LECTURE NOTES - II

« FLUID MECHANICS »

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CHAPTER 2

FLUID STATICS

Fluid statics is the study of fluid problems in which there is no relative motion between fluid elements. With no relative motion between individual elements (and thus no velocity gradients), no shear can exist, whatever the viscosity of the fluid is. Accordingly, viscosity has no effect in static problems and exact analytical solutions to such problems are relatively easy to obtain. Hence, all free bodies in fluid statics have only normal pressure forces acting on them.

2.1. PRESSURE AT A POINT

The average pressure is calculated by dividing the normal force pushing against a plate area by the area. The pressure at a point is the limit of the ratio of normal force to area, as the area approaches zero size at the point.

Fig. 2.1 shows a small wedge of fluid at rest of size Δx by Δz by Δs and depth b into the paper. Since there can be no shear forces, the only forces are the normal surface forces and gravity. Summation of forces must equal zero (no acceleration) in both the x and z directions.

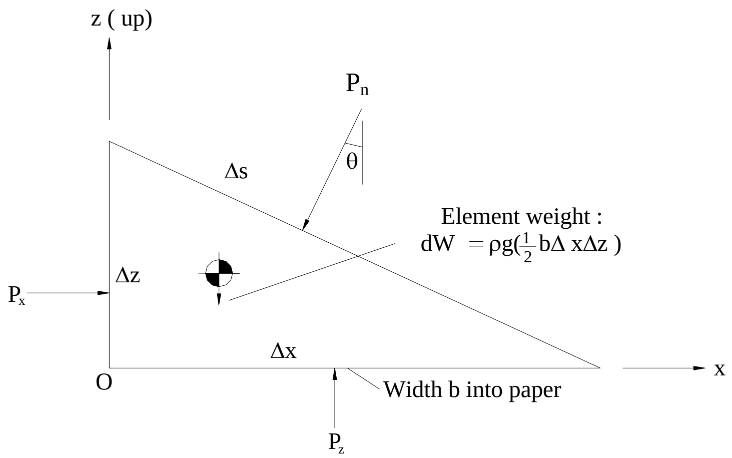


Fig. 2.1

$$\sum F_x = p_x b\Delta z - p_n b\Delta s Sin\theta = \frac{\Delta x\Delta z}{2}\rho a_x = 0$$

$$\sum F_z = p_z b\Delta x - p_n b\Delta s Sin\theta - \frac{1}{2}\gamma b\Delta x\Delta z = \frac{\Delta x\Delta z}{2}\rho a_z = 0 \qquad (2.1)$$

In which p_x, p_z, p_n are the average pressures on the three faces, γ is the specific weight of the fluid, ρ is the specific mass, and a_x and a_z are the acceleration components of the wedge in the x and z direction respectively. The geometry of the wedge is such that

$$\Delta z = \Delta s Sin\theta \quad , \ \Delta x = \Delta s Cos\theta$$

Substitution into Equ. (2.1) and rearrangement give

$$p_x = p_n \quad , \ p_z = p_n + \frac{1}{2}\gamma\Delta z \qquad (2.2)$$

These relations illustrate two important principles of the hydrostatic, or shear free, condition:

1) There is no pressure change in the horizontal direction,
2) There is a vertical change in pressure proportional to the specific mass, gravity and depth change.

In the limit as the fluid wedge shrinks to a point, $\Delta z \to 0$ and Equ. (2.2) becomes

$$p_x = p_z = p_n = p \qquad (2.3)$$

Since θ is arbitrary, we conclude that the pressure p at a point in a static fluid is independent of orientation, and has the same value in all directions.

2.2. PRESSURE VARIATION IN A STATIC FLUID

The fundamental equation of fluid statics is that relating pressure, specific mass and vertical distance in a fluid. This equation may be derived by considering the static equilibrium of a typical differential element of fluid (Fig. 2.2). The z-axis is in the direction parallel to the gravitational force field (vertical). Applying Newton's first law ($\Sigma F_x = 0$ and $\Sigma F_z = 0$) to the element

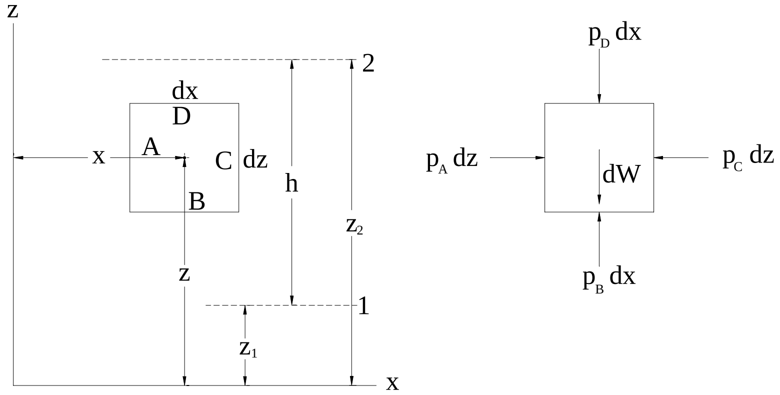


Fig. 2.2

And using the average pressure on each face to closely approximate the actual pressure distribution on the differential element (recall dx and dz are very small), give

$$\sum F_x = p_A dz - p_C dz = 0$$
$$\sum F_z = p_B dx - p_D dx - dW = 0 \qquad (2.4)$$

In which p and γ are functions of x and z. In partial derivation notation the pressures on the faces of the element are, in terms of pressure p in the center

$$p_A = p - \frac{\partial p}{\partial x}\frac{dx}{2} \quad , \; p_B = p - \frac{\partial p}{\partial z}\frac{dz}{2}$$

$$p_C = p + \frac{\partial p}{\partial x}\frac{dx}{2} \quad , \; p_D = p + \frac{\partial p}{\partial z}\frac{dz}{2}$$

The weight of the small element is dW = γdxdz (as dx and dz approach zero in the limiting process for partial differentiation, any variations in γ over the element will vanish). Thus, Eqs. (2.4) become

$$\left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)dz - \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)dz = -\frac{\partial p}{\partial x}dxdz = 0$$

And similarly

$$-\frac{\partial p}{\partial z}dzdx - \gamma dxdz = 0$$

Canceling the dxdz in both cases gives

$$\frac{\partial p}{\partial x} = 0 \quad \text{and} \quad \frac{\partial p}{\partial z} = \frac{dp}{dz} = -\gamma \qquad (2.5)$$

The first of these equations shows there is no variation of pressure with horizontal distance, that is, pressure is constant in a horizontal plane in a static fluid; therefore pressure is a function of z only and the total derivative may replace the partial derivative in the second equation, which is the basic equation of fluid statics.

Equ. (2.5) may be integrated directly to find

$$z_2 - z_1 = \int_{p_2}^{p_1} \frac{dp}{\gamma} \qquad (2.6)$$

For a fluid of constant specific weight, the integration yields

$$z_2 - z_1 = h = \frac{p_1 - p_2}{\gamma} \quad \text{or} \quad p_1 - p_2 = \gamma(z_2 - z_1) = \gamma h \tag{2.7}$$

Permitting ready calculation of the increase of pressure with depth in a fluid of constant specific weight. Equ. (2.7) also shows that pressure differences ($p_1 - p_2$) may be readily expressed as a *head h* of fluid of specific weight γ. Thus pressures are often quoted as heads in millimeters of mercury, meters of water. The open *manometer* and *piezometer columns* of Fig. 2.3 illustrate the relation of pressure to head.

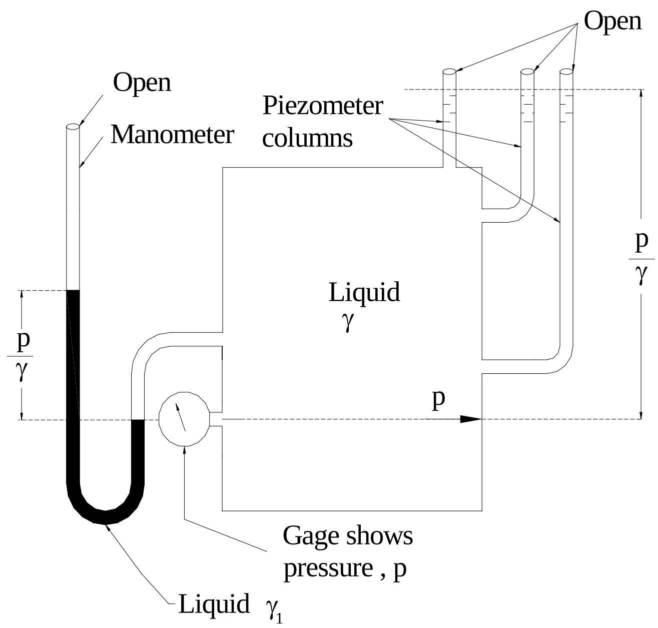


Fig. 2.3

Equ. (2.7) may be arranged fruitfully to

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2 = \text{Constant} \tag{2.8}$$

for later comparison with equations of fluid flow. Taking points 1 and 2 as typical, it is evident from Equ. (2.8) that the quantity (z + p/γ) is the same for all points in a static fluid. This may be visualized geometrically as shown on Fig. 2.4.

