# LECTURE 11/12

LONGITUDINAL AND LATERAL
LINEARIZED EQUATIONS OF MOTION
AND
CHARACTERISTIC MODES

WE WILL COVER THIS TOPIC

- MAINLY FROM MIT AA 16-333
  (PROF. J.P. HOW) [my advisor (M.S. / Postdoc)]

- BASED ON
  - STEVEN'S (A LITTLE)
  - ETKIN (MOSTLY)
  - NELSON (PARTIALLY)

## CLASS LOGISTICS

# HW 2 out today!

- HW 2 due : Dec 19 [Class Time]

# HW 3 will be out on Dec 19th!

- HW 3 due : Dec 26 [Class Time]

**MIDTERM # 2 is on Dec 27th 6pm**

# HW 4 will be out the last week of classes. Due before beginning of final week TRAINING

## Reading Assignment

- NELSON Chp 3 (was already due....)
- NELSON <u>Chp 4, 5</u>

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Before Final....

- NELSON <u>Chp 7, 8</u>

## 16.333 Lecture 4

Aircraft Dynamics

- Aircraft nonlinear EOM
- Linearization – dynamics
- Linearization – forces & moments
- Stability derivatives and coefficients

# Aircraft Dynamics

- Note can develop good approximation of key aircraft motion (Phugoid) using simple balance between kinetic and potential energies.
- Consider an aircraft in steady, level flight with speed $U_0$ and height $h_0$. The motion is perturbed slightly so that

$$U_0 \rightarrow U = U_0 + u \tag{1}$$
$$h_0 \rightarrow h = h_0 + \Delta h \tag{2}$$

- Assume that $E = \frac{1}{2}mU^2 + mgh$ is constant before and after the perturbation. It then follows that $u \approx -\frac{g\Delta h}{U_0}$
- From Newton's laws we know that, in the vertical direction

$$m\ddot{h} = L - W$$

where weight $W = mg$ and lift $L = \frac{1}{2}\rho S C_L U^2$ ($S$ is the wing area). We can then derive the equations of motion of the aircraft:

$$m\ddot{h} = L - W = \frac{1}{2}\rho S C_L (U^2 - U_0^2) \tag{3}$$
$$= \frac{1}{2}\rho S C_L((U_0+u)^2 - U_0^2) \approx \frac{1}{2}\rho S C_L(2uU_0) \tag{4}$$
$$\approx -\rho S C_L \left(\frac{g\Delta h}{U_0}U_0\right) = -(\rho S C_L g)\Delta h \tag{5}$$

Since $\ddot{h} = \Delta\ddot{h}$ and for the original equilibrium flight condition $L = W = \frac{1}{2}(\rho S C_L)U_0^2 = mg$, we get that

$$\frac{\rho S C_L g}{m} = 2\left(\frac{g}{U_0}\right)^2$$

Combine these result to obtain:

$$\Delta\ddot{h} + \Omega^2 \Delta h = 0 \quad , \quad \Omega \approx \frac{g}{U_0}\sqrt{2}$$

- These equations describe an oscillation (called the phugoid oscillation) of the altitude of the aircraft about it nominal value.
  - Only approximate natural frequency (Lanchester), but value close.

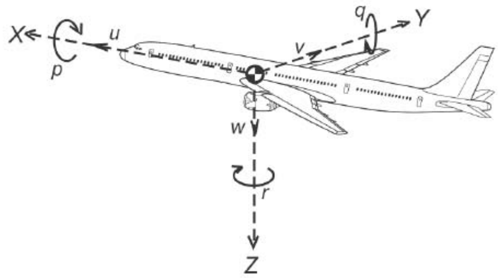
- The basic dynamics are:

$$\vec{F} = m\dot{\vec{v}}_c^{\,I} \quad \text{and} \quad \vec{T} = \dot{\vec{H}}^{I}$$

$$\Rightarrow \ \frac{1}{m}\vec{F} = \dot{\vec{v}}_c^{\,B} + \ {}^{BI}\vec{\omega} \times \vec{v}_c \qquad \textbf{Transport Thm.}$$

$$\Rightarrow \ \vec{T} = \dot{\vec{H}}^{B} + \ {}^{BI}\vec{\omega} \times \vec{H}$$

- Basic assumptions are:




1. Earth is an inertial reference frame
2. A/C is a rigid body
3. Body frame **B** fixed to the aircraft ($\vec{i}, \vec{j},$

- Instantaneous mapping of $\vec{v}_c$ and ${}^{BI}\vec{\omega}$ into the body frame:

$${}^{BI}\vec{\omega} = P\vec{i} + Q\vec{j} + R\vec{k} \qquad \vec{v}_c = U\vec{i} + V\vec{j} + W\vec{k}$$

$$\Rightarrow \ {}^{BI}\omega_B = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \qquad \Rightarrow (v_c)_B = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

- By symmetry, we can show that $I_{xy} = I_{yz} = 0$, but value of $I_{xz}$ depends on specific frame selected. Instantaneous mapping of the angular momentum

$$\vec{H} = H_x\vec{i} + H_y\vec{j} + H_z\vec{k}$$

into the Body Frame given by

$$H_B = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_{yy} & 0 \\ I_{xz} & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

- The overall equations of motion are then:

$$\frac{1}{m}\vec{F} = \dot{\vec{v}}_c^{\,B} + {}^{BI}\vec{\omega} \times \vec{v}_c$$

$$\Rightarrow \frac{1}{m}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{bmatrix} + \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix}\begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

$$= \begin{bmatrix} \dot{U} + QW - RV \\ \dot{V} + RU - PW \\ \dot{W} + PV - QU \end{bmatrix}$$

$$\vec{T} = \dot{\vec{H}}^{B} + {}^{BI}\vec{\omega} \times \vec{H}$$

$$\Rightarrow \begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{P} + I_{xz}\dot{R} \\ I_{yy}\dot{Q} \\ I_{zz}\dot{R} + I_{xz}\dot{P} \end{bmatrix} + \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix}\begin{bmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_{yy} & 0 \\ I_{xz} & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$= \begin{bmatrix} I_{xx}\dot{P} + I_{xz}\dot{R} & +QR(I_{zz} - I_{yy}) + PQI_{xz} \\ I_{yy}\dot{Q} & +PR(I_{xx} - I_{zz}) + (R^2 - P^2)I_{xz} \\ I_{zz}\dot{R} + I_{xz}\dot{P} & +PQ(I_{yy} - I_{xx}) - QRI_{xz} \end{bmatrix}$$

- Clearly these equations are very nonlinear and complicated, and we have not even said where $\vec{F}$ and $\vec{T}$ come from. $\Longrightarrow$ Need to linearize!!
  - Assume that the aircraft is flying in an *equilibrium condition* and we will linearize the equations about this nominal flight condition.

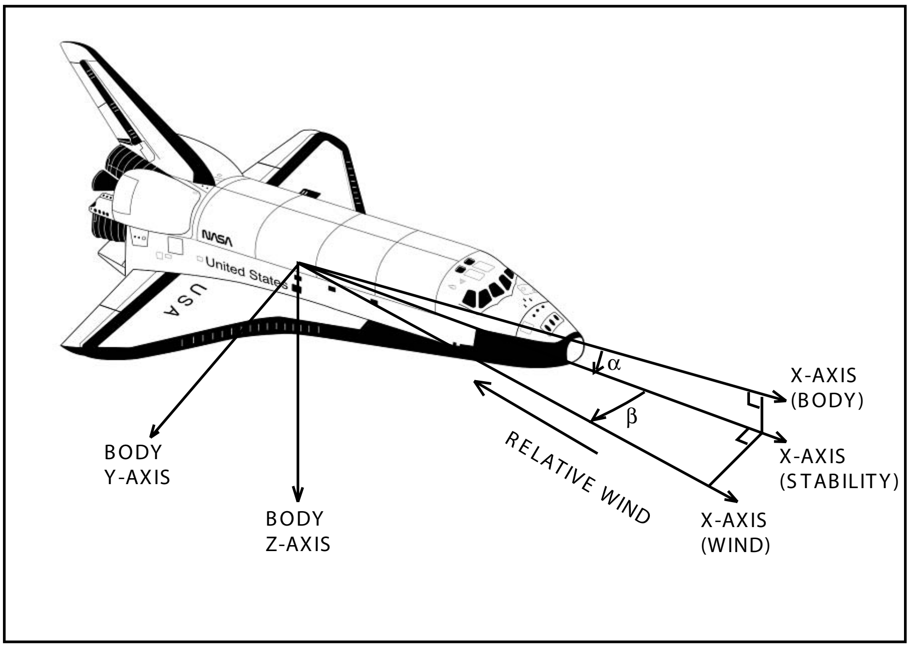
# Axes

- But first we need to be a little more specific about which *Body Frame* we are going use. Several standards:

  1. **Body Axes** - X aligned with fuselage nose. Z perpendicular to X in plane of symmetry (down). Y perpendicular to XZ plane, to the right.

  2. **Wind Axes** - X aligned with $\vec{v}_c$. Z perpendicular to X (pointed down). Y perpendicular to XZ plane, off to the right.

  3. **Stability Axes** - X aligned with projection of $\vec{v}_c$ into the fuselage plane of symmetry. Z perpendicular to X (pointed down). Y same.




- Advantages to each, but typically use the **stability axes**.

  – In different *flight equilibrium conditions*, the axes will be oriented differently with respect to the A/C principal axes ⇒ need to transform (rotate) the principal inertia components between the frames.

