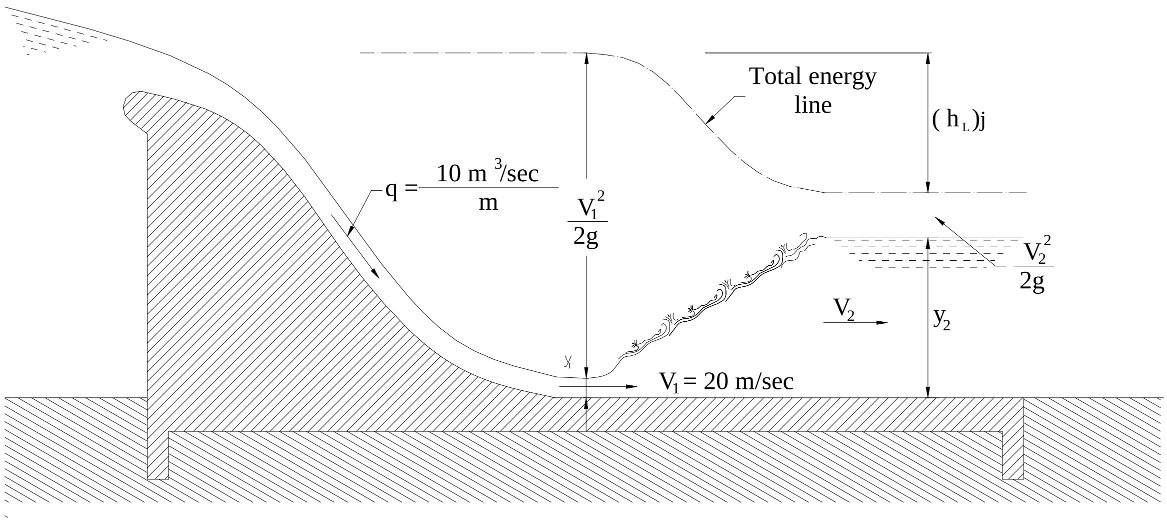

Fig. 4.30

SOLUTION: The continuity equation of flow is,

$$q = 10 m^3 / \sec / m = 20 y_1 = V_2 y_2$$

Hence,

$$y_1 = 0.5 m$$

The momentum equation relating flow conditions at two end sections of the jump is,

$$\frac{1 \times 0.5^2}{2} - \frac{1 \times y_2^2}{2} = \frac{1}{9.81} \times y_2 \times V_2^2 - \frac{1}{9.81} \times 0.5 \times 20^2$$

Solving these equations simultaneously yield,

$V_2 = 1.63\, m/\sec$ and $y_2 = 6.14m$

From Equ. (4.27) the loss of energy of flow in the hydraulic jump is,

$$h_L = \left(0.5 + \frac{20^2}{2g}\right) - \left(6.14 + \frac{1.63^2}{2g}\right)$$

$h_L = 14.60m$ per meter width

Horsepower dissipation $= \frac{\gamma Q h_L}{75} = \frac{1000 \times 10 \times 14.60}{75} = 1946Hp$