Frequently, in engineering problems the liquid surface is exposed to atmospheric pressure; if the latter is taken to be zero, the dashed line of Fig. 2.4 will necessarily coincide with the liquid surface.

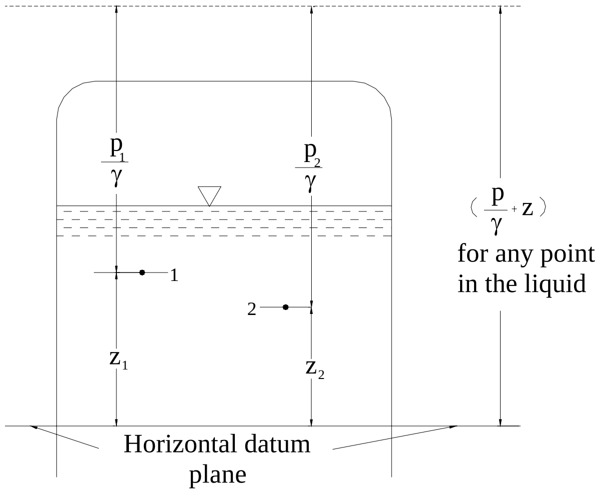


Fig. 2.4

2.3. THE HYDROSTATIC PARADOX

From Equ. (2.7) it can be seen that the pressure exerted by a fluid is dependent only on the vertical head of fluid and its specific weight; it is not affected by the weight of the fluid present. Thus, in Fig. 2.5 the four vessels all have the same base area A and filled to the same height with the same liquid of specific weight γ.

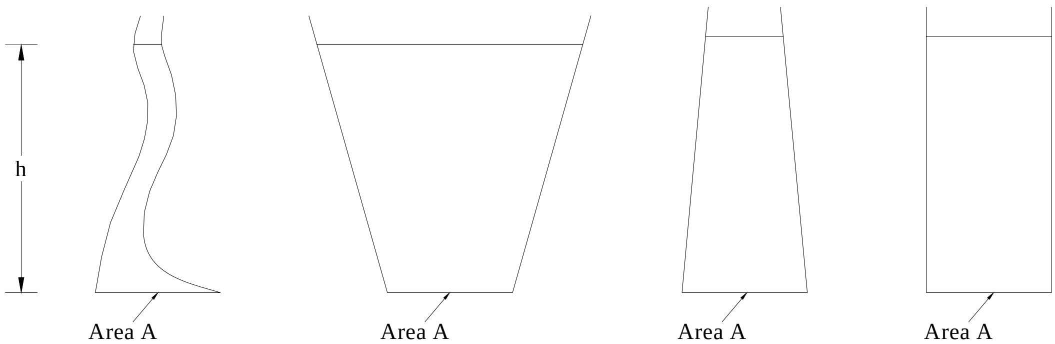


Fig. 2.5

Pressure on bottom in each case = $p = \gamma \times h$

Force on bottom = Pressure×Area = $p \times A = \gamma \times h \times A$

Thus, although the weight of fluid is obviously different in the four cases, the force on the bases of the vessels is the same, depending on the depth h and the base area A.

2.4. ABSOLUTE AND GAGE PRESSURES

Pressures are measured and quoted in two different systems, one *relative (gage)*, and the other *absolute*; no confusion results if the relation between the systems and the common methods of measurement is completely understood.

Liquid devices that measure gage and absolute pressures are shown on Fig. 2.6; these are the open U-tube and conventional mercury barometer. With the U-tube open, atmospheric pressure will act on the upper liquid surface; if this pressure is taken to be zero, the applied gage pressure p will equal γh and h will thus be a direct measure of gage pressure.

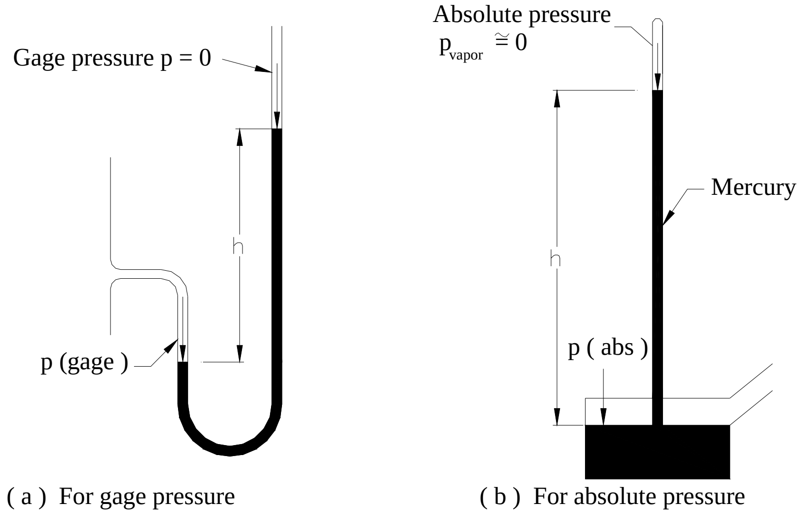


Fig. 2.6

The mercury barometer (invented by Toricelli, 1643) is constructed by filling the tube with air-free mercury and inverting it with its open end beneath the mercury surface in the receptacle. Ignoring the small pressure of the mercury vapor, the pressure in the space above the mercury will be absolute zero and again p = γh; here the height h is direct measure of the absolute pressure, p.

From the foregoing descriptions an equation relating (gage) and absolute pressures may now be written,

$$\text{Absolute pressure} = \text{Atmospheric pressure} \begin{matrix} -\text{Vacuum} \\ \\ +\text{Gage pressure} \end{matrix} \qquad (2.9)$$

Which allows easy conversion from one system to the other. Possibly a better picture of these relationships can be gained from a diagram such as that of Fig. 2.7 in which are shown two typical pressures, A and B, one above, the other below, atmospheric pressure, with all the relationships indicated graphically.

At sea level standard atmosphere $p_{atm} = p_0 = 10.33$ t/m^2, a pieozometer column of mercury will stand at a height of 0.76 m. However, if water were used, a reading of about 10.33 m would be obtained.

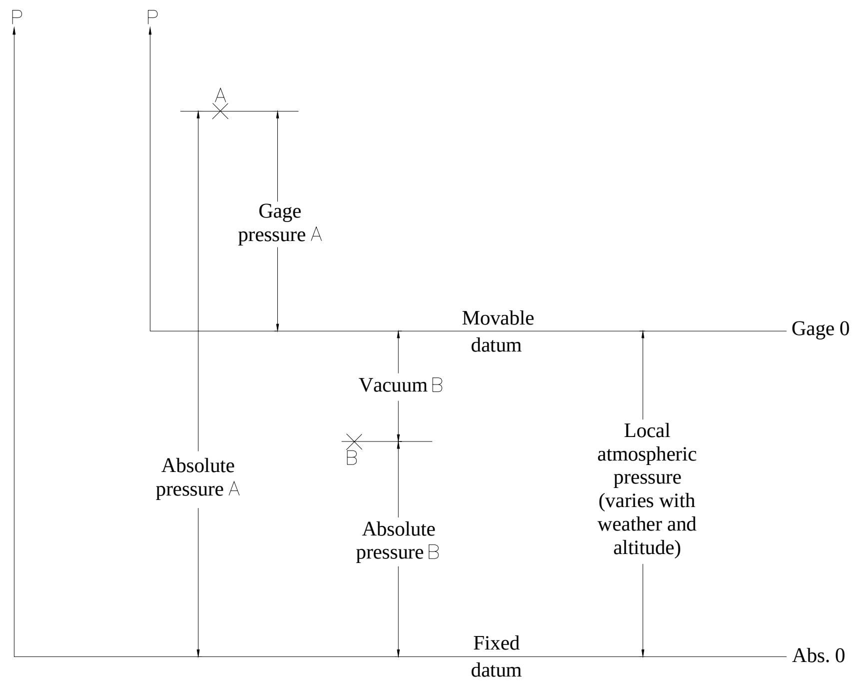


Fig. 2.7

EXAMPLE 2.1: A cylinder contains a fluid at a relative (gage) pressure of 35 t/m^2. Express this pressure in terms of a head of, a) water (γ_{water} = 1000 kg/m^3), b) mercury (γ_{Hg} = 13.6 t/m^3).

What would be the absolute pressure in the cylinder if the atmospheric pressure is 10.33 t/m^2?