  – When vehicle undergoes motion with respect to the equilibrium, **Stability Axes remain fixed to airplane as if painted on.**

- Can linearize about various steady state conditions of flight.
  - For steady state flight conditions must have

    $$\vec{F} = \vec{F}_{\text{aero}} + \vec{F}_{\text{gravity}} + \vec{F}_{\text{thrust}} = 0 \;\; \text{and} \; \vec{T} = 0$$

    ◇ So for equilibrium condition, forces balance on the aircraft $L = W$ and $T = D$

  - Also assume that $\dot{P} = \dot{Q} = \dot{R} = \dot{U} = \dot{V} = \dot{W} = 0$

  - Impose additional constraints that depend on **flight condition:**

    ◇ Steady wings-level flight $\rightarrow \Phi = \dot{\Phi} = \dot{\Theta} = \dot{\Psi} = 0$

- **Key Point:** While nominal forces and moments balance to zero, motion about the equilibrium condition results in perturbations to the forces/moments.
  - Recall from basic flight dynamics that lift $L_0^f = C_{L_\alpha}\alpha_0$ where:

    ◇ $C_{L_\alpha}$ = *lift curve slope* – function of the equilibrium condition

    ◇ $\alpha_0$ = nominal *angle of attack* (angle that wing meets air flow)

  - But, as the vehicle moves about the equilibrium condition, would expect that the angle of attack will change

    $$\alpha = \alpha_0 + \Delta\alpha$$

  - Thus the lift forces will also be perturbed

    $$L^f = C_{L_\alpha}(\alpha_0 + \Delta\alpha) = L_0^f + \Delta L^f$$

- Can extend this idea to all dynamic variables and how they influence all aerodynamic forces and moments

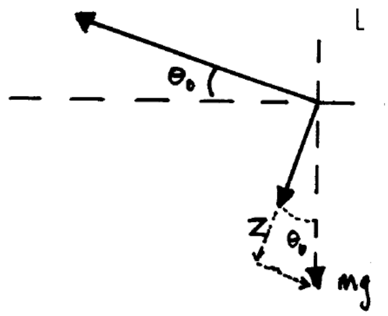
# Gravity Forces

- Gravity acts through the CoM in vertical direction (inertial frame +Z)
  - Assume that we have a non-zero pitch angle $\Theta_0$
  - Need to map this force into the body frame
  - Use the Euler angle transformation (2–15)

$$F_B^g = T_1(\Phi)T_2(\Theta)T_3(\Psi)\begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = mg\begin{bmatrix} -\sin\Theta \\ \sin\Phi\cos\Theta \\ \cos\Phi\cos\Theta \end{bmatrix}$$

- For symmetric steady state flight equilibrium, we will typically assume that $\Theta \equiv \Theta_0$, $\Phi \equiv \Phi_0 = 0$, so

$$F_B^g = mg\begin{bmatrix} -\sin\Theta_0 \\ 0 \\ \cos\Theta_0 \end{bmatrix}$$




- Use Euler angles to specify vehicle rotations with respect to the Earth frame

$$\begin{aligned} \dot{\Theta} &= Q\cos\Phi - R\sin\Phi \\ \dot{\Phi} &= P + Q\sin\Phi\tan\Theta + R\cos\Phi\tan\Theta \\ \dot{\Psi} &= (Q\sin\Phi + R\cos\Phi)\sec\Theta \end{aligned}$$

  - Note that if $\Phi \approx 0$, then $\dot{\Theta} \approx Q$

- **Recall:** $\Phi \approx$ Roll, $\Theta \approx$ Pitch, and $\Psi \approx$ Heading.

# Linearization

- Define the **trim** angular rates and velocities

$$ {}^{BI}\omega_B^o = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \qquad (v_c)_B^o = \begin{bmatrix} U_o \\ 0 \\ 0 \end{bmatrix} $$

which are associated with the flight condition. In fact, these define the type of equilibrium motion that we linearize about. **Note:**

– $W_0 = 0$ since we are using the stability axes, and

– $V_0 = 0$ because we are assuming symmetric flight

- Proceed with linearization of the dynamics for various flight conditions

| | Nominal Velocity | Perturbed Velocity | $\Rightarrow$ | Perturbed Acceleration |
|---|---|---|---|---|
| Velocities | $U_0,$ | $U = U_0 + u$ | $\Rightarrow$ | $\dot{U} = \dot{u}$ |
| | $W_0 = 0,$ | $W = w$ | $\Rightarrow$ | $\dot{W} = \dot{w}$ |
| | $V_0 = 0,$ | $V = v$ | $\Rightarrow$ | $\dot{V} = \dot{v}$ |
| Angular Rates | $P_0 = 0,$ | $P = p$ | $\Rightarrow$ | $\dot{P} = \dot{p}$ |
| | $Q_0 = 0,$ | $Q = q$ | $\Rightarrow$ | $\dot{Q} = \dot{q}$ |
| | $R_0 = 0,$ | $R = r$ | $\Rightarrow$ | $\dot{R} = \dot{r}$ |
| Angles | $\Theta_0,$ | $\Theta = \Theta_0 + \theta$ | $\Rightarrow$ | $\dot{\Theta} = \dot{\theta}$ |
| | $\Phi_0 = 0,$ | $\Phi = \phi$ | $\Rightarrow$ | $\dot{\Phi} = \dot{\phi}$ |
| | $\Psi_0 = 0,$ | $\Psi = \psi$ | $\Rightarrow$ | $\dot{\Psi} = \dot{\psi}$ |

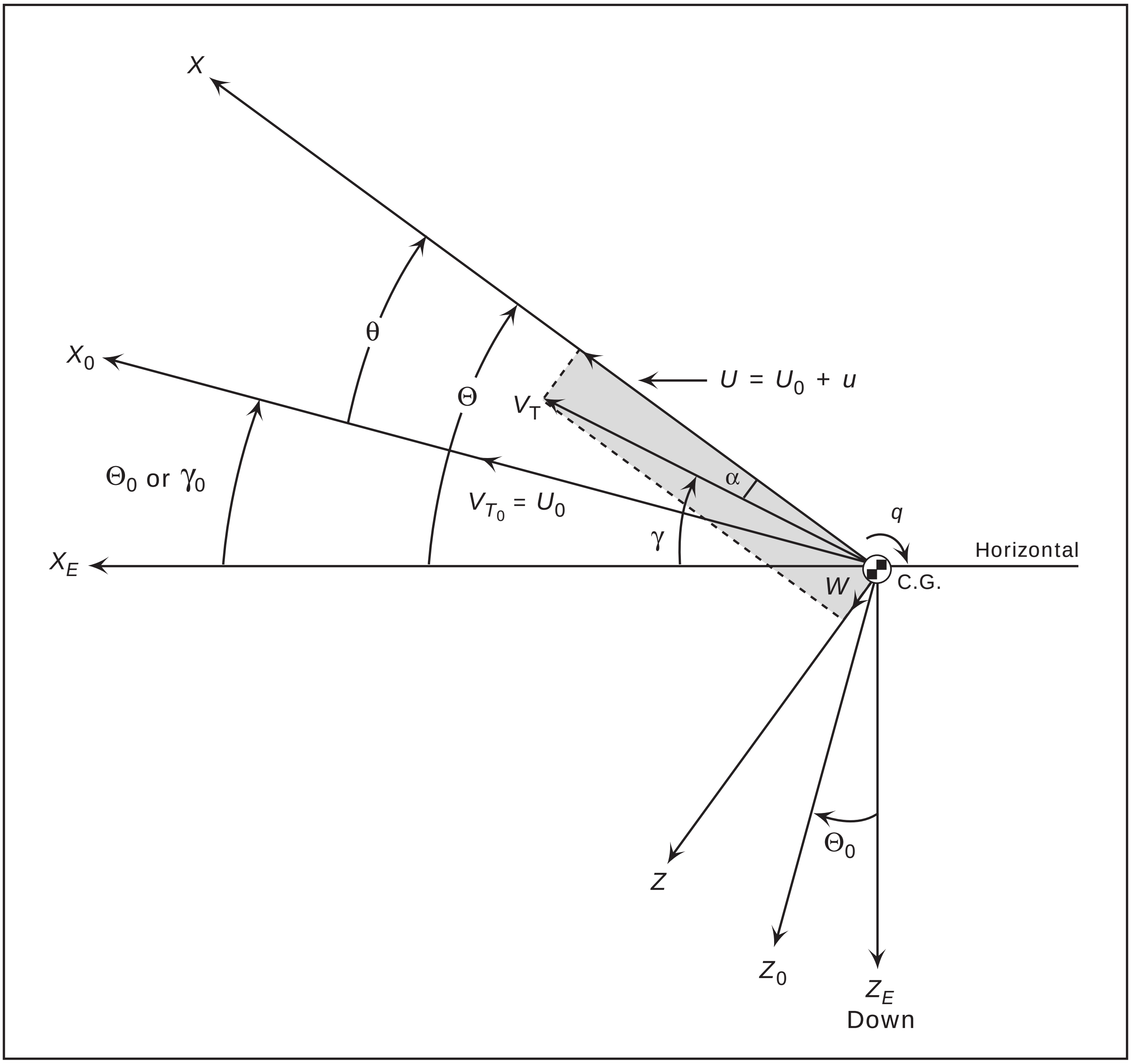



Figure 1: Perturbed Axes. The equilibrium condition was that the aircraft was angled up by $\Theta_0$ with velocity $V_{T0} = U_0$. The vehicle's motion has been perturbed $(X_0 \rightarrow X)$ so that now $\Theta = \Theta_0 + \theta$ and the velocity is $V_T \neq V_{T0}$. Note that $V_T$ is no longer aligned with the $X$-axis, resulting in a non-zero $u$ and $w$. The angle $\gamma$ is called the **flight path angle,** and it provides a measure of the angle of the velocity vector to the inertial horizontal axis.

- **Linearization for symmetric flight**
  $U = U_0 + u$, $V_0 = W_0 = 0$, $P_0 = Q_0 = R_0 = 0$.

