

LECTURE NOTES - III

« **FLUID MECHANICS** »

Prof. Dr. Atıl BULU

Istanbul Technical University
College of Civil Engineering
Civil Engineering Department
Hydraulics Division

CHAPTER 3

KINEMATICS OF FLUIDS

3.1. FLUID IN MOTION

Fluid motion observed in nature, such as the flow of waters in rivers is usually rather chaotic. However, the motion of fluid must conform to the general principles of mechanics. Basic concepts of mechanics are the tools in the study of fluid motion.

Fluid, unlike solids, is composed of particles whose relative motions are not fixed from time to time. Each fluid particle has its own velocity and acceleration at any instant of time. They change both respects to time and space. For a complete description of fluid motion it is necessary to observe the motion of fluid particles at various points in space and at successive instants of time.

Two methods are generally used in describing fluid motion for mathematical analysis, the *Lagrangian* method and the *Eulerian* method.

The Lagrangian method describes the behavior of the individual fluid during its course of motion through space. In rectangular Cartesian coordinate system, Lagrange adopted a , b , c , and t as independent variables. The motion of fluid particle is completely specified if the following equations of motion in three rectangular coordinates are determined:

$$\begin{aligned}x &= F_1(a, b, c, t) \\y &= F_2(a, b, c, t) \\z &= F_3(a, b, c, t)\end{aligned}\tag{3.1}$$

Eqs. (3.1) describe the exact spatial position (x, y, z) of any fluid particle at different times in terms of its *initial position* ($x_0 = a, y_0 = b, z_0 = c$) at the given initial time $t = t_0$. They are usually referred to as parametric equations of the path of fluid particles. The attention here is focused on the paths of different fluid particles as time goes on. After the equations describing the paths of fluid particles are determined, the instantaneous velocity components and acceleration components at any instant of time can be determined in the usual manner by taking derivatives with respect to time.

$$\begin{aligned}
u &= \frac{dx}{dt} \quad , \quad a_x = \frac{du}{dt} = \frac{d^2x}{dt^2} \\
v &= \frac{dy}{dt} \quad , \quad a_y = \frac{dv}{dt} = \frac{d^2y}{dt^2} \\
w &= \frac{dz}{dt} \quad , \quad a_z = \frac{dw}{dt} = \frac{d^2z}{dt^2}
\end{aligned} \tag{3.2}$$

In which u , v , and w , and a_x , a_y , and a_z are respectively the x , y , and z components of velocity and acceleration.

In the Eulerian method, the individual fluid particles are not identified. Instead, a fixed position in space is chosen, and the velocity of particles at this position as a function of time is sought. Mathematically, the velocity of particles at any point in the space can be written,

$$\begin{aligned}
u &= f_1(x, y, z, t) \\
v &= f_2(x, y, z, t) \\
w &= f_3(x, y, z, t)
\end{aligned} \tag{3.3}$$

Euler chose x , y , z , and t as independent variables in his method.

The relationship between Eulerian and Lagrangian methods can be shown. According to the Lagrangian method, we have a set of Eqs. (3.2) for each particle which can be combined with Eqs. (3.3) as follows:

$$\begin{aligned}
\frac{dx}{dt} &= u(x, y, z, t) \\
\frac{dy}{dt} &= v(x, y, z, t) \\
\frac{dz}{dt} &= w(x, y, z, t)
\end{aligned} \tag{3.4}$$

The integration of Eqs. (3.4) leads to three constants of integration, which can be considered as initial coordinates a , b , c of the fluid particle. Hence the solutions of Eqs. (3.4) give the equations of Lagrange (Eqs. 3.1).

Although the solution of Lagrangian equations yields the complete description of paths of fluid particles, the mathematical difficulty encountered in solving these equations makes the Lagrangian method impractical. In most fluid mechanics problems, knowledge of the behavior of each particle is not essential. Rather the general state of motion expressed in terms of velocity components of flow and the change of velocity with respect to time at various points in the flow field are of greater practical significance. Therefore the Eulerian method is generally adopted in fluid mechanics.