SOLUTION:

From Equ. (2.7), head, h = p/γ.

a) Putting p = 35 t/m^2, γ = 1 t/m^3,

Equivalent head of water = $\frac{35}{1} = 35m.$

b) For mercury γ_{Hg} = 13.6 t/m^3,

Equivalent head of mercury = $\frac{35}{13.6} = 2.57m.$

Absolute pressure = Gage pressure + Atmospheric pressure

$$p_{abs} = 35 + 10.33 = 45.33 t/m^2$$

2.5. MANOMETER

From the hydrostatic Equ. (2.7), a change in elevation ($z_2 - z_1$) of a liquid is equivalent to a change in pressure ($p_2 - p_1$)/γ. Thus a static column of one or more liquids can be used to measure differences between two points. Such a device is called a *manometer*. If multiple fluids are used, we must change the specific weight in the equation as move from one fluid to another. Fig. 2.8 illustrates the use of the equation with a column of multiple fluids. The pressure change through each fluid is calculated separately. If we wish to know the total change ($p_5 - p_1$), we add successive changes ($p_2 - p_1$), ($p_3 - p_2$), ($p_4 - p_3$), and ($p_5 - p_4$). The intermediate values of p cancel, and we have, for the example of Fig. 2.8,

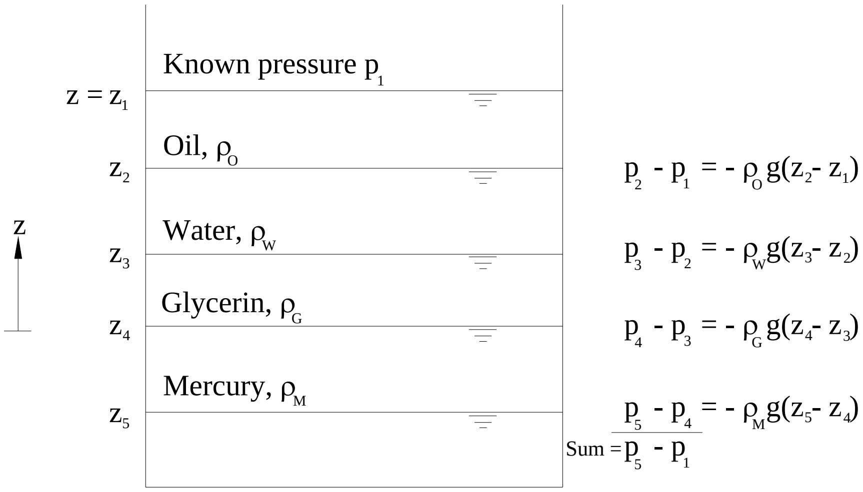


Fig. 2.8

$$p_5 - p_1 = -\gamma_o(z_2 - z_1) - \gamma_w(z_3 - z_2) - \gamma_G(z_4 - z_3) - \gamma_{Hg}(z_5 - z_4) \qquad (2.10)$$

No additional simplification is possible on the right-hand side because of the different specific weights. Notice that we have placed the fluids in order from the lightest on top to the heaviest at bottom.

When calculating hydrostatic pressure changes, engineers work instinctively by simply having the pressure increase downward and decrease upward.

$$p_{down} = p^{up} + \gamma|\Delta z| \qquad (2.11)$$

Thus, without worrying too much about which point is z_1 and which is z_2, the equation simply increases or decreases the pressure according to whether one is moving down or up. For example, Equ. (2.10) could be written in the following "multiple increase" mode:

$$p_5 = p_1 + \gamma_o|z_1 - z_2| + \gamma_w|z_2 - z_3| + \gamma_G|z_3 - z_4| + \gamma_{Hg}|z_4 - z_5|$$

That is, keep adding on pressure increments as you move down through the layered fluid. A different application is a manometer, which involves both "up" and "down" calculations.

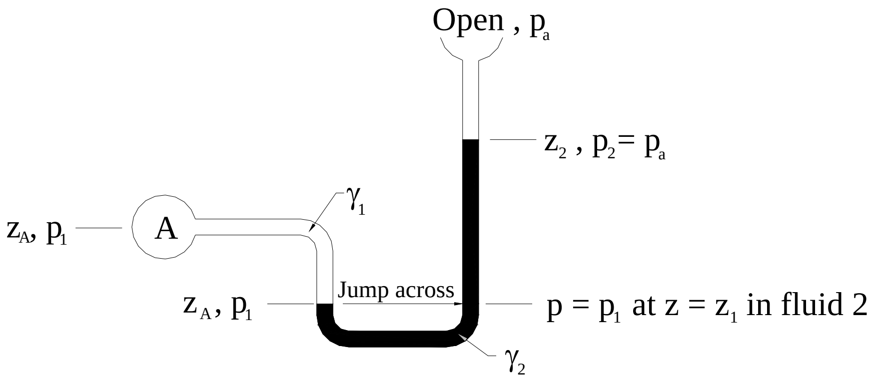


Fig. 2.9

Fig. 2.9 shows a simple manometer for measuring p_A in a closed chamber relative to atmospheric pressure p_0, in other words, measuring gage (relative) pressure. The chamber fluid γ_1 is combined with a second fluid γ_2, perhaps for two reasons: 1) To protect the environment from a corrosive chamber fluid or, 2) Because a heavier fluid γ_2 will keep z_2 small and the open tube can be shorter. One can apply the basic hydrostatic Equ. (2.7). Or, more simply, one can begin at A, apply Equ. (2.11) "down" to z_1, jump across fluid 2 (see Fig. 2.9) to the same pressure p_1, and then use Equ. (2.11) "up" to level z_2:

$$p_A + \gamma_1 |z_A - z_1| - \gamma_2 |z_1 - z_2| = p_2 = p_{atm} = p_0 \qquad (2.12)$$

The physical reason that we can "jump across" at section 1 in that a continuous length of the same fluid connects these two elevations. The hydrostatic relation (Equ. 2.7) requires this equality as a form of Pascal's law:

Any two points at the same elevation in a continuous mass of the same static fluid will be at the same pressure.

This idea of jumping across to equal pressures facilitates multiple-fluid problems.

EXAMPLE 2.2: A U-tube manometer in Fig. 2.10 is used to measure the gage pressure of a fluid P of specific weight γ_P = 800 kg/m^3. If the specific weight of the liquid Q is γ_Q = 13.6×10^3 kg/m^3, what will be the gage pressure at A if, a) h_1 = 0.5 m and D is 0.9 m above BC, b) h_1 = 0.1 m and D is 0.2 m below BC?

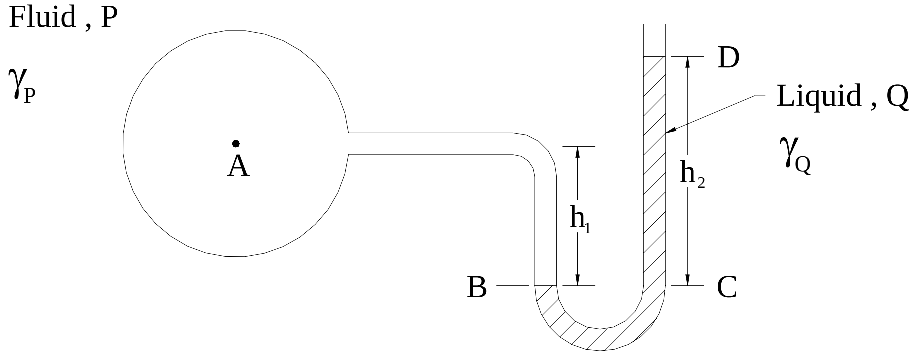


Fig. 2.10

SOLUTION:

a) In Equ. (2.12), γ_1 = 0.8 t/m^3, γ_2 = 13.6 t/m^3, $(z_A - z_1)$ = 0.5 m, $(z_1 - z_2)$ = 0.9 m.

$$p_A + 0.8\times 0.5 - 13.6\times 0.9 = p_0 = 0$$
$$p_A = 13.6\times 0.9 - 0.8\times 0.5 = 11.84\, t/m^2$$

b) Putting $|z_A - z_1|$ =0.1 m and $|z_1 - z_2|$ =-0.2 m into Equ. (2.12) gives,

$$p_A + 0.8\times 0.1 - 13.6\times(-0.2) = p_0 = 0$$
$$p_A = -0.08 - 2.72 = -2.8\, t/m^2$$

The negative sign indicating that p_A is below atmospheric pressure. The absolute pressure at A is according to Equ. (2.9),

$$p_{A_{abs}} = p_0 + p_A = 10.33 - 2.8 = 7.53\, t/m^2$$

Fig. 2.11 illustrates a multiple-fluid manometer problem for finding the difference in pressure between two chambers A and B. We repeatedly apply Equ. (2.7) jumping across at equal pressures when we come to a continuous mass of the same fluid. Thus, in Fig. 2.11, we compute four pressure differences while making three jumps:

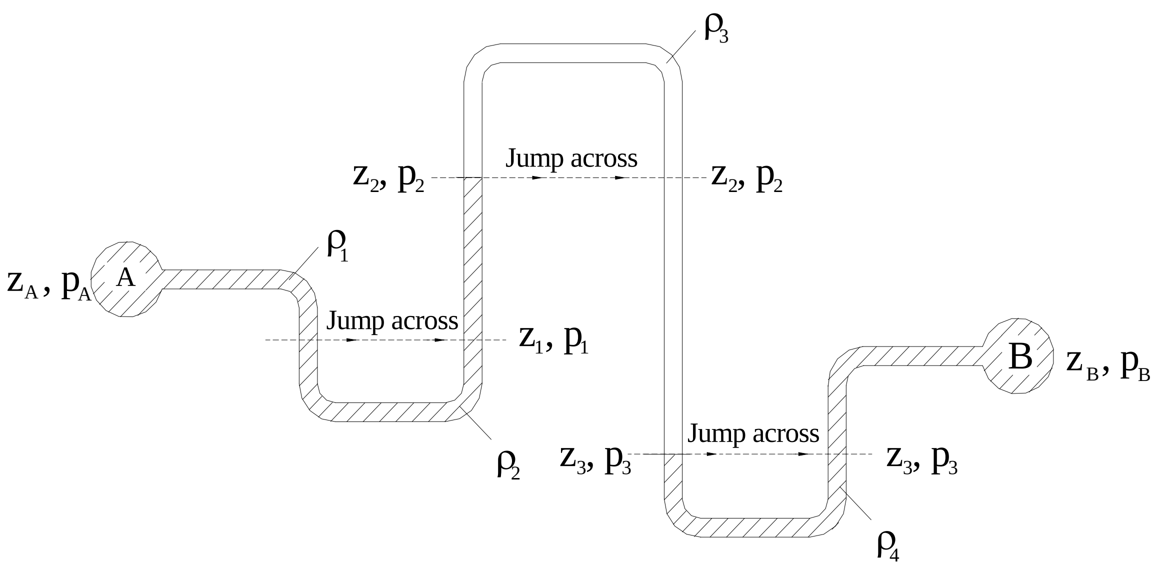


Fig. 2.11

$$\begin{aligned} p_A - p_B &= (p_A - p_1) + (p_1 - p_2) + (p_2 - p_3) + (p_3 - p_B) \\ &= -\gamma_1(z_A - z_1) - \gamma_2(z_1 - z_2) - \gamma_3(z_2 - z_3) - \gamma_4(z_3 - z_B) \end{aligned} \qquad (2.13)$$

The intermediate pressures $p_{1,2,3}$ cancel. It looks complicated, but it is merely sequential. One starts at A, goes down to 1, jump across, goes down to 3, jumps across, and finally goes up to B.

EXAMPLE 2.3: Pressure gage B is to measure the pressure at point A in a water flow. If the pressure at B is 9 t/m², estimate the pressure at A. γ_{water} = 1000 kg/m³, γ_{Hg} = 13600 kg/m³, γ_{oil}= 900 kg/m³.

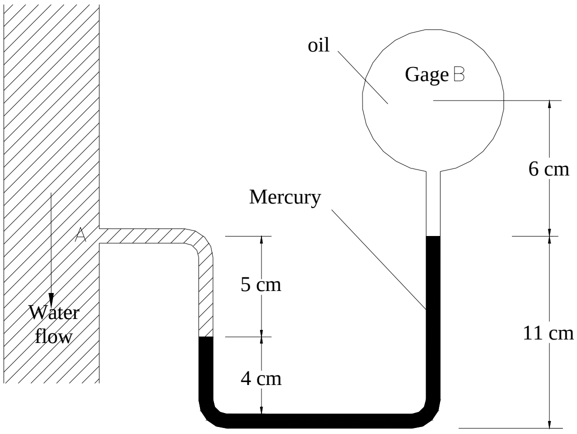


Fig. 2.12

SOLUTION: Proceed from A to B, calculating the pressure change in each fluid and adding:

$$p_A - \gamma_w (\Delta z)_w - \gamma_{Hg} (\Delta z)_{Hg} - \gamma_o (\Delta z)_o = p_B$$

or

$$p_A - 1000 \times (-0.05) - 13600 \times 0.07 - 900 \times 0.06$$

$$= p_A + 50 - 952 - 54 = p_B = 9000\, kg/m^2$$

$$p_A = 9956\, kg/m^2 \cong 9.96\, t/m^2$$

2.6. FORCES ON SUBMERGED PLANE SURFACES

The calculation of the magnitude, direction, and location of the total forces on surfaces submerged in a liquid is essential in the design of dams, bulkheads, gates, ships, and the like.

For a submerged plane, horizontal area the calculation of these force properties is simple because the pressure does not vary over the area; for nonhorizontal planes the problem is complicated by pressure variation. Pressure in constant specific weight liquids has been shown to vary linearly with depth (Equ. 2.7), producing the typical pressure distributions and resultant forces on the walls of a container of Fig. 2.13.

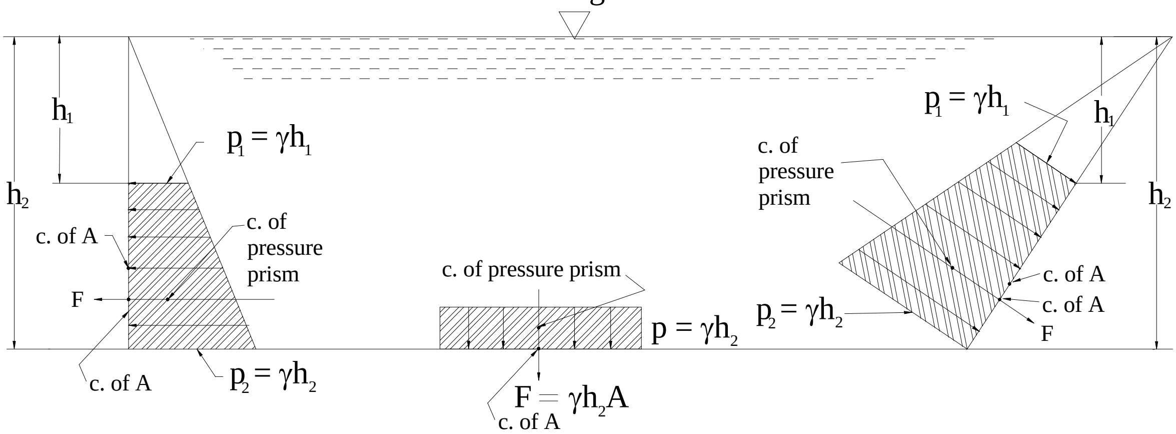


Fig. 2.13

The shaded areas, appearing as trapezoids are really volumes, known as *pressure prisms*. In mechanics it has been shown that the resultant force, F, is equal to the volume of the pressure prism and passes through its centroid.

Now consider the general case of a plane submerged area, A' B', such that of Fig. 2.14. It is inclined θ^0 from the horizontal.

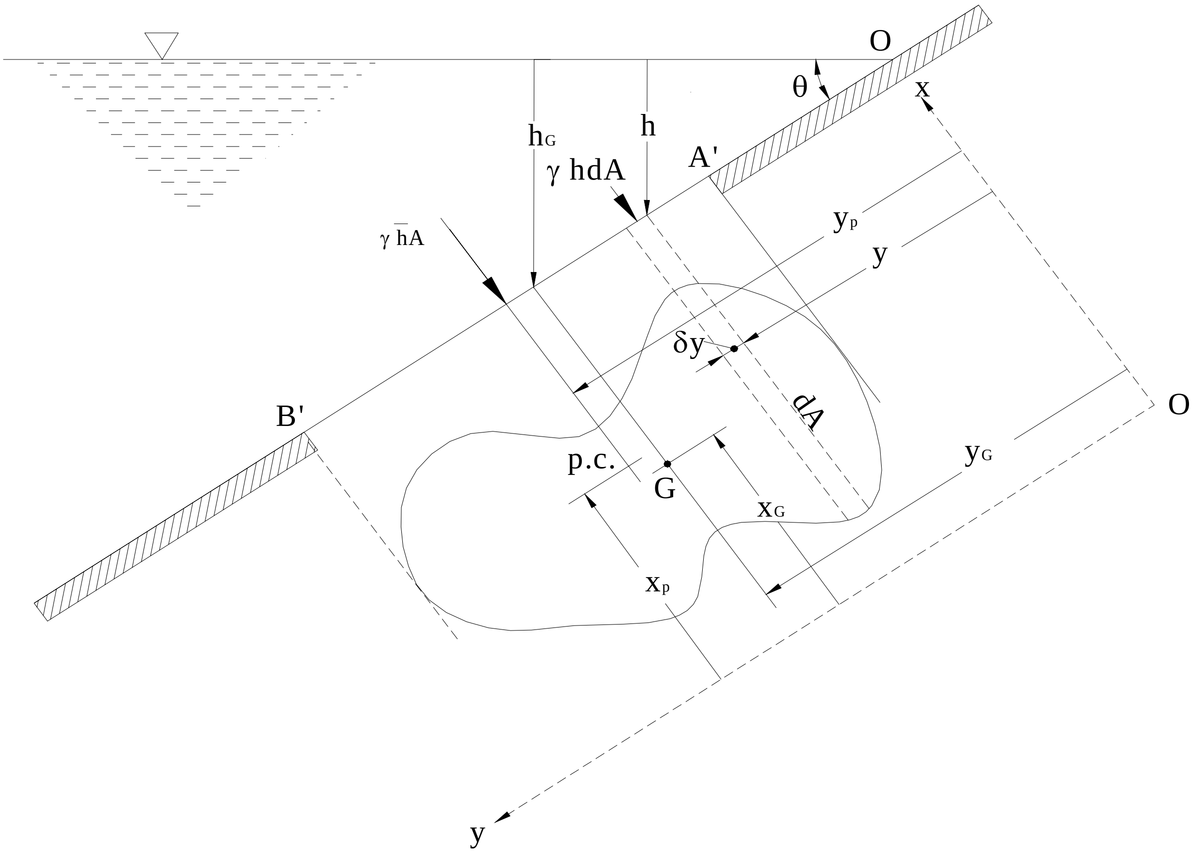


Fig. 2.14

The intersection of the plane of the area and the free surface is taken as the x-axis. The y-axis is taken in the plane of the area, with origin O, as shown, in the free surface. The xy-plane portrays the arbitrary inclined area. The magnitude, direction, and line of action of the resultant force due to liquid, acting on one side of the area are sought.