  Note that the forces and moments are also perturbed.

$$\begin{aligned}
\frac{1}{m}\left[X_0 + \Delta X\right] &= \dot{U} + QW - RV \;\approx\; \dot{u} + qw - rv \approx \dot{u} \\
\frac{1}{m}\left[Y_0 + \Delta Y\right] &= \dot{V} + RU - PW \\
&\approx\; \dot{v} + r(U_0 + u) - pw \;\approx\; \dot{v} + rU_0 \\
\frac{1}{m}\left[Z_0 + \Delta Z\right] &= \dot{W} + PV - QU \;\approx\; \dot{w} + pv - q(U_0 + u) \\
&\approx\; \dot{w} - qU_0
\end{aligned}$$

$$\Rightarrow \frac{1}{m}\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \dot{v} + rU_0 \\ \dot{w} - qU_0 \end{bmatrix} \quad \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix}$$

- Attitude motion:

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{P} + I_{xz}\dot{R} & +QR(I_{zz} - I_{yy}) + PQI_{xz} \\ I_{yy}\dot{Q} & +PR(I_{xx} - I_{zz}) + (R^2 - P^2)I_{xz} \\ I_{zz}\dot{R} + I_{xz}\dot{P} & +PQ(I_{yy} - I_{xx}) - QRI_{xz} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \Delta L \\ \Delta M \\ \Delta N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{p} + I_{xz}\dot{r} \\ I_{yy}\dot{q} \\ I_{zz}\dot{r} + I_{xz}\dot{p} \end{bmatrix} \quad \begin{matrix} \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \end{matrix}$$

- Key aerodynamic parameters are also perturbed:

$$\begin{aligned}
&\text{Total Velocity} \\
V_T &= ((U_0+u)^2+v^2+w^2)^{1/2} \approx U_0+u \\
&\text{Perturbed Sideslip angle} \\
\beta &= \sin^{-1}(v/V_T) \approx v/U_0 \\
&\text{Perturbed Angle of Attack} \\
\alpha_x &= \tan^{-1}(w/U) \approx w/U_0
\end{aligned}$$

- To understand these equations in detail, and the resulting impact on the vehicle dynamics, we must investigate the terms $\Delta X \ldots \Delta N$.
  - We must also address the left-hand side ($\vec{F}$, $\vec{T}$)
  - **Net** forces and moments must be zero in equilibrium condition.
  - Aerodynamic and Gravity forces are a function of equilibrium condition **AND** the perturbations about this equilibrium.

- Predict the changes to the aerodynamic forces and moments using a first order expansion in the key flight parameters

$$\begin{aligned}
\Delta X &= \frac{\partial X}{\partial U}\mathbf{\Delta U} + \frac{\partial X}{\partial W}\mathbf{\Delta W} + \frac{\partial X}{\partial \dot{W}}\mathbf{\Delta \dot{W}} + \frac{\partial X}{\partial \Theta}\Delta\Theta + \ldots + \frac{\partial X^g}{\partial \Theta}\Delta\Theta + \Delta X^c \\
&= \frac{\partial X}{\partial U}\mathbf{u} + \frac{\partial X}{\partial W}\mathbf{w} + \frac{\partial X}{\partial \dot{W}}\mathbf{\dot{w}} + \frac{\partial X}{\partial \Theta}\theta + \ldots + \frac{\partial X^g}{\partial \Theta}\theta + \Delta X^c
\end{aligned}$$

- $\frac{\partial X}{\partial U}$ called **stability derivative** – evaluated at eq. condition.

- Gives dimensional form; non-dimensional form available in tables.

- Clearly approximation since ignores lags in the aerodynamics forces (assumes that forces only function of instantaneous values)

# Stability Derivatives

- First proposed by Bryan (1911) – has proven to be a **very** effective way to analyze the aircraft flight mechanics – well supported by numerous flight test comparisons.
- The forces and torques acting on the aircraft are very complex nonlinear functions of the flight equilibrium condition and the perturbations from equilibrium.
  - Linearized expansion can involve many terms $u, \dot{u}, \ddot{u}, \ldots, w, \dot{w}, \ddot{w}, \ldots$
  - Typically only retain a few terms to capture the dominant effects.
- Dominant behavior most easily discussed in terms of the:
  - Symmetric variables: $U$, $W$, $Q$ & forces/torques: $X$, $Z$, and $M$
  - Asymmetric variables: $V$, $P$, $R$ & forces/torques: $Y$, $L$, and $N$
- Observation – for truly symmetric flight $Y$, $L$, and $N$ will be exactly **zero** for any value of $U$, $W$, $Q$

  $\Rightarrow$ Derivatives of asymmetric forces/torques with respect to the symmetric motion variables are **zero**.
- Further (convenient) assumptions:
  1. Derivatives of symmetric forces/torques with respect to the asymmetric motion variables are small and can be neglected.
  2. We can neglect derivatives with respect to the derivatives of the motion variables, but keep $\partial Z/\partial \dot{w}$ and $M_{\dot{w}} \equiv \partial M/\partial \dot{w}$ (aerodynamic lag involved in forming new pressure distribution on the wing in response to the perturbed angle of attack)
  3. $\partial X/\partial q$ is negligibly small.

| $\partial()/\partial()$ | **X** | **Y** | **Z** | **L** | **M** | **N** |
|---|---|---|---|---|---|---|
| **u** | $\bullet$ | 0 | $\bullet$ | 0 | $\bullet$ | 0 |
| **v** | 0 | $\bullet$ | 0 | $\bullet$ | 0 | $\bullet$ |
| **w** | $\bullet$ | 0 | $\bullet$ | 0 | $\bullet$ | 0 |
| **p** | 0 | $\bullet$ | 0 | $\bullet$ | 0 | $\bullet$ |
| **q** | $\approx 0$ | 0 | $\bullet$ | 0 | $\bullet$ | 0 |
| **r** | 0 | $\bullet$ | 0 | $\bullet$ | 0 | $\bullet$ |

- Note that we must also find the perturbation gravity and thrust forces and moments

$$\left.\frac{\partial X^g}{\partial \Theta}\right|_0 = -mg\cos\Theta_0 \qquad \left.\frac{\partial Z^g}{\partial \Theta}\right|_0 = -mg\sin\Theta_0$$

- **Aerodynamic summary:**

**1A** $\Delta X = \left(\frac{\partial X}{\partial U}\right)_0 u + \left(\frac{\partial X}{\partial W}\right)_0 w \Rightarrow \Delta X \sim u,\, \alpha_x \approx w/U_0$

**2A** $\Delta Y \sim \beta \approx v/U_0,\, p,\, r$

**3A** $\Delta Z \sim u,\, \alpha_x \approx w/U_0,\, \dot{\alpha}_x \approx \dot{w}/U_0,\, q$

**4A** $\Delta L \sim \beta \approx v/U_0,\, p,\, r$

**5A** $\Delta M \sim u,\, \alpha_x \approx w/U_0,\, \dot{\alpha}_x \approx \dot{w}/U_0,\, q$

**6A** $\Delta N \sim \beta \approx v/U_0,\, p,\, r$

- Result is that, with these force, torque approximations, equations **1**, **3**, **5** decouple from **2**, **4**, **6**

  – **1**, **3**, **5** are the **longitudinal dynamics** in $u$, $w$, and $q$

$$\begin{bmatrix} \Delta X \\ \Delta Z \\ \Delta M \end{bmatrix} = \begin{bmatrix} m\dot{u} \\ m(\dot{w} - qU_0) \\ I_{yy}\dot{q} \end{bmatrix}$$

$$\approx \begin{bmatrix} \left(\frac{\partial X}{\partial U}\right)_0 u + \left(\frac{\partial X}{\partial W}\right)_0 w + \left(\frac{\partial X^g}{\partial \Theta}\right)_0 \theta + \Delta X^c \\ \left(\frac{\partial Z}{\partial U}\right)_0 u + \left(\frac{\partial Z}{\partial W}\right)_0 w + \left(\frac{\partial Z}{\partial \dot{W}}\right)_0 \dot{w} + \left(\frac{\partial Z}{\partial Q}\right)_0 q + \left(\frac{\partial Z^g}{\partial \Theta}\right)_0 \theta + \Delta Z^c \\ \left(\frac{\partial M}{\partial U}\right)_0 u + \left(\frac{\partial M}{\partial W}\right)_0 w + \left(\frac{\partial M}{\partial \dot{W}}\right)_0 \dot{w} + \left(\frac{\partial M}{\partial Q}\right)_0 q + \Delta M^c \end{bmatrix}$$

  – **2**, **4**, **6** are the **lateral dynamics** in $v$, $p$, and $r$

$$\begin{bmatrix} \Delta Y \\ \Delta L \\ \Delta N \end{bmatrix} = \begin{bmatrix} m(\dot{v} + rU_0) \\ I_{xx}\dot{p} + I_{xz}\dot{r} \\ I_{zz}\dot{r} + I_{xz}\dot{p} \end{bmatrix}$$

$$\approx \begin{bmatrix} \left(\frac{\partial Y}{\partial V}\right)_0 v + \left(\frac{\partial Y}{\partial P}\right)_0 p + \left(\frac{\partial Y}{\partial R}\right)_0 r + +\Delta Y^c \\ \left(\frac{\partial L}{\partial V}\right)_0 v + \left(\frac{\partial L}{\partial P}\right)_0 p + \left(\frac{\partial L}{\partial R}\right)_0 r + \Delta L^c \\ \left(\frac{\partial N}{\partial V}\right)_0 v + \left(\frac{\partial N}{\partial P}\right)_0 p + \left(\frac{\partial N}{\partial R}\right)_0 r + \Delta N^c \end{bmatrix}$$

# Basic Stability Derivative Derivation

- Consider changes in the drag force with forward speed $U$

$$
\begin{aligned}
D &= 1/2\rho V_T^2 S C_D \\
V_T^2 &= (u_0+u)^2 + v^2 + w^2 \\
\frac{\partial V_T^2}{\partial u} &= 2(u_0+u) \Rightarrow \left(\frac{\partial V_T^2}{\partial u}\right)_0 = 2u_0
\end{aligned}
$$

Note: $\left(\frac{\partial V_T^2}{\partial v}\right)_0 = 0$ and $\left(\frac{\partial V_T^2}{\partial w}\right)_0 = 0$

At reference condition:

$$
\begin{aligned}
\Rightarrow D_u \equiv \left(\frac{\partial D}{\partial u}\right)_0 &= \left\{\frac{\partial}{\partial u}\left(\frac{\rho V_T^2 S C_D}{2}\right)\right\}_0 \\
&= \frac{\rho S}{2}\left(u_0^2\left(\frac{\partial C_D}{\partial u}\right)_0 + C_{D_0}\left(\frac{\partial V_T^2}{\partial u}\right)_0\right) \\
&= \frac{\rho S}{2}\left(u_0^2\left(\frac{\partial C_D}{\partial u}\right)_0 + 2u_0 C_{D_0}\right)
\end{aligned}
$$

– Note $\frac{\partial D}{\partial u}$ is the **stability derivative**, which is **dimensional**.