With the Eulerian concept of describing fluid motion, Eqs. (3.3) give a specific velocity field in which the velocity at every point is known. In using the velocity field, and noting that x, y, z are functions of time, we may establish the acceleration components $a_x, a_y,$ and a_z by employing the chain rule of partial differentiation,

$$\begin{aligned}
 a_x &= \frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt} + \frac{\partial u}{\partial t} \frac{dt}{dt} \\
 u &= f_1(x, y, z, t) \quad , \quad a_x = \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) + \left(\frac{\partial u}{\partial t} \right) \\
 v &= f_2(x, y, z, t) \quad , \quad a_y = \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) + \left(\frac{\partial v}{\partial t} \right) \quad (3.5) \\
 w &= f_3(x, y, z, t) \quad , \quad a_z = \left(u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) + \left(\frac{\partial w}{\partial t} \right)
 \end{aligned}$$

The acceleration of fluid particles in a flow field may be imagined as the superposition of two effects:

- 1) At a given time t , the field is assumed to become and remain steady. The particle, under such circumstances, is in the process of changing position in this steady field. It is thus undergoing a change in velocity because the velocity at various positions in this field will be different at any time t . This time rate of change of velocity due to changing position in the field is called *convective acceleration*, and is given the first parentheses in the preceding acceleration equations.
- 2) The term within the second parentheses in the acceleration equations does not arise from the change of particle, but rather from the rate of change of the velocity field itself at the position occupied by the particle at time t . It is called *local acceleration*.

3.2. UNIFORM FLOW AND STEADY FLOW

Conditions in a body of fluid can vary from point to point and, at any given point, can vary from one moment of time to the next. Flow is described as *uniform* if the velocity at a given instant is the same in magnitude and direction at every point in the fluid. If, at the given instant, the velocity changes from point to point, the flow is described as *non-uniform*.

A *steady* flow is one in which the velocity and pressure may vary from point to point but do not change with time. If, at a given point, conditions do change with time, the flow is described as *unsteady*.

For example, in the pipe of Fig. 3.1 leading from an infinite reservoir of fixed surface elevation, unsteady flow exits while the valve A is being opened or closed; with the valve opening fixed, steady flow occurs under the former condition, pressures, velocities, and the like, vary with time and location; under the latter they may vary only with location.

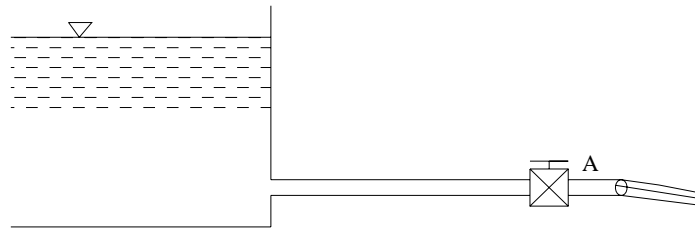


Fig. 3.1

There are, therefore, four possible types of flow.

- 1) *Steady uniform flow*. Conditions do not change with position or time. The velocity of fluid is the same at each cross-section; e.g. flow of a liquid through a pipe of constant diameter running completely full at constant velocity.
- 2) *Steady non-uniform flow*. Conditions change from point to point but not with time. The velocity and cross-sectional area of the stream may vary from cross-section to cross-section, they will not vary with time; e.g. flow of a liquid at a constant rate through a conical pipe running completely full.
- 3) *Unsteady uniform flow*. At a given instant of time the velocity at every point is the same, but this velocity will change with time; e.g. accelerating flow of a liquid through a pipe of uniform diameter running full, such as would occur when a pump is started up.
- 4) *Unsteady non-uniform flow*. The cross-sectional area and velocity vary from point to point and also change with time; a wave travelling along a channel.