The force, dF, on the area, dA, is given by,

$$dF = pdA = \gamma hdA = \gamma ySin\theta dA \qquad (2.14)$$

The integral over the area yields the magnitude of force F, acting on one side of the area,

$$F = \int_A pdA = \gamma Sin\theta \int_A ydA = \gamma Sin\theta y_G A = \gamma h_G A \qquad (2.15)$$

With the relations from Fig. 2.14, $y_G \times Sin\theta = h_G$, and $p_G = \gamma \times h_G$, the pressure at the centroid of the area. In words, the magnitude of force exerted on one side of the plane area submerged in a liquid is the product of the area and the pressure at its centroid. As all force elements are normal to the surface, the line of action of the resultant is also normal to the surface.

The line of action of the resultant force has its piercing point in the surface at a point called the *pressure center,* with coordinates (x_P, y_P). Unlike that for the horizontal surface, the center of pressure of an inclined surface is not at the centroid. To find the pressure center, the moments of the resultant $x_P \times F$, $y_p \times F$ are equated to the moment of the forces about the y-axis and x-axis, respectively; thus

$$x_P F = \int_A x dF = \int_A x p dA \tag{2.16}$$

$$y_P F = \int_A y dF = \int_A y p dA \tag{2.17}$$

After solving for the coordinates of pressure center,

$$x_P = \frac{1}{F} \int_A x p dA \tag{2.18}$$

$$y_P = \frac{1}{F} \int_A y p dA \tag{2.19}$$

Eqs. (2.18) and (2.19) may be transformed into general formulas as follows:

$$x_P = \frac{1}{\gamma y_G A Sin\theta} \int_A x \gamma y Sin\theta dA = \frac{1}{y_G A} \int_A x y dA = \frac{I_{xy}}{y_G A} \tag{2.20}$$

Since the products of inertia $\bar{I}_{xy}$ about centroidal axes parallel to the xy-axes produces,

$$I_{xy} = \bar{I}_{xy} + x_G y_G A \tag{2.21}$$

Equ. (2.20) takes the form of,

$$x_P = \frac{\bar{I}_{xy}}{y_G A} + x_G \tag{2.22}$$

When either of the centroidal axes, $x = x_G$ or $y = y_G$, is an axis of symmetry for the surface, $\bar{I}_{xy}$ vanishes and pressure center lies on $x = x_G$. Since $\bar{I}_{xy}$ may be either positive or negative, the pressure center may lie on either side of the line $x = x_G$. To determine y_P by formula, with Eqs. (2.14) and (2.19)

$$y_P = \frac{1}{\gamma y_G A Sin\theta}\int_A y\gamma y Sin\theta dA = \frac{1}{y_G A}\int_A y^2 dA = \frac{I_x}{y_G A} \qquad (2.23)$$

In the parallel-axis theorem for moments of inertia

$$I_x = I_G + y_G^2 A \qquad (2.24)$$

If I_x is eliminated from Equ. (2.23)

$$y_P = \frac{I_G}{y_G A} + y_G \qquad (2.25)$$

or

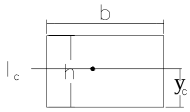
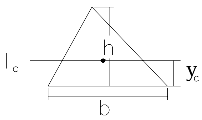
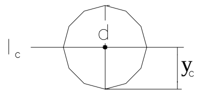
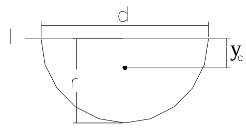
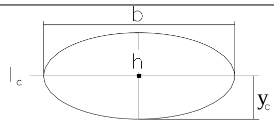
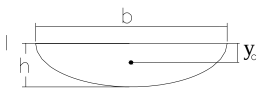
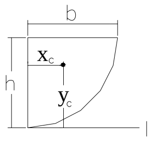
$$y_P - y_G = \frac{I_G}{y_G A} \qquad (2.26)$$

I_G is always positive; hence (y_P – y_G) is always positive, and the pressure center is always below the centroid of the surface. It should be emphasized that y_G and (y_P–y_G) are distances in the plane of the surface. A summary of I_G's for common areas is given in Table 2.1.

The areas of irregular forms may be divided into simple areas, the forces being located on them, and the location of their resultant being found by the methods of statics. The point where the line of action of the resultant force intersects the area is the center of pressure for the composite area.

Table 2.1

Properties of Areas

	Sketch	Area	Location of Centroid	I or I_c
Rectangle		bh	$y_c = \frac{h}{2}$	$I_c = \frac{bh^3}{12}$
Triangle		$\frac{bh}{2}$	$y_c = \frac{h}{3}$	$I_c = \frac{bh^3}{36}$
Circle		$\frac{\pi d^2}{4}$	$y_c = \frac{d}{2}$	$I_c = \frac{\pi d^4}{64}$
Semicircle		$\frac{\pi d^2}{8}$	$y_c = \frac{4r}{3\pi}$	$I_c = \frac{\pi d^4}{128}$
Ellipse		$\frac{\pi bd}{4}$	$y_c = \frac{h}{2}$	$I_c = \frac{\pi bh^3}{64}$
Semiellipse		$\frac{\pi bd}{4}$	$y_c = \frac{4h}{3\pi}$	$I_c = \frac{\pi bh^3}{16}$
Parabola		$\frac{2}{2}bh$	$y_c = \frac{3h}{5}$ $x_c = \frac{3b}{8}$	$I_c = \frac{2bh^3}{7}$

EXAMPLE 2.4: The gate in Fig. 2.15 is 5 m wide, is hinged at point B, and rests against a smooth wall at point A. Compute,

a) The force on the gate due to the water pressure,
b) The horizontal force P exerted by the wall at point A,
c) The reactions at hinge B.

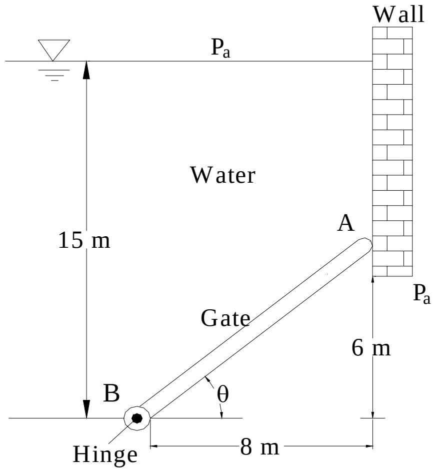


Fig. 2.15

SOLUTION:

a) By geometry the gate is 10 m long from A to B, and its centroid is halfway between, or at elevation 3 m above point B. The depth h_C is thus 15 – 3 = 12 m. The gate area is 5×10 = 50 m^2. Neglect p_0 (atmospheric pressure) as acting on both sides of the gate. From Equ. (2.15) the hydrostatic force on the gate is

$$F = p_C A = \gamma h_C A = 1 \times 12 \times 50 = 600 ton$$

b) First we must find the center of pressure of F. A free-body diagram of the gate is shown in Fig. 2.16. The gate is rectangle, hence

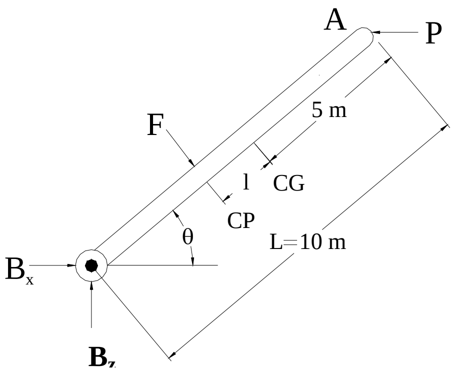


Fig. 2.16

$$I_{xy} = 0 \qquad \text{and} \qquad I_G = \frac{bL^3}{12} = \frac{5 \times 10^3}{12} = 417 m^4$$

The distance l from the CG to the CP is given by Equ. (2.26) since p_0 is neglected.

$$Sin\theta = \frac{6}{10} = 0.6 \quad , \ \theta = 37^0$$

$$y_G = \frac{h_G}{Sin\theta} = \frac{12}{0.6} = 20m$$
$$l = y_P - y_G = \frac{I_G}{y_G A} = \frac{417}{20 \times 50} = 0.417m$$