- Define nondimensional **stability coefficient** $C_{D_u}$ as derivative of $C_D$ with respect to a **nondimensional velocity** $u/u_0$

$$
C_D = \frac{D}{\frac{1}{2}\rho V_T^2 S} \quad \Rightarrow \quad C_{D_u} \equiv \left(\frac{\partial C_D}{\partial u/u_0}\right)_0 \quad \text{and} \quad \mathbf{C_{D_0} \equiv (C_D)_0}
$$

– So $(\bullet)_0$ corresponds to the variable at its equilibrium condition.

- Nondimensionalize:

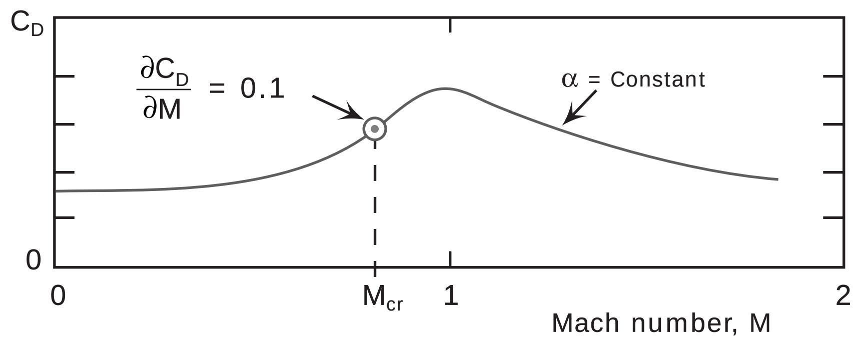
$$\begin{aligned}\left(\frac{\partial D}{\partial u}\right)_0 &= \frac{\rho S u_0}{2}\left(u_0\left(\frac{\partial C_D}{\partial u}\right)_0 + 2C_{D_0}\right)\\ &= \frac{QS}{u_0}\left(\left(\frac{\partial C_D}{\partial u/u_0}\right)_0 + 2C_{D_0}\right)\\ \left(\frac{u_0}{QS}\right)\left(\frac{\partial D}{\partial u}\right)_0 &= (C_{D_u} + 2C_{D_0})\end{aligned}$$

  So given stability coefficient, can compute the drag force increment.

- Note that Mach number has a significant effect on the drag:

$$C_{D_u} = \left(\frac{\partial C_D}{\partial u/u_0}\right)_0 = \left(\frac{u_0}{a}\frac{\partial C_D}{\partial\left(\frac{u}{a}\right)}\right)_0 = \mathbf{M}\frac{\partial C_D}{\partial \mathbf{M}}$$

  where $\frac{\partial C_D}{\partial \mathbf{M}}$ can be estimated from empirical results/tables.



Aerodynamic Principles

- Thrust forces

$$C_{T_u} = \left(\frac{\partial C_T}{\partial u/u_0}\right)_0 \quad\Rightarrow\quad \left(\frac{\partial T}{\partial u}\right)_0 = C_{T_u}\frac{1}{u_0}QS$$

  – For a glider, $C_{T_u} = 0$
  – For a jet, $C_{T_u} \approx 0$
  – For a prop plane, $C_{T_u} = -C_{D_0}$

- Lift forces similar to drag

$$L = 1/2\rho V_T^2 S C_L$$

$$\begin{aligned}\Rightarrow \left(\frac{\partial L}{\partial u}\right)_0 &= \frac{\rho S u_0}{2}\left(u_0\left(\frac{\partial C_L}{\partial u}\right)_0 + 2C_{L_0}\right)\\ &= \frac{QS}{u_0}\left(\left(\frac{\partial C_L}{\partial u/u_0}\right)_0 + 2C_{L_0}\right)\end{aligned}$$

$$\left(\frac{u_0}{QS}\right)\left(\frac{\partial L}{\partial u}\right)_0 = (C_{L_u} + 2C_{L_0})$$

where $C_{L_0}$ is the lift coefficient for the eq. condition and $C_{L_u} = \mathbf{M}\frac{\partial C_L}{\partial \mathbf{M}}$ as before. From aerodynamic theory, we have that

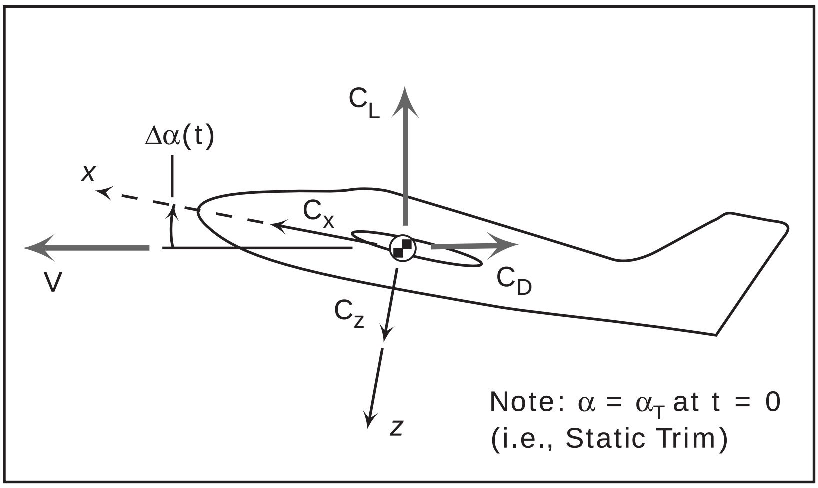
$$C_L = \frac{C_L|_{\mathbf{M=0}}}{\sqrt{1-\mathbf{M^2}}} \quad \Rightarrow \quad \frac{\partial C_L}{\partial \mathbf{M}} = \frac{\mathbf{M}}{1-\mathbf{M^2}}C_L$$

$$\Rightarrow C_{L_u} = \frac{\mathbf{M^2}}{1-\mathbf{M^2}}C_{L_0}$$

- $\alpha$ Derivatives: Now consider what happens with changes in the angle of attack. Take derivatives and evaluate at the reference condition:
  - Lift: $\Rightarrow C_{L_\alpha}$
  - Drag: $C_D = C_{D_{\min}} + \dfrac{C_L^2}{\pi e AR} \Rightarrow C_{D_\alpha} = \dfrac{2C_{L_0}}{\pi e AR}C_{L_\alpha}$

- Combine into $X$, $Z$ Forces
  - At equilibrium, forces balance.
  - Use stability axes, so $\alpha_0 = 0$
  - Include the effect in the force balance of a change in $\alpha$ on the force rotations so that we can see the perturbations.
  - Assume perturbation $\alpha$ is small, so rotations are by $\cos\alpha \approx 1$, $\sin\alpha \approx \alpha$

$$
\begin{aligned}
X &= T - D + L\alpha \\
Z &= -(L + D\alpha)
\end{aligned}
$$




- So, now consider the $\alpha$ derivatives of these forces:

$$
\frac{\partial X}{\partial \alpha} = \frac{\partial T}{\partial \alpha} - \frac{\partial D}{\partial \alpha} + L + \alpha \frac{\partial L}{\partial \alpha}
$$

  - Thrust variation with $\alpha$ very small $\left(\frac{\partial T}{\partial \alpha}\right)_0 \approx 0$
  - Apply at the reference condition ($\alpha = 0$), i.e. $C_{X_\alpha} = \left(\frac{\partial C_X}{\partial \alpha}\right)_0$

- Nondimensionalize and apply reference condition:

$$
\begin{aligned}
C_{X_\alpha} &= -C_{D_\alpha} + C_{L_0} \\
&= C_{L_0} - \frac{2C_{L_0}}{\pi e A\!\!R} C_{L_\alpha}
\end{aligned}
$$

- And for the $Z$ direction
$$\frac{\partial Z}{\partial \alpha} = -D - \alpha \frac{\partial D}{\partial \alpha} - \frac{\partial L}{\partial \alpha}$$
Giving
$$C_{Z_\alpha} = -C_{D_0} - C_{L_\alpha}$$

- Recall that $C_{M_\alpha}$ was already found during the static analysis

- Can repeat this process for the other derivatives with respect to the forward speed.

- Forward speed:
$$\frac{\partial X}{\partial u} = \frac{\partial T}{\partial u} - \frac{\partial D}{\partial u} + \alpha \frac{\partial L}{\partial u}$$
So that
$$\left(\frac{u_0}{QS}\right)\left(\frac{\partial X}{\partial u}\right)_0 = \left(\frac{u_0}{QS}\right)\left(\frac{\partial T}{\partial u}\right)_0 - \left(\frac{u_0}{QS}\right)\left(\frac{\partial D}{\partial u}\right)_0$$
$$\Rightarrow C_{X_u} \equiv C_{T_u} - (C_{D_u} + 2C_{D_0})$$

- Similarly for the $Z$ direction:
$$\frac{\partial Z}{\partial u} = -\frac{\partial L}{\partial u} - \alpha \frac{\partial D}{\partial u}$$
So that
$$\left(\frac{u_0}{QS}\right)\left(\frac{\partial Z}{\partial u}\right)_0 = -\left(\frac{u_0}{QS}\right)\left(\frac{\partial L}{\partial u}\right)_0$$
$$\begin{aligned} C_{Z_u} &\equiv -(C_{L_u} + 2C_{L_0}) \\ &= -\frac{\mathbf{M^2}}{1-\mathbf{M^2}} C_{L_0} - 2C_{L_0} \end{aligned}$$

- Many more derivatives to consider !

# Summary

- Picked a specific Body Frame (stability axes) from the list of alternatives

  $\Rightarrow$ Choice simplifies some of the linearization, but the inertias now change depending on the equilibrium flight condition.

- Since the nonlinear behavior is too difficult to analyze, we needed to consider the linearized dynamic behavior around a specific flight condition

  $\Rightarrow$ Enables us to linearize RHS of equations of motion.