3.3. STREAMLINES AND STREAM TUBES

If curves are drawn in a steady flow in such a way that the tangent at any point is in the direction of the velocity vector at that point, such curves are called *streamlines*. Individual fluid particles must travel on paths whose tangent is always in the direction of the fluid velocity at any point. Thus, path lines are the same as streamlines in steady flows.

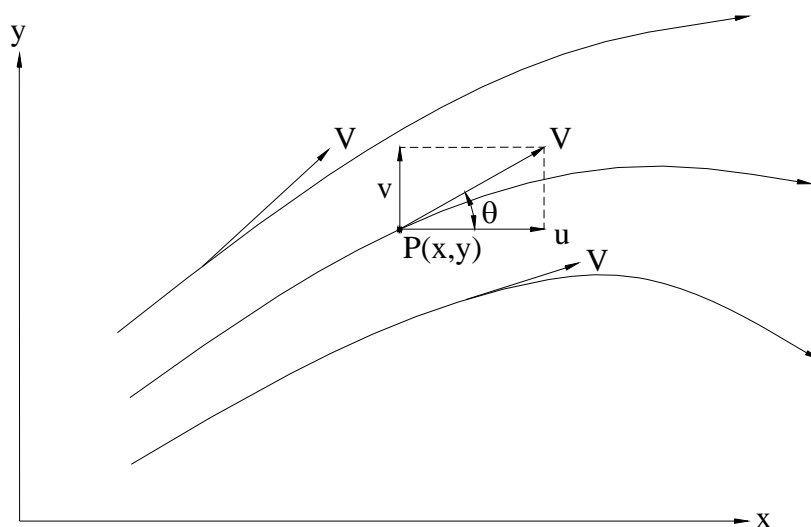


Fig. 3.2

Streamlines for a flow pattern in the xy -plane are shown in Fig. 3.2, in which a streamline passing through the point $P(x, y)$ is tangential to the velocity vector \vec{V} at P . If u and v are the x and y components of \vec{V} ,

$$\frac{v}{u} = \tan\theta = \frac{dy}{dx}$$

Where dy and dx are the y and x components of the differential displacement ds along the streamline in the immediate vicinity of P . Therefore, the differential equation for streamlines in the xy -plane may be written as

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad udy - vdx = 0 \quad (3.6)$$

The differential equation for streamlines in space is,

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad (3.7)$$

Obviously, a streamline is everywhere tangential to the velocity vector; *there can be no flow occurring across a streamline*. In steady flow the pattern of streamlines remains invariant with time.

A *stream tube* such as that shown in Fig. 3.3 may be visualized as formed by a bundle of streamlines in a steady flow field. *No flow crosses the wall of a stream tube*. Often times in simpler flow problems, such as fluid flow in conduits, the solid boundaries may serve as the periphery of a stream tube since they satisfy the condition of having no flow crossing the wall of the tube.