The distance from point B to force F is thus 10 – l –5 = 10 – 0.417 – 5 =4.583 m. Summing moments counterclockwise about B gives

$$PLSin\theta - F(5-l) = 0$$
$$P = \frac{F(5-l)}{LSin\theta} = \frac{600 \times (5 - 0.417)}{10 \times 0.6} = 458.3ton$$

c) With F and P known, the reactions B_x and B_z are found by summing forces on the gate.

$$\sum F_x = B_x + FSin\theta - P = 0$$
$$B_x = 458.3 - 600 \times 0.6 = 98.3ton$$

$$\sum F_z = B_z - FCos\theta = 0$$
$$B_z = 600 \times 0.8 = 480ton$$

2.7. DIFFERENT PRESSURES ON TWO SIDES

The pressure has been considered as varying from zero at M to NK at N. In reality there is some pressure on the surface of the liquid, which might be represented by an equivalent height MO, and the absolute pressure on the left-hand side of the plane will vary as shown by ODE (Fig. 2.17)

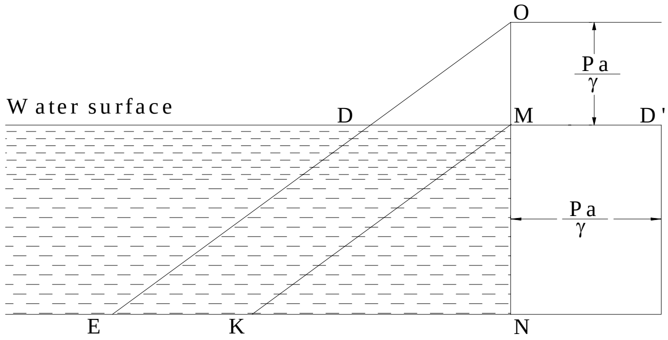


Fig. 2.17

In most practical cases it is the difference between the forces on the two sides is desired. The pressure of the air upon the surface of the liquid also produces a uniform pressure over the right-hand side of the area and thus MO = MD = MD'; and as the same air pressure acts a like on both sides, it mat be neglected altogether.

In a case such as that in Fig. 2.18, where surface represented by the trace MN is submerged by a liquid at two different heights on the two sides, the pressure variations are represented by CDE and LK. If the liquids are of the same specific weight, triangles DEN and LNK are equal. Thus the net pressure difference on the two sides is the uniform value DL, which is equal to γh.

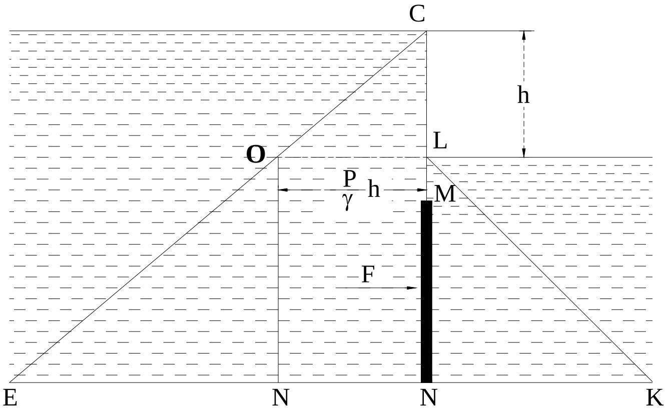


Fig. 2.18

Hence on any area by the same specific weight liquid on both sides but with a difference in level h as in Fig. (2.18), the resultant force is

$$F = \gamma h A \qquad (2.27)$$

and it will be applied at the centroid of the area.

2.8.FORCE ON A CURVED SURFACE

If a surface is curved, it is convenient to calculate the horizontal and vertical components of the resultant force.

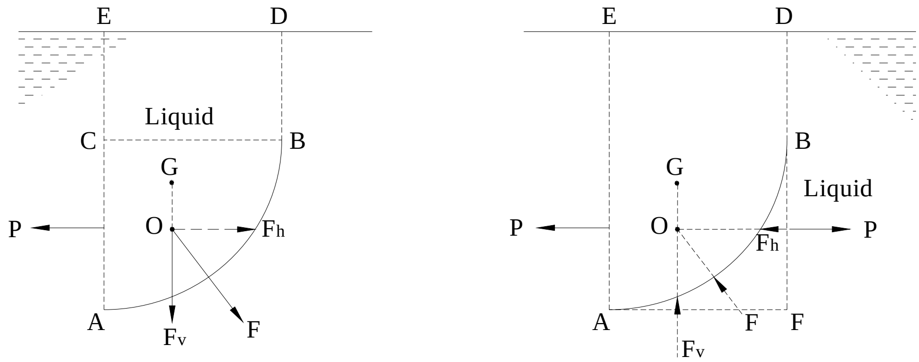


Fig. 2.19

In Fig. 2.19 (a) and (b), AB is the immersed surface and F_h and F_v are the horizontal and vertical components of the resultant force F of the liquid on one side of the surface. In Fig. 2.19. (a) the liquid lies above the immersed surface, while in Fig. 2.19 (b) it acts below the surface.

In Fig. 2.19 (a), if ACE is a vertical plane through A, and BC is a horizontal plane, then, since element ACB is in equilibrium, the resultant force F on AC must equal the horizontal component F_h of the force exerted by the fluid on AB because there are no other horizontal forces acting. But AC is the projection of AB on a vertical plane, therefore,

Horizontal component, F_h = Resultant force on the projection of AB on a vertical plane

Also, for equilibrium, P and F_h must act in the same straight line; therefore, the horizontal component F_h acts through the center of pressure of the projection of AB on a vertical plane.

Similarly, in Fig. 2.19 (b), element ABF is in equilibrium, and the horizontal component F_h is equal to the resultant force on the projection BF of the curved surface AB on a vertical plane, and acts through the center of pressure of this projection.

In Fig. 2.19 (a), the vertical component F_v will be entirely due to the weight of the fluid in the area ABDE lying vertically above AB. There are no other vertical forces, since there can be no shear forces on AE and BD because the fluid is at rest. Thus,

Vertical component, F_v = Weight of fluid vertically above AB

and will act vertically downwards through the center of gravity G of ABDE.

In Fig. 2.19 (b), if the liquid is on the right side of the surface AB, this liquid would be in equilibrium under its own weight and the vertical force on the boundary AB. Therefore,

Vertical component, F_v = Weight of the volume of the same fluid which would lie vertically above AB

and will act vertically upwards through the center of gravity G of this imaginary volume of fluid.

The resultant force F is found by combining the components vertically. If the surface is of uniform with perpendicular to the diagram, F_h and F_v will intersect at O. Thus,

Resultant force, $F = \sqrt{F_h^2 + F_v^2}$

and acts through O at an angle θ given by $Tan\theta = F_v / F_h$.

In the special case of a cylindrical surface, all the forces on each small element of area acting normal to the surface will be radial and will pass through the center of the curvature O (Fig. 2.20). The resultant force F must, therefore, also pass through the center of curvature O.

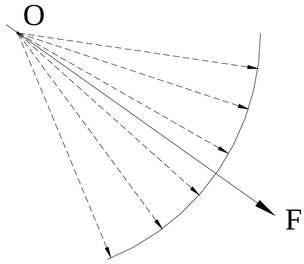


Fig. 2.20

EXAMPLE 2.5: A sluice gate is in the form of a circular arc of radius 6 m as shown in Fig. 2.21. Calculate the magnitude and direction of the resultant force on the gate, and the location with respect to O of a point of its line of action.

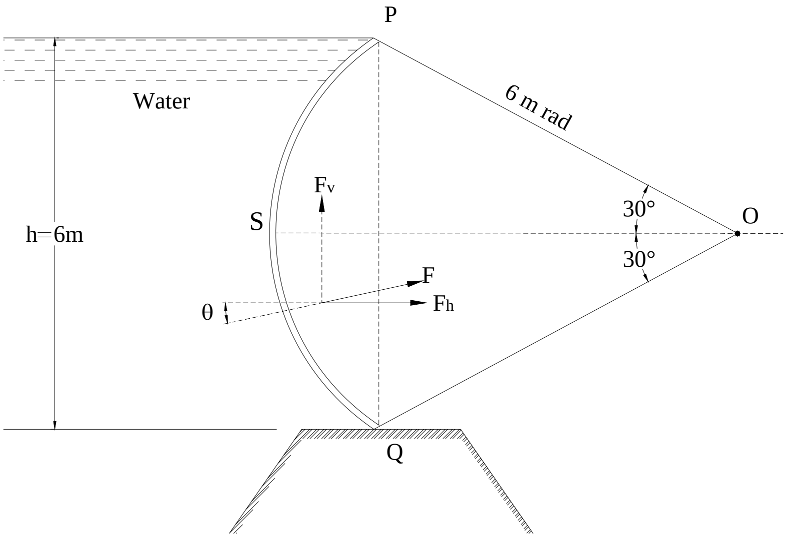


Fig. 2.21

SOLUTION:

Since the water reaches the top of the gate,

Depth of water, $h = 2 \times 6 \times Sin30^0 = 6m$

Horizontal component of force on gate = F_h (per unit length)

F_h = Resultant force on PQ per unit length

$$F_h = \gamma \times h \times \frac{h}{2} = \frac{\gamma h^2}{2}$$

$$F_h = \frac{1 \times 6^2}{2} = 18 ton$$

Vertical component of force on gate = F_v (per unit length)

F_v = Weight of water displaced by segment PSQ

F_v = (Sector OPSQ – Triangle OPQ)×γ

$$F_v = \left(\frac{60}{360} \times \pi \times 6^2 - 6 \times Sin30^0 \times 6 \times Cos30^0 \right) \times 1$$
$$F_v = 3.26 ton$$

Resultant force on gate,

$$F = \sqrt{F_h^2 + F_v^2}$$
$$F = \sqrt{18^2 + 3.26^2} = 18.29 ton/m$$

$$Tan\theta = \frac{F_v}{F_h} = \frac{3.26}{18} = 0.18$$

$\theta = 10^0 27'$ to the horizontal

Since the surface of the gate is cylindrical, the resultant force F must pass through O.