- Forces and moments also complicated nonlinear functions, so we linearized the LHS as well

  $\Rightarrow$ Enables us to write the perturbations of the forces and moments in terms of the motion variables.

  – Engineering insight allows us to argue that many of the stability derivatives that couple the longitudinal (symmetric) and lateral (asymmetric) motions are small and can be ignored.

- Approach requires that you have the stability derivatives.

  – These can be measured or calculated from the aircraft plan form and basic aerodynamic data.

# 16.333: Lecture # 6

Aircraft Longitudinal Dynamics

- Typical aircraft open-loop motions
- Longitudinal modes
- Impact of actuators
- **Linear Algebra in Action!**

# Longitudinal Dynamics

- Recall: $X$ denotes the force in the $X$-direction, and similarly for $Y$ and $Z$, then (as on 4–13)

$$X_u \equiv \left(\frac{\partial X}{\partial u}\right)_0, \ldots$$

- Longitudinal equations (see 4–13) can be rewritten as:

$$\begin{aligned} m\dot{u} &= X_u u + X_w w - mg\cos\Theta_0\theta + \Delta X^c \\ m(\dot{w} - qU_0) &= Z_u u + Z_w w + Z_{\dot{w}}\dot{w} + Z_q q - mg\sin\Theta_0\theta + \Delta Z^c \\ I_{yy}\dot{q} &= M_u u + M_w w + M_{\dot{w}}\dot{w} + M_q q + \Delta M^c \end{aligned}$$

- There is no roll/yaw motion, so $q = \dot{\theta}$.

- Control commands $\Delta X^c$, $\Delta Z^c$, and $\Delta M^c$ have not yet been specified.

- Rewrite in **state space** form as

$$\begin{bmatrix} m\dot{u} \\ (m - Z_{\dot{w}})\dot{w} \\ -M_{\dot{w}}\dot{w} + I_{yy}\dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -mg\cos\Theta_0 \\ Z_u & Z_w & Z_q + mU_0 & -mg\sin\Theta_0 \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta X^c \\ \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m - Z_{\dot{w}} & 0 & 0 \\ 0 & -M_{\dot{w}} & I_{yy} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} X_u & X_w & 0 & -mg\cos\Theta_0 \\ Z_u & Z_w & Z_q + mU_0 & -mg\sin\Theta_0 \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta X^c \\ \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix}$$

$$E\dot{\mathcal{X}} = \bar{A}\mathcal{X} + \hat{\mathbf{c}} \qquad \text{descriptor state space form}$$

$$\Rightarrow \dot{\mathcal{X}} = E^{-1}(\bar{A}\mathcal{X} + \hat{\mathbf{c}}) = A\mathcal{X} + \mathbf{c}$$

- Write out in state space form:

$$A = \left[\begin{array}{c|c|c|c} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\Theta_0 \\ \\ \dfrac{Z_u}{m-Z_{\dot{w}}} & \dfrac{Z_w}{m-Z_{\dot{w}}} & \dfrac{Z_q+mU_0}{m-Z_{\dot{w}}} & \dfrac{-mg\sin\Theta_0}{m-Z_{\dot{w}}} \\ \\ I_{yy}^{-1}\left[M_u+Z_u\Gamma\right] & I_{yy}^{-1}\left[M_w+Z_w\Gamma\right] & I_{yy}^{-1}\left[M_q+(Z_q+mU_0)\Gamma\right] & -I_{yy}^{-1}mg\sin\Theta_0\Gamma \\ \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Gamma = \frac{M_{\dot{w}}}{m-Z_{\dot{w}}}$$

- Note: slight savings if we defined symbols to embed the mass/inertia $\hat{X}_u = X_u/m$, $\hat{Z}_u = Z_u/m$, and $\hat{M}_q = M_q/I_{yy}$ then A matrix collapses to:

$$\hat{A} = \left[\begin{array}{c|c|c|c} \hat{X}_u & \hat{X}_w & 0 & -g\cos\Theta_0 \\ \\ \dfrac{\hat{Z}_u}{1-\hat{Z}_{\dot{w}}} & \dfrac{\hat{Z}_w}{1-\hat{Z}_{\dot{w}}} & \dfrac{\hat{Z}_q+U_0}{1-\hat{Z}_{\dot{w}}} & \dfrac{-g\sin\Theta_0}{1-\hat{Z}_{\dot{w}}} \\ \\ \left[\hat{M}_u+\hat{Z}_u\hat{\Gamma}\right] & \left[\hat{M}_w+\hat{Z}_w\hat{\Gamma}\right] & \left[\hat{M}_q+(\hat{Z}_q+U_0)\hat{\Gamma}\right] & -g\sin\Theta_0\hat{\Gamma} \\ \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\hat{\Gamma} = \frac{\hat{M}_{\dot{w}}}{1-\hat{Z}_{\dot{w}}}$$

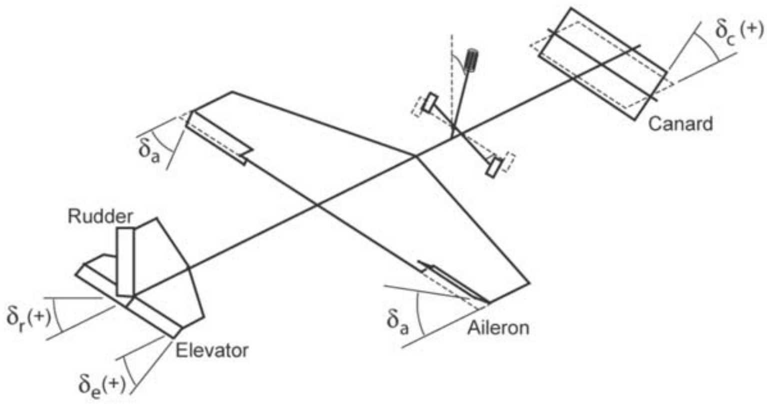
- Check the notation that is being used very carefully

- To figure out the $\mathbf{c}$ vector, we have to say a little more about how the control inputs are applied to the system.

# Longitudinal Actuators

- Primary actuators in longitudinal direction are the elevators and thrust.
  - Clearly the thrusters/elevators play a key role in defining the steady-state/equilibrium flight condition
  - Now interested in determining how they also influence the aircraft motion about this equilibrium condition

$$\text{deflect elevator} \quad \rightarrow \quad u(t), w(t), q(t), \ldots$$




- Recall that we defined $\Delta X^c$ as the perturbation in the total force in the $X$ direction as a result of the actuator commands
  - Force change due to an actuator deflection from trim

- Expand these aerodynamic terms using same perturbation approach

$$\Delta X^c = X_{\delta_e}\delta_e + X_{\delta_p}\delta_p$$

  - $\delta_e$ is the deflection of the elevator from trim (down positive)
  - $\delta_p$ change in thrust
  - $X_{\delta_e}$ and $X_{\delta_p}$ are the **control stability derivatives**

- Now we have that

$$\mathbf{c} \;=\; E^{-1}\begin{bmatrix} \Delta X^c \\ \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix} = E^{-1}\begin{bmatrix} X_{\delta_e} & X_{\delta_p} \\ Z_{\delta_e} & Z_{\delta_p} \\ M_{\delta_e} & M_{\delta_p} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \delta_e \\ \delta_p \end{bmatrix} = Bu$$

- For the longitudinal case

$$B = \left[\begin{array}{c|c} \dfrac{X_{\delta_e}}{m} & \dfrac{X_{\delta_p}}{m} \\ \\ \dfrac{Z_{\delta_e}}{m - Z_{\dot{w}}} & \dfrac{Z_{\delta_p}}{m - Z_{\dot{w}}} \\ \\ I_{yy}^{-1}\left[M_{\delta_e} + Z_{\delta_e}\Gamma\right] & I_{yy}^{-1}\left[M_{\delta_p} + Z_{\delta_p}\Gamma\right] \\ \\ 0 & 0 \end{array}\right]$$

- Typical values for the B747

$$\begin{array}{ll} X_{\delta_e} = -16.54 & X_{\delta_p} = 0.3mg = 849528 \\ Z_{\delta_e} = -1.58 \cdot 10^6 & Z_{\delta_p} \approx 0 \\ M_{\delta_e} = -5.2 \cdot 10^7 & M_{\delta_p} \approx 0 \end{array}$$

- Aircraft response $y = G(s)u$

$$\begin{array}{l} \dot{\mathcal{X}} = A\mathcal{X} + Bu \;\rightarrow\; G(s) = C(sI - A)^{-1}B \\ y = C\mathcal{X} \end{array}$$

- We now have the means to modify the dynamics of the system, but first let's figure out what $\delta_e$ and $\delta_p$ really do.

# Longitudinal Response

- **Final response** to a step input $u = \hat{u}/s$, $y = G(s)u$, use the **FVT**

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\left(G(s)\frac{\hat{u}}{s}\right)$$

$$\Rightarrow \lim_{t\to\infty} y(t) = G(0)\hat{u} = -(CA^{-1}B)\hat{u}$$

- **Initial response** to a step input, use the **IVT**

$$y(0^+) = \lim_{s\to\infty} s\left(G(s)\frac{\hat{u}}{s}\right) = \lim_{s\to\infty} G(s)\hat{u}$$

  – For your system, $G(s) = C(sI - A)^{-1}B + D$, but $D \equiv 0$, so

$$\lim_{s\to\infty} G(s) \to 0$$

  – **Note: there is NO immediate change** in the output of the motion variables in response to an elevator input $\Rightarrow y(0^+) = 0$

- Consider the *rate of change* of these variables $\dot{\mathbf{y}}(\mathbf{0}^+)$

$$\dot{y}(t) = C\dot{\mathcal{X}} = CA\mathcal{X} + CBu$$

and normally have that $CB \neq 0$. Repeat process above[1] to show that $\dot{y}(0^+) = CB\hat{u}$, and since $C \equiv I$,

$$\dot{y}(0^+) = B\hat{u}$$

- Looks good. Now compare with numerical values computed in Matlab. Plot $u$, $\alpha$, and flight path angle $\gamma = \theta - \alpha$ (since $\Theta_0 = \gamma_0 = 0$ – see picture on 4–8)

[1] Note that $C(sI - A)^{-1}B + D = D + \frac{CB}{s} + \frac{CA^{-1}B}{s^2} + \ldots$

### Elevator ($1^\circ$ elevator down – stick forward)

- See very rapid response that decays quickly (mostly in the first 10 seconds of the $\alpha$ response)
- Also see a very lightly damped long period response (mostly $u$, some $\gamma$, and very little $\alpha$). Settles in $>$600 secs
- Predicted **steady state** values from code:

| | | | |
|---|---|---|---|
| 14.1429 | m/s | $u$ | (speeds up) |
| -0.0185 | rad | $\alpha$ | (slight reduction in AOA) |
| -0.0000 | rad/s | $q$ | |
| -0.0161 | rad | $\theta$ | |
| 0.0024 | rad | $\gamma$ | |

  – Predictions appear to agree well with the numerical results.