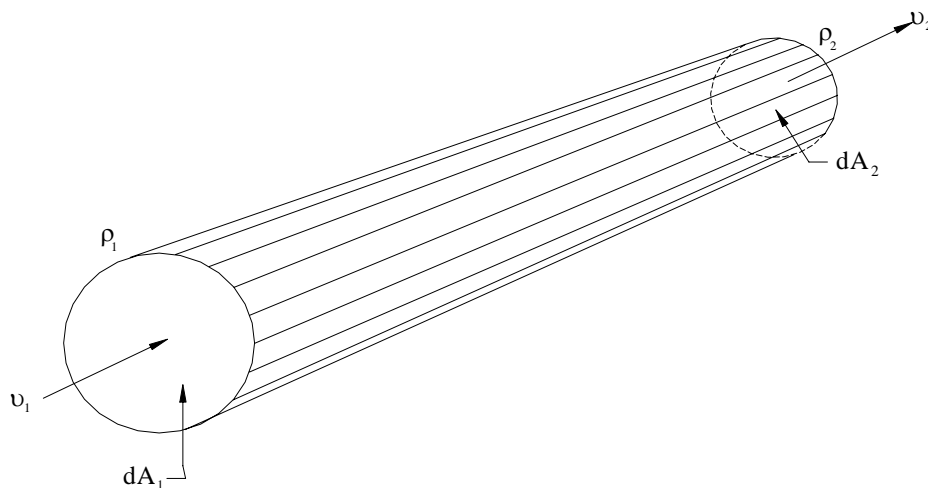


Fig. 3.3

In general, the cross-sectional area may vary along a stream tube since streamlines are generally curvilinear. Only in the steady flow field with uniform velocity will streamlines be straight and parallel. By definition, the velocities of all fluid particles in a uniform flow are the same in both magnitude and direction. If either the magnitude or direction of the velocity changes along any one streamline, the flow is then considered *non-uniform*.

3.4. ONE, TWO AND THREE-DIMENSIONAL FLOW

Although, in general, all fluid flow occurs in three dimensions, so that, velocity, pressure and other factors vary with reference to three orthogonal axes, in some problems the major changes occur in two directions or even in only one direction. Changes along the other axis or axes can, in such cases, be ignored without introducing major errors, thus simplifying the analysis.

Flow is described as *one-dimensional* if the factors, or parameters, such as velocity, pressure and elevation, describing the flow at a given instant, vary only along the direction of flow and not across the cross-section at any point. If the flow is unsteady, these parameters may vary with time. The one dimension is taken as the distance along the streamline of the flow, even though this may be a curve in space, and the values of velocity, pressure and elevation at each point along this streamline will be the average values across a section normal to the streamline (Fig.3.4).

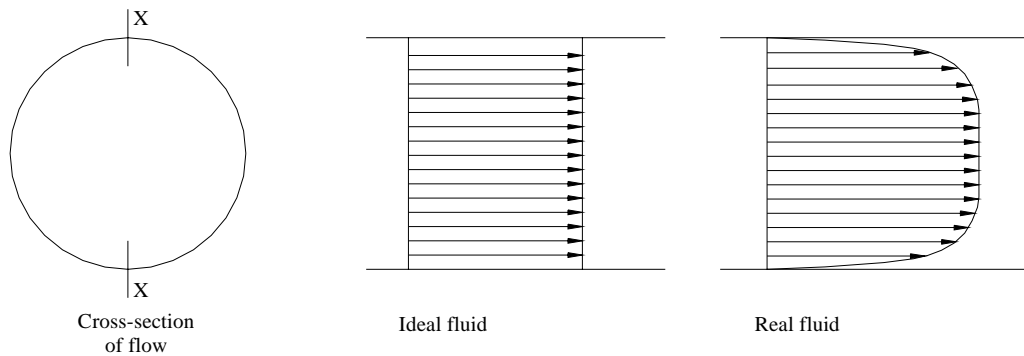


Fig. 3.4

In *two-dimensional* flow it is assumed that the flow parameters may vary in the direction of flow and in one direction at right angles, so that the streamlines are curves lying in a plane and identical in all planes parallel to this plane.

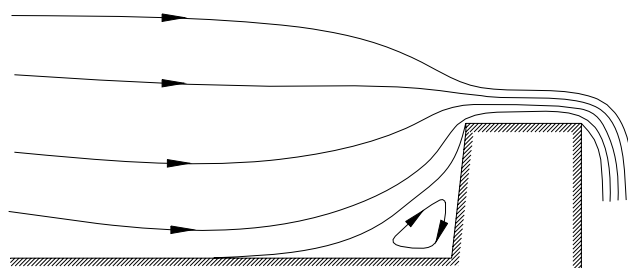


Fig. 3.5

Thus, the flow over a weir of constant cross-section (Fig.3.5) and infinite width perpendicular to the plane of the diagram can be treated as two-dimensional.

In *three-dimensional* flow it is assumed that the flow parameters may vary in space, x in the direction of motion, y and z in the plane of the cross-section.