2.9. BUOYANCY AND FLOTATION

The familiar laws of buoyancy (Archimedes' principle) and flotation are usually stated:

1) A body immersed in a fluid is buoyed up by a force equal to the weight of fluid displaced,
2) A floating body displaces its own weight of the fluid in which it floats.

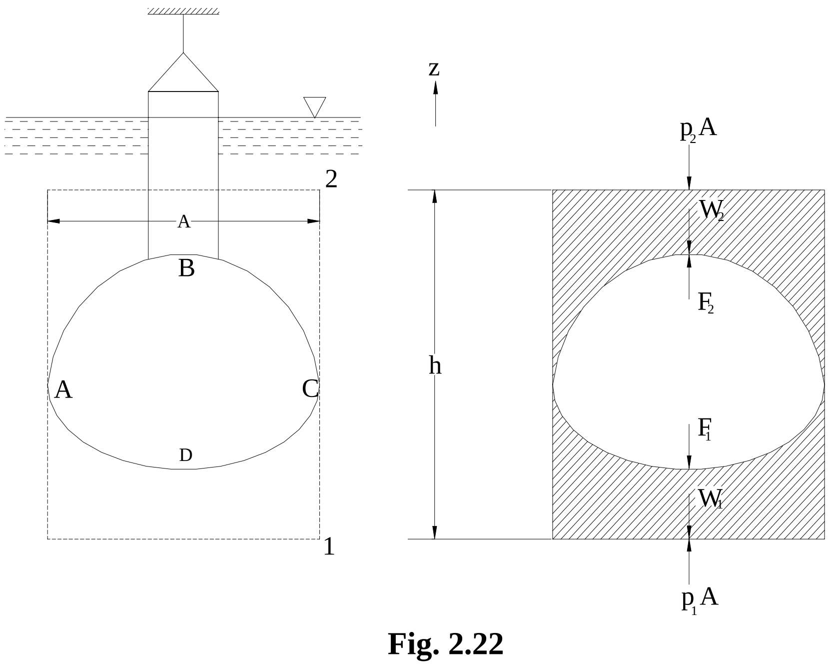


Fig. 2.22

A body ABCD suspended in a fluid of specific weight γ is illustrated in Fig. 2.22. Isolating a free body of fluid with vertical sides tangent to the body allows identification of the vertical forces exerted by the lower (ADC) and upper (ABC) surfaces of the body surrounding fluid. These are F_1 and F_2 with (F_1 - F_2) the buoyant force on the body. For the upper portion of the free body

$$\sum F_z = F_2 - W_2 - p_2 A = 0$$

and for the lower portion

$$\sum F_z = F_1 + W_1 - p_1 A = 0$$

Whence (by subtracting of these equations)

$$F_B = F_1 - F_2 = (p_1 - p_2)A - (W_1 + W_2)$$

However, (p_1 – p_2) = γh and γhA is the weight of a cylinder of fluid extending between horizontal planes 1 and 2, and the right side of the equation for F_B is identified as the weight of volume of fluid exactly equal to the of the body

$$F_B = \gamma \times (\text{Volume of object}) \qquad (2.28)$$

For the floating object of Fig. 2.23 a similar analysis will show that

$$F_B = \gamma \times (\text{Volume displaced}) \qquad (2.29)$$

and, from static equilibrium of the object, its weight must be equal to this buoyant force; thus the object displaces its own weight of the liquid in which it floats.

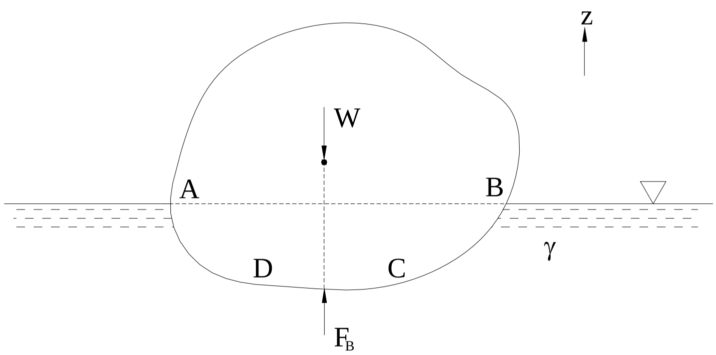


Fig. 2.23

EXAMPLE 2.6: A block of concrete weighs 100 kg in air and weighs 60 kg when immersed in water. What is the average specific weight of the block?

SOLUTION: The buoyant force is,

$$\sum F_z = 60 + F_B - 100 = 0$$

$$F_B = 40kg = \gamma_{water} \times (\text{Volume of the block})$$

$$V = \frac{40}{1000} = 0.04m^3$$

Therefore the specific weight of the block is,

$$\gamma = \frac{100}{0.04} = 2500\,kg/m^3 = 2.5\,ton/m^3$$

2.10. FLUIDS IN RELATIVE EQUILIBRIUM

If a fluid is contained in a vessel which is at rest, or moving with constant linear velocity, it is not affected by the motion of the vessel; but if the container is given a continuous acceleration, this will be transmitted to the fluid and affect the pressure distribution in it. Since the fluid remains at rest relative to the container, there is no relative motion of the particles of the fluid and, therefore, no shear stresses, fluid pressure being everywhere normal to the surface on which it acts. Under these conditions the fluid is said to be in relative equilibrium.

2.10.1. Pressure Distribution in a Liquid Subject to Horizontal Acceleration

Fig. 2.24 shows a liquid contained in a tank which has an acceleration a. A particle of mass m on the free surface at O will have the same acceleration as the tank and will be subjected to an accelerating force F. From Newton's second law,

$$F = ma \tag{2.30}$$

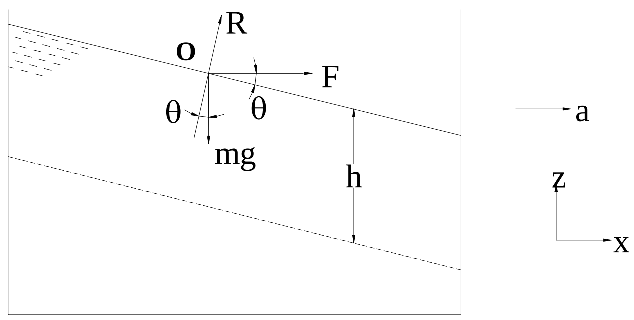


Fig. 2.24

Also, F is the resultant of the fluid pressure force R, acting normally to the free surface at O, and the weight of the particle mg, acting vertically. Therefore,

$$F = mgTan\theta \tag{2.31}$$

Comparing Eqs. (2.30) and (2.31)

$$Tan\theta = \frac{a}{g} \tag{2.32}$$

and is constant for all points on the free surface. Thus, the free surface is a plane inclined at a constant angle θ to the horizontal.

Since the acceleration is horizontal, vertical forces are not changed and the pressure at any depth h below the surface will be γh. Planes of equal pressure lie parallel to the free surface.