  – **Primary result** is a **slightly lower angle of attack and a higher speed**

- Predicted **initial rates** of the output values from code:

| | | |
|---|---|---|
| -0.0001 | m/s$^2$ | $\dot{u}$ |
| -0.0233 | rad/s | $\dot{\alpha}$ |
| -1.1569 | rad/s$^2$ | $\dot{q}$ |
| 0.0000 | rad/s | $\dot{\theta}$ |
| 0.0233 | rad/s | $\dot{\gamma}$ |

  – All outputs are zero at $t = 0^+$, but see rapid changes in $\alpha$ and $q$.

  – Changes in $u$ and $\gamma$ (also a function of $\theta$) are much more gradual – not as easy to see this aspect of the prediction

- **Initial impact** Change in $\alpha$ and $q$ (pitches aircraft)
- **Long term impact** Change in $u$ (determines speed at new equilibrium condition)

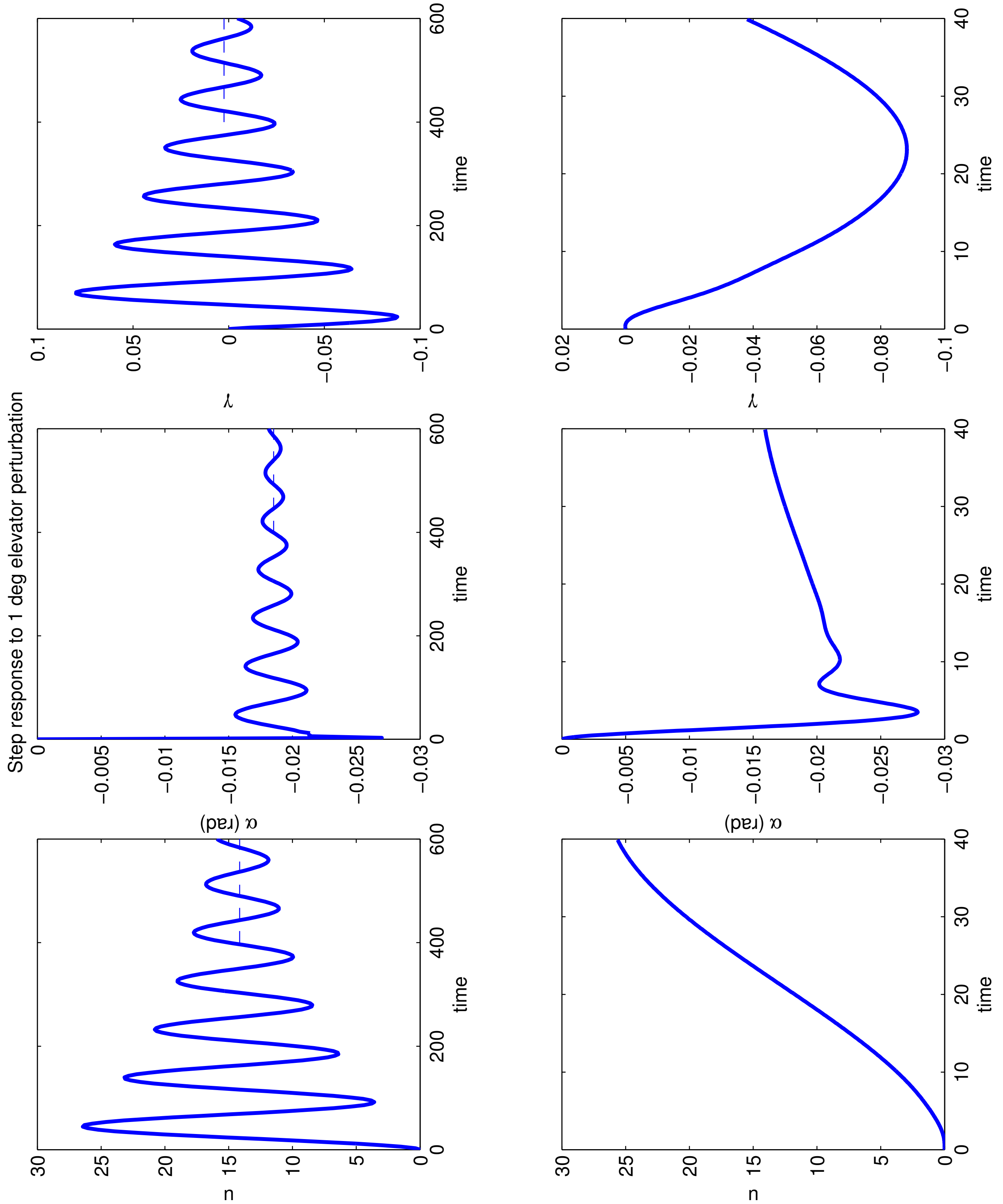



Figure 1: Step Response to 1 deg elevator perturbation – B747 at M=0.8

## Thrust (1/6 input)

- Motion now dominated by the lightly damped long period response
- Short period motion barely noticeable at beginning.
- Predicted **steady state** values from code:

| | | |
|---|---|---|
| 0 | m/s | $u$ |
| 0 | rad | $\alpha$ |
| 0 | rad/s | $q$ |
| 0.05 | rad | $\theta$ |
| 0.05 | rad | $\gamma$ |

  – Predictions appear to agree well with the simulations.
  – **Primary result – now climbing with a flight path angle of 0.05 rad at the same speed we were going before.**

- Predicted **initial rates** of the output values from code:

| | | |
|---|---|---|
| 2.9430 | m/s$^2$ | $\dot{u}$ |
| 0 | rad/s | $\dot{\alpha}$ |
| 0 | rad/s$^2$ | $\dot{q}$ |
| 0 | rad/s | $\dot{\theta}$ |
| 0 | rad/s | $\dot{\gamma}$ |

  – Changes to $\alpha$ are very small, and $\gamma$ response initially flat.
  – Increase power, and the aircraft initially speeds up

- **Initial impact** Change in $u$ (accelerates aircraft)
- **Long term impact** Change in $\gamma$ (determines climb rate)

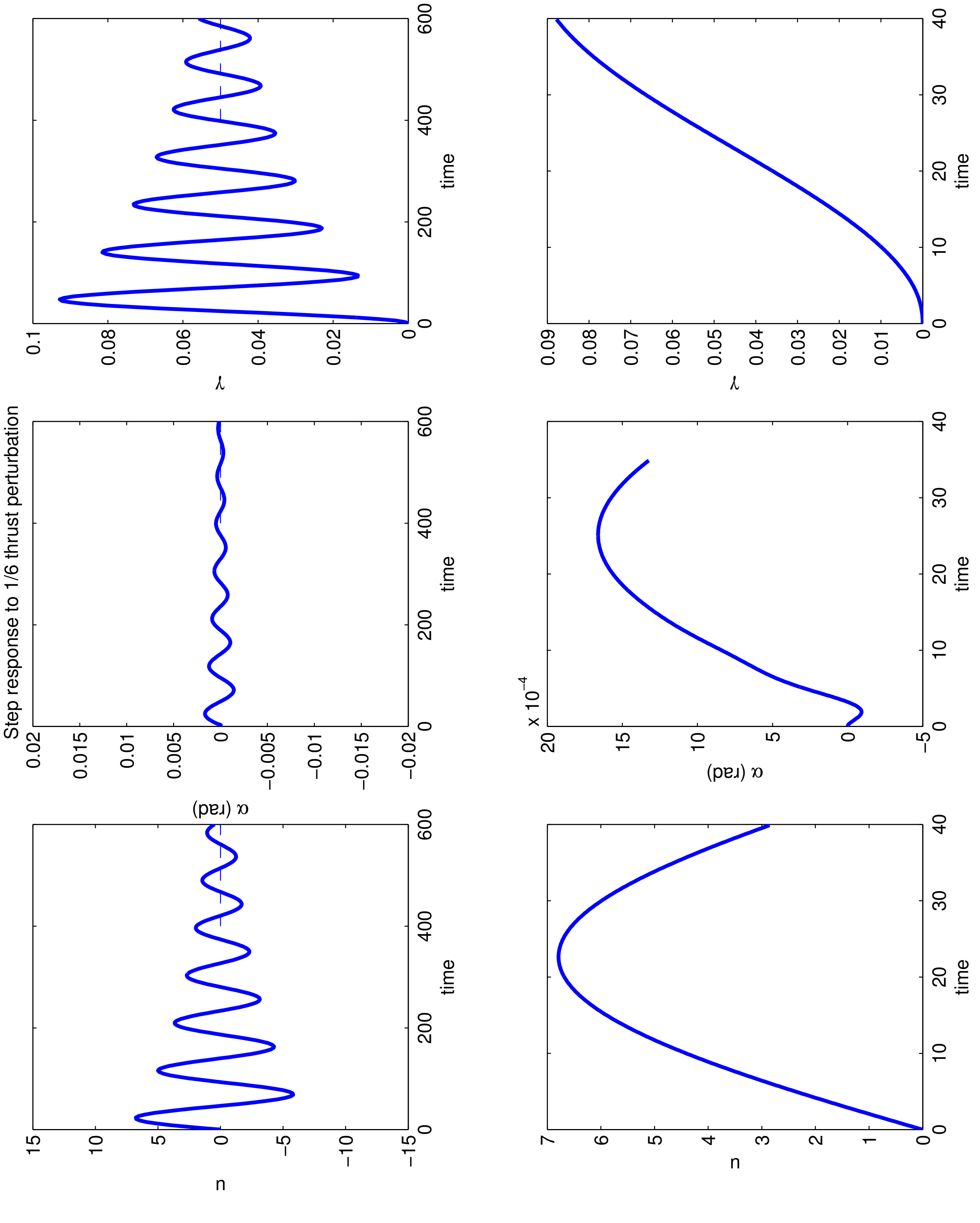



Figure 2: Step Response to 1/6 thrust perturbation – B747 at M=0.8

# Frequency Domain Response

- Plot and inspect transfer functions from $\delta_e$ and $\delta_p$ to $u$, $w$, and $\gamma$
  - See following pages

- **From elevator:**
  - Huge response at the phugoid mode for both $u$ and $\gamma$ (very lightly damped)
  - Short period mode less pronounced
  - Response falls off very rapidly
  - Response to $w$ shows a pole/zero cancelation (almost) of the phugoid mode. So the magnitude level is essentially constant out to the frequency of the short period mode

    Why would we expect that?

- **From thrust:**
  - Phugoid peaks present, but short period mode is very weak (not in $u$, low in $\gamma$, $w$). $\Rightarrow$ entirely consistent with the step response.
  - Thrust controls speed (initially), so we would expect to see a large response associated with the phugoid mode (speed variations are a key component of this mode)

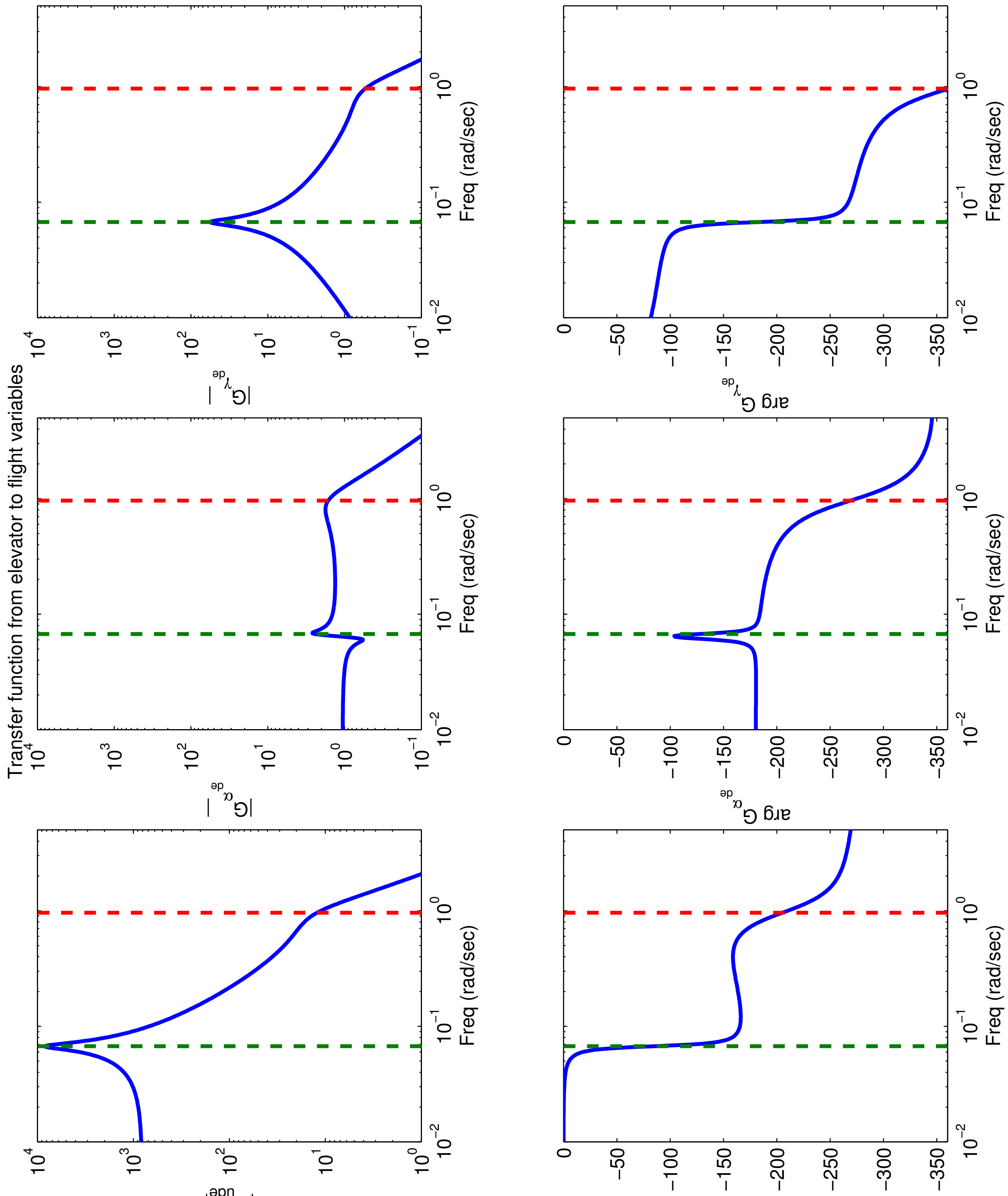



Figure 3: TF's from elevator to flight variables – B747 at M=0.8

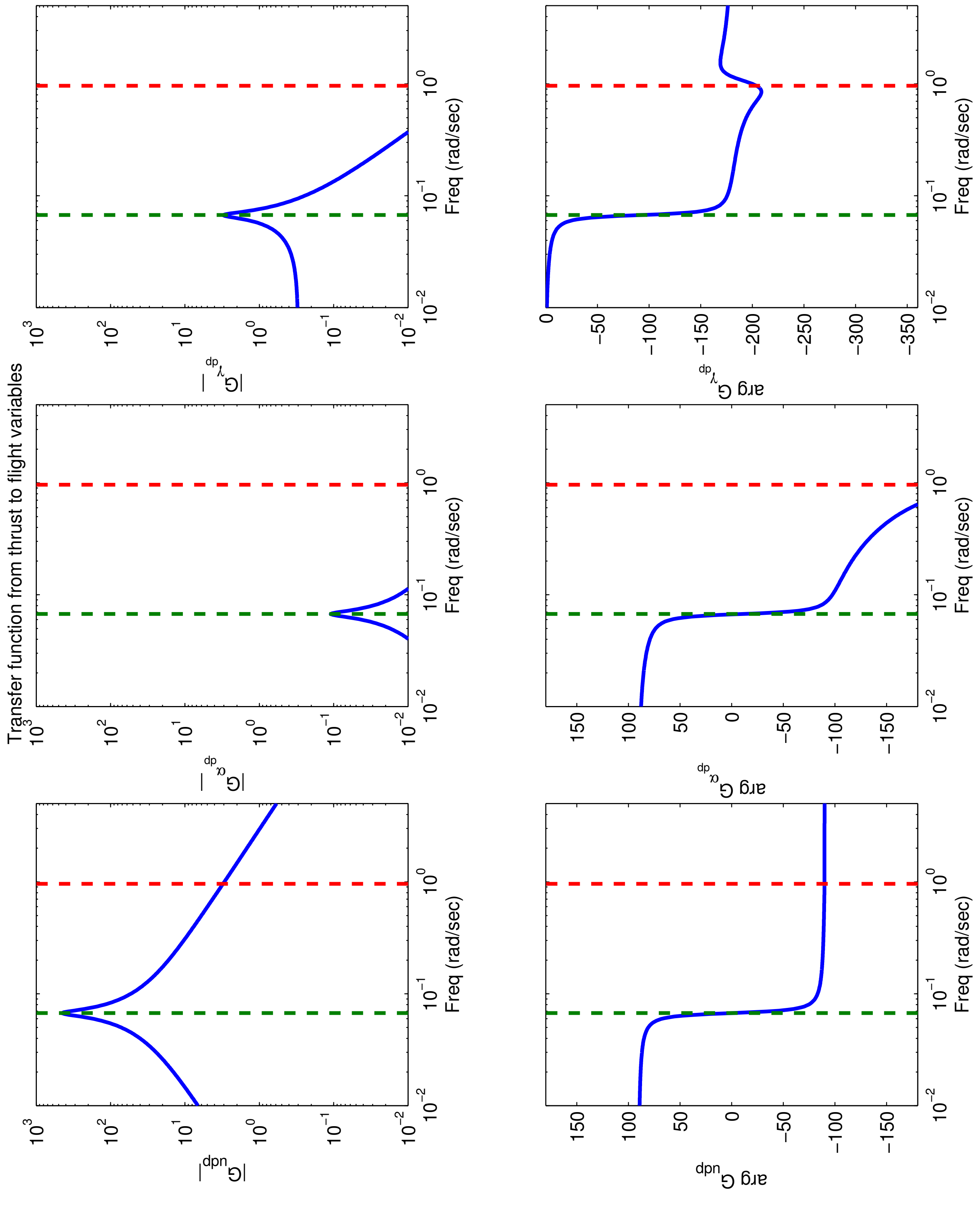



Figure 4: TF's from thrust to flight variables– B747 at M=0.8

- **Summary:**
  - **To increase equilibrium climb rate, add power.**
  - **To increase equilibrium speed, increase $\delta_e$ (move elevator further down).**
  - Transient (initial) effects are the opposite **and tend to be more consistent with what you would intuitively expect to occur**