3.5. EQUATION OF CONTINUITY: ONE-DIMENSIONAL STEADY FLOW

The application of the principle of conservation of mass to a steady flow in a stream tube results in the *equation of continuity*, which expresses the continuity of flow from section to section of the stream. Consider a physical system that is a particular collection of matter and is identified and viewed as being separated from everything external to the system by an imagined or real closed boundary. The fluid system retains its mass, but not its position or shape. This suggests needs to define a more convenient object for analysis. This object is a volume fixed in space and is called a *control volume*, through whose boundary matter, mass, momentum, energy, and the like may flow. The boundary of the control volume is the *control surface*. The fixed control volume can be of any useful size (finite or infinitesimal) and shape, provided only that the bounding control surface is a closed (completely surrounding) boundary. Neither the control volume nor the control shape changes shape or position with time.

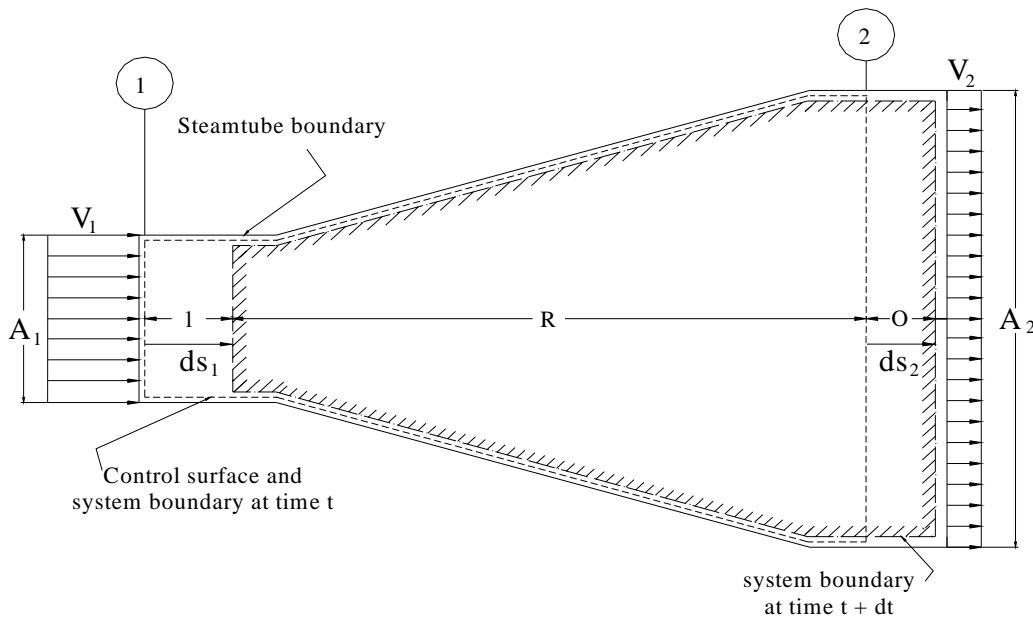


Fig. 3.6

Now consider the element of a finite stream tube in Fig. 3.6 through which passes a steady, one-dimensional flow of an incompressible fluid (note the uniform velocities at sections 1 and 2). In the tube near section 1 the cross-sectional area is A_1 and near section 2, A_2 . With the control surface shown coinciding with the stream tube walls and the cross sections at 1 and 2, the control volume comprises volumes I and R. Let a fluid system be defined as the fluid within the control volume (I + R) at time t . The control volume is fixed in space, but in time dt the system moves downstream as shown. From the conservation of system mass

$$(m_I + m_R)_t = (m_R + m_O)_{t+dt}$$

$$(\text{Mass of fluid in zones I and R at time } t) = (\text{Mass of fluid in zones O and R at time } t+dt)$$