2.10.2. Effect of Vertical Acceleration

If the acceleration is vertical, the free surface will remain horizontal. Considering a vertical prism of cross-sectional area A (Fig. 2.25), subject to an upward acceleration a, then at depth h below the surface, where the pressure is p,

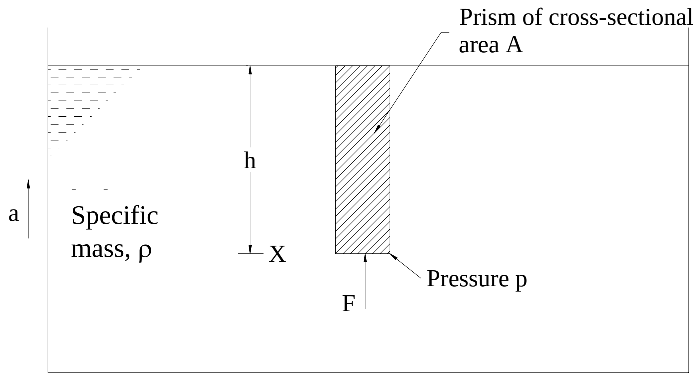


Fig. 2.25

Upward accelerating force, F = Force due to p – Weight of prism

$$F = pA - \gamma hA$$

By Newton's second law,

F = Mass of prism × Acceleration

$$F = \rho hA \times a$$

Therefore,

$$pA - \gamma hA = \rho hAa$$

$$p = \gamma h\left(1 + \frac{a}{g}\right) \qquad (2.33a)$$

If the acceleration a is downward towards to the center of the earth as gravitational acceleration, Equ. (2.33a) will take the form of,

$$p = \gamma h\left(1 - \frac{a}{g}\right) \qquad (2.33b)$$

2.10.3. General Expression for the Fluid in Relative Equilibrium

If ∂p/∂x, ∂p/∂y and ∂p/∂z are the rates of change of pressure p in the x, y and z directions (Fig. 2.26) and a_x, a_y and a_z the accelerations,

Force in x direction, $F_x = p\Delta y\Delta z - \left(p + \frac{\partial p}{\partial x}\Delta x\right)\Delta y\Delta z$

$$F_x = -\frac{\partial p}{\partial x}\Delta x\Delta y\Delta z$$

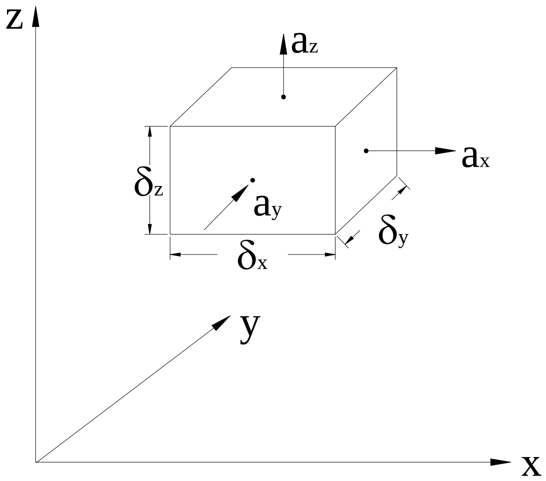


Fig. 2.26

By Newton's second law, $F_x = \rho\Delta x\Delta y\Delta z \times a_x$, therefore

$$-\frac{\partial p}{\partial x} = \rho a_x \qquad (2.34)$$

Similarly, in the y direction

$$-\frac{\partial p}{\partial y} = \rho a_y \qquad (2.35)$$

In the vertical z direction, the weight of the element $W = \rho g\Delta x\Delta y\Delta z$ must be considered:

$$F_z = p\Delta x\Delta y - \left(p + \frac{\partial p}{\partial z}\Delta z\right)\Delta x\Delta y - \rho g\Delta x\Delta y\Delta z$$

$$F_x = -\frac{\partial p}{\partial z}\Delta x\Delta y\Delta z - \rho g\Delta x\Delta y\Delta z$$

By Newton's second law, $F_z = \rho\Delta x\Delta y\Delta z \times a_z$, therefore,

$$-\frac{\partial p}{\partial z} = \rho(g + a_z) \qquad (2.36)$$

For an acceleration a_s in any direction in the x-z plane making an angle ϕ with the horizontal, the components of the acceleration are

$a_x = a_s Cos\phi$ and $a_z = a_s Sin\phi$

Now

$$\frac{dp}{ds} = \frac{\partial p}{\partial x}\frac{dx}{ds} + \frac{\partial p}{\partial z}\frac{dz}{ds} \qquad (2.37)$$

For the free surface and all other planes of constant pressure, dp/ds = 0. If θ is the inclination of the planes of constant pressure to the horizontal, Tanθ = dz/dx. Putting dp/ds = 0 in Equ. (2.37)

$$\frac{\partial p}{\partial x}\frac{dx}{ds} + \frac{\partial p}{\partial z}\frac{dz}{ds} = 0$$

$$\frac{dz}{dx} = Tan\theta = -\frac{\partial p/\partial x}{\partial p/\partial z}$$

Substituting from Eqs. (2.34) and (2.36)

$$Tan\theta = -\frac{a_x}{g + a_z} \qquad (2.38)$$

Or, in terms of a_s,

$$Tan\theta = -\frac{a_s Cos\phi}{(g + a_s Sin\phi)} \qquad (2.39)$$

For the case of horizontal acceleration, $\phi = 0$ and Equ. (2.39) gives Tanθ = $-a_s/g$, which agrees with Equ. (2.32). For vertical acceleration, $\phi = 90^0$ giving Tanθ = 0, indicating that the free surface remains horizontal.

EXAMPLE 2.7: A rectangular tank 1.2 m deep and 2 m long is used to convey water up a ramp inclined at an angle ϕ of 30^0 to the horizontal (Fig. 2.27).

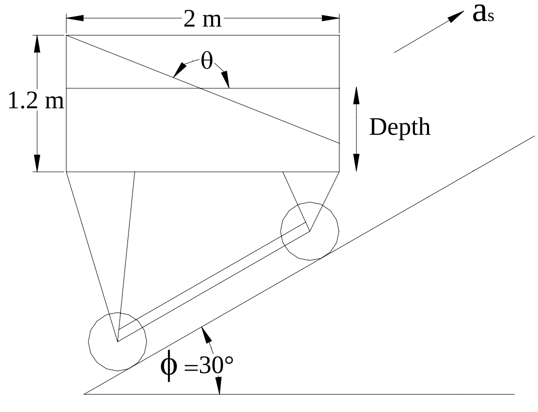


Fig. 2.27

Calculate the inclination of the water surface to the horizontal when,

a) The acceleration parallel to the slope on starting from bottom is 4 m/sec^2,
b) The deceleration parallel to the slope on reaching the top is 4.5 m/sec^2.

If no water is to be spilt during the journey what is the greatest depth of water permissible in the tank when it is at rest?

SOLUTION: The slope of the water surface is given by Equ. (2.39). During acceleration, a_s = 4 m/sec^2

$$Tan\theta_A = -\frac{a_s Cos\phi}{g + a_s Sin\phi} = -\frac{4 \times Cos30^0}{9.81 + 4 \times Sin30^0} = -0.2933$$

$$\theta_A = 163^0 39'$$

During retardation, a_s = - 4.5 m/sec^2,

$$Tan\theta_R = -\frac{(-4.5) \times Cos30^0}{9.81 - 4.5 \times Sin30^0} = 0.5154$$

$$\theta_R = 27^0 16'$$

Since $180^0 - \theta_R > \theta_A$, the worst case for spilling will be during retardation. When the water surface is inclined, the maximum depth at the tank wall will be

Depth + 0.5×Length×Tanθ

Which must not exceed 1.2 m if the water is not to be spilt. Putting length = 2 m, Tanθ = Tan θ_R = 0.5154,

Depth + 0.5×2×0.5154 = 1.2

Depth = 1.2 – 0.5154 = 0.6846 m

2.10.4. Forced Vortex

A body of fluid, contained in a vessel, which is rotating about a vertical axis with uniform angular velocity, will eventually reach relative equilibrium and rotate with the same angular velocity w as the vessel, forming a forced vortex. The acceleration of any particular of fluid at radius r due to rotation will be ($-w^2r$) perpendicular to the axis of rotation, taking the direction of r as positive outward from the axis. Thus, from Equ. (2.34),

$$\frac{dp}{dr} = -\rho w^2 r \qquad (2.40)$$

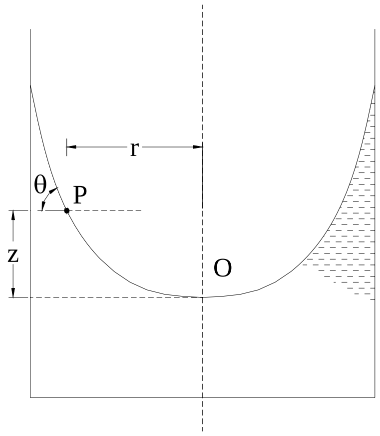


Fig. 2.28

Fig. 2.28 shows a cylindrical vessel containing liquid rotating about its axis, which is vertical. At any P on free surface, the inclination θ of the free surface is given by Equ. (2.38),

$$Tan\theta = -\frac{a_x}{g + a_z} = \frac{w^2 r}{g} = \frac{dz}{dr} \tag{2.41}$$

The inclination of the free surface varies with r and, if z is the height of P above O, the surface profile is given by integrating Equ. (2.41):

$$z = \int_0^x \frac{w^2 r}{g} dr = \frac{w^2 r^2}{2g} + C \tag{2.42}$$

Thus, the profile of the water is a paraboloid. Similarly, other surfaces of equal pressure will also be paraboloids.

The value of integration constant is found by specifying the pressure at one point. If z = 0 at point O, then the integration constant is zero. Then the Equ. (2.42) becomes,

$$z = \frac{w^2 r^2}{2g} \tag{2.43}$$