# Modal Behavior

- Analyze model of vehicle dynamics to quantify the responses seen.
  - Homogeneous dynamics of the form $\dot{X} = AX$, so the response is

    $$X(t) = e^{At}X(0) \text{ – a matrix exponential.}$$

- To simplify the investigation of the system response, find the **modes** of the system using the *eigenvalues* and *eigenvectors*
  - $\lambda$ is an **eigenvalue** of $A$ if $\det(\lambda I - A) = 0$ which is true iff there exists a nonzero $v$ (**eigenvector**) for which

    $$(\lambda I - A)v = 0 \quad \Rightarrow \quad Av = \lambda v$$

  - If A $(n \times n)$, typically get $n$ eigenvalues/eigenvectors $Av_i = \lambda_i v_i$
  - Assuming that eigenvectors are **linearly independent**, can form

$$A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$AT = T\Lambda$$

$$\Rightarrow T^{-1}AT = \Lambda \quad , \quad A = T\Lambda T^{-1}$$

  - Given that $e^{At} = I + At + \frac{1}{2!}(At)^2 + \ldots$, and that $A = T\Lambda T^{-1}$, then it is easy to show that

$$X(t) = e^{At}X(0) = Te^{\Lambda t}T^{-1}X(0) = \sum_{i=1}^{n} v_i e^{\lambda_i t}\beta_i$$

  - State solution is a linear combination of **system modes** $v_i e^{\lambda_i t}$

  $e^{\lambda_i t}$ – determines **nature** of the time response

  $v_i$ – gives extent to which each state **participates** in that mode

  $\beta_i$ – determines extent to which initial condition **excites** the mode

- The total behavior of the system can be found from the system modes
- Consider numerical example of B747

$$A = \begin{bmatrix} -0.0069 & 0.0139 & 0 & -9.8100 \\ -0.0905 & -0.3149 & 235.8928 & 0 \\ 0.0004 & -0.0034 & -0.4282 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}$$

which gives two sets of complex eigenvalues

$$\lambda = -0.3717 \pm 0.8869\ \mathbf{i}, \quad \omega = 0.962, \quad \zeta = 0.387, \qquad \text{short period}$$

$$\lambda = -0.0033 \pm 0.0672\ \mathbf{i}, \quad \omega = 0.067, \quad \zeta = 0.049, \quad \text{Phugoid - long period}$$

  - **Result is consistent with step response** - heavily damped fast response, and a lightly damped slow one.

- To understand eigenvectors, must do some normalization (scales each element appropriately so that we can compare relative sizes)
  - $\hat{u} = u/U_0$, $\alpha = w/U_0$, $\hat{q} = q/(2U_0/\bar{c})$
  - Then divide through so that $\theta \equiv 1$

| | Short Period | Phugoid |
|---|---|---|
| $\hat{u}$ | $0.0156 + 0.0244\ \mathbf{i}$ | $-0.0254 + 0.6165\ \mathbf{i}$ |
| $\alpha$ | $1.0202 + 0.3553\ \mathbf{i}$ | $0.0045 + 0.0356\ \mathbf{i}$ |
| $\hat{q}$ | $-0.0066 + 0.0156\ \mathbf{i}$ | $-0.0001 + 0.0012\ \mathbf{i}$ |
| $\theta$ | $1.0000$ | $1.0000$ |

- **Short Period** – primarily $\theta$ and $\alpha = \hat{w}$ in the same phase. The $\hat{u}$ and $\hat{q}$ response is very small.
- **Phugoid** – primarily $\theta$ and $\hat{u}$, and $\theta$ lags by about 90°. The $\alpha$ and $\hat{q}$ response is very small.
- Dominant behavior agrees with time step responses – note how initial conditions were formed.

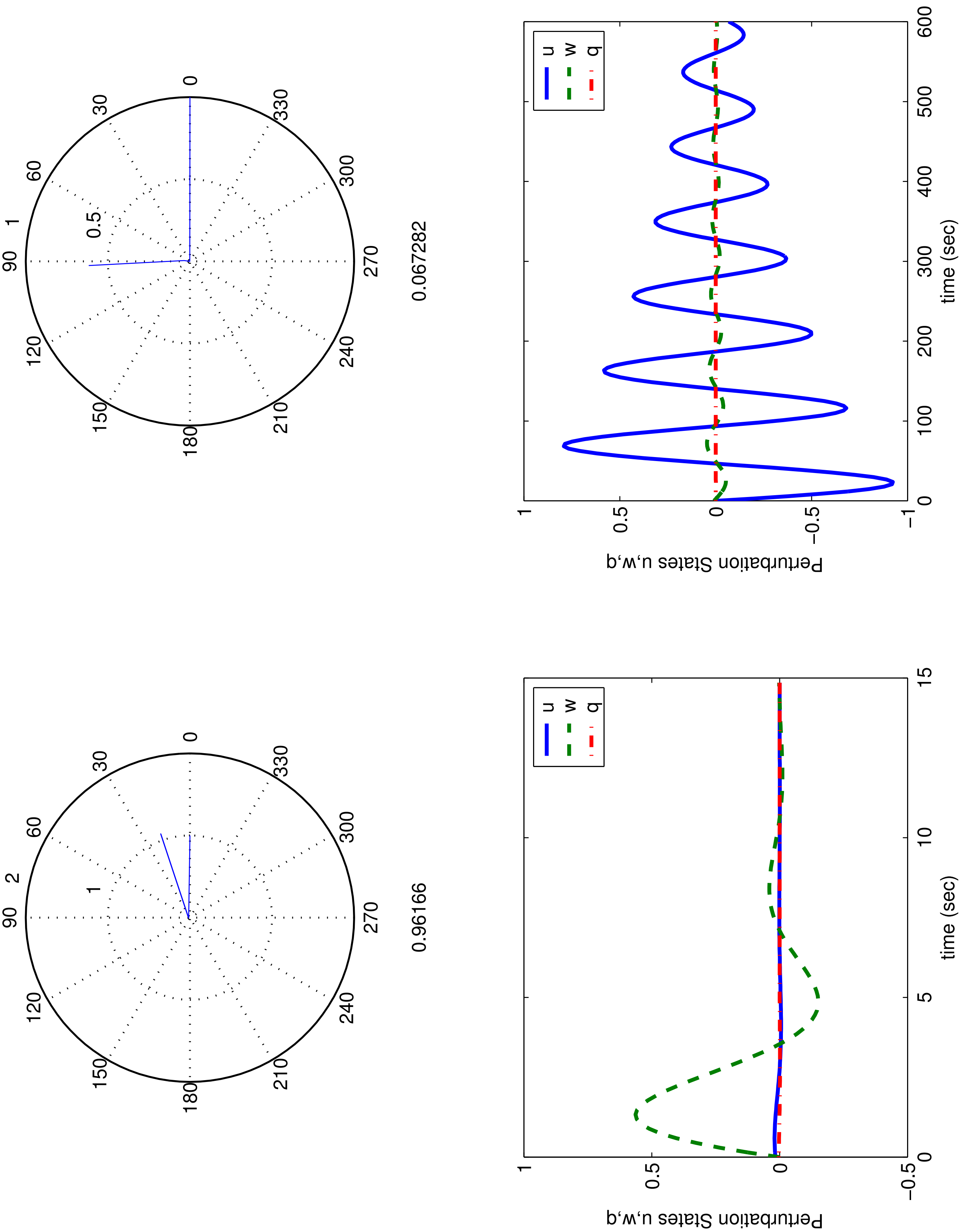



Figure 5: Mode Response – B747 at M=0.8

- Relative motion between aircraft and an observer flying at a constant speed $U_0 t$

(Image removed for copyright considerations.)

- Motion of perturbed aircraft with respect to an unperturbed one
- Note phasing of the forward velocity $\dot{x}_e$ with respect to altitude $z_e$
  – aircraft faster than observer at the bottom, slower at the top
  – The aircraft speeds up and slows down – leads and lags the observer.

- Consistent with flight path?
- Consistent with Lanchester's approximation on 4–1?

# Summary

- Two primary longitudinal modes: phugoid and short-period
  - Have versions from the full model – but can develop good approximations that help identify the aerodynamic features that determine the mode frequencies and damping

  Impact of the various actuators clarified:
  - Short time-scale
  - Long time-scale

# Matrix Diagonalization

- Suppose $A$ is diagonizable with independent eigenvectors

$$V = [v_1, \ldots, v_n]$$

  – use similarity transformations to diagonalize dynamics matrix

$$\begin{aligned} \dot{x} &= Ax \Rightarrow \dot{x}_d = A_d x_d \\ V^{-1}AV &= \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \triangleq \Lambda = A_d \end{aligned}$$

  – Corresponds to change of state from $x$ to $x_d = V^{-1}x$

- System response given by $e^{At}$, look at power series expansion

$$\begin{aligned} At &= V\Lambda t V^{-1} \\ (At)^2 &= (V\Lambda t V^{-1})V\Lambda t V^{-1} = V\Lambda^2 t^2 V^{-1} \\ \Rightarrow (At)^n &= V\Lambda^n t^n V^{-1} \end{aligned}$$

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2}(At)^2 + \ldots \\ &= V\left\{ I + \Lambda + \frac{1}{2}\Lambda^2 t^2 + \ldots \right\} V^{-1} \\ &= Ve^{\Lambda t}V^{-1} = V \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} V^{-1} \end{aligned}$$

- Taking Laplace transform,

$$
\begin{aligned}
(sI - A)^{-1} &= V \begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix} V^{-1} \\
&= \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i}
\end{aligned}
$$

where the residue $R_i = v_i w_i^T$, and we define

$$
V = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} \quad , V^{-1} = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}
$$

- Note that the $w_i$ are the left eigenvectors of $A$ associated with the right eigenvectors $v_i$

$$
AV = V \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \Rightarrow V^{-1}A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} V^{-1}
$$

$$
\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}
$$

where $w_i^T A = \lambda_i w_i^T$

- So, if $\dot{x} = Ax$, the time domain solution is given by

$$\begin{aligned} x(t) &= \sum_{i=1}^{n} e^{\lambda_i t} v_i w_i^T x(0) \qquad \text{dyad} \\ x(t) &= \sum_{i=1}^{n} [w_i^T x(0)] e^{\lambda_i t} v_i \end{aligned}$$

- The part of the solution $v_i e^{\lambda_i t}$ is called a **mode** of a system
  – solution is a weighted sum of the system modes
  – weights depend on the components of $x(0)$ along $w