In a steady flow the fluid properties at points in space are not functions of time so $(m_R)_t = (m_R)_{t+dt}$ and consequently

$$(m_I)_t = (m_O)_{t+dt}$$

These two terms are easily in terms of the mass of fluid moving across the control surface in time dt. The volume of I is $A_1 ds_1$, and that of O is $A_2 ds_2$; accordingly,

$$(m_I)_t = \rho A_1 ds_1$$

$$(m_O)_{t+dt} = \rho A_2 ds_2$$

and

$$\rho A_1 ds_1 = \rho A_2 ds_2$$

Dividing by dt,

$$\rho A_1 \frac{ds_1}{dt} = \rho A_2 \frac{ds_2}{dt}$$

However, ds_1/dt and ds_2/dt are recognized as the velocities past sections 1 and 2, respectively, therefore, if $m = \rho AV$ is the mass flow rate, then

$$m = \rho A_1 V_1 = \rho A_2 V_2$$

$$Q = A_1 V_1 = A_2 V_2 \quad (3.8)$$

Which is the *equation of continuity*. Thus for incompressible fluids, along a stream tube the product of velocity and cross-sectional area will be constant. This product, Q , is designated as (*flowrate*) *discharge* and has dimensions of $[L^3 T^{-1}]$ and units of cubic meters per second (m^3/sec).

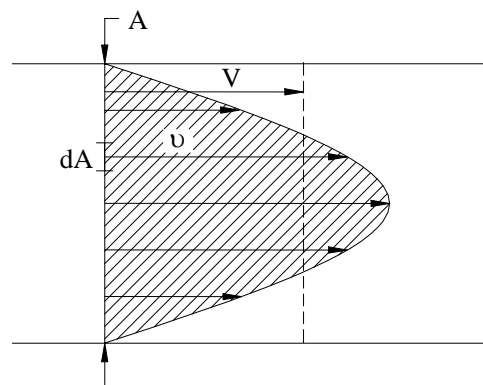


Fig. 3.7

Frequently in fluid flows the velocity distribution through a flow cross-section may be non-uniform, as shown in Fig. 3.7. From consideration of mass, it is evident at once that non-uniformity of velocity distribution does not invalidate the continuity principle. Thus, for steady flow of the incompressible fluid, Equ. (3.8) applies as before. Here, however, the velocity V in the equation is the *mean velocity* defined by $V = Q/A$ in which the discharge Q is obtained from the summation of the differential discharges, dQ , passing through the differential areas, dA . Thus, V is a fictitious uniform velocity that will transport the same amount of mass through the cross-section as will the actual the velocity distribution;

$$V = \frac{1}{A} \int_A v dA \quad (3.9)$$

From which the mean velocity may be obtained by performing the indicated integration (Equ. 3.9). With the velocity mathematically defined, formal integration may be employed; when the velocity profile is known but not mathematically defined, graphical or numerical methods may be used to evaluate integral.

The fact that the product AV remains constant along a stream tube allows a partial physical interpretation of streamline pictures. As the cross-sectional area of a stream tube increases, the velocity must decrease; hence the conclusion; streamlines widely spaced indicate regions of low velocity, streamlines closely spaced indicate regions of high velocity.

The continuity of flow equation is one of the major tools of fluid mechanics, providing a means of calculating velocities at different points in a system.

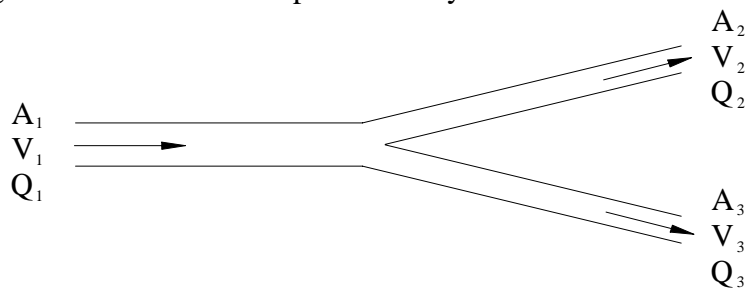


Fig. 3.8

The continuity equation can also be applied to determine the relation between the flows into and out of a junction. In Fig. 3.8, for steady conditions,

Total inflow to junction = Total outflow from junction

$$Q_1 = Q_2 + Q_3$$

or

$$A_1 V_1 = A_2 V_2 + A_3 V_3$$

In general, if we consider flow towards the junction as positive and flow away from the junction as negative, then for steady flow at any junction the algebraic sum of the discharges must be zero.

$$\sum Q = 0$$

EXAMPLE 3.1: Water flows through a pipe AB (Fig.3.9) of diameter $d_1 = 50$ mm, which is in series with a pipe BC of diameter $d_2 = 75$ mm in which the mean velocity $V_2 = 2$ m/sec.

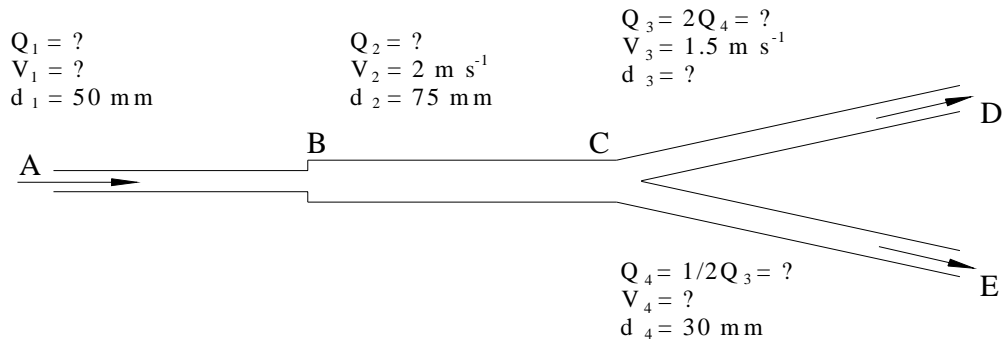


Fig. 3.9

At C the pipe forks and one branch CD is of diameter d_3 such that the mean velocity $V_3 = 1.5 \text{ m/sec}$. The other branch CE is of diameter $d_4 = 30 \text{ mm}$ and conditions are such that the discharge Q_2 from BC divides so that $Q_4 = 0.5Q_3$. Calculate the values of Q_1 , V_1 , Q_2 , Q_3 , d_3 , Q_4 and V_4 .

SOLUTION: Since pipes AB and BC in series, the volume rate of flow (discharge) will be the same in each pipe, $Q_1 = Q_2$.

$$Q_2 = \text{Area of pipe} \times \text{Mean velocity} = \frac{\pi d_2^2}{4} V_2$$

$$Q_1 = Q_2 = \frac{\pi}{4} \times 0.075^2 \times 2 = 8.836 \times 10^{-3} \text{ m}^3/\text{sec}$$

$$\text{Mean velocity in AB} = V_1 = \frac{4Q_1}{\pi d_1^2} = \frac{4 \times 8.836 \times 10^{-3}}{\pi \times 0.05^2}$$

$$V_1 = 4.5 \text{ m/sec}$$

Considering pipes BC, CD and CE, the discharge from BC must be equal to the sum of discharges through CD and CE. Therefore, $Q_2 = Q_3 + Q_4$, and since $Q_4 = 0.5Q_3$, we have $Q_2 = 1.5Q_3$, from which

$$Q_3 = \frac{Q_2}{1.5} = \frac{8.836 \times 10^{-3}}{1.5} = 5.891 \times 10^{-3} \text{ m}^3/\text{sec}$$

and

$$Q_4 = \frac{Q_3}{2} = 2.945 \times 10^{-3} \text{ m}^3/\text{sec}$$

Also, since

$$Q_3 = \frac{\pi d_3^2}{4} V_3, \quad d_3 = \sqrt{\frac{4Q_3}{\pi V_3}}$$

$$d_3 = \sqrt{\frac{4 \times 5.891 \times 10^{-3}}{\pi \times 1.5}} = 0.071 \text{ m}$$

$$V_4 = \frac{4Q_4}{\pi d_4^2} = \frac{4 \times 2.945 \times 10^{-3}}{\pi \times 0.03^2} = 4.17 \text{ m/sec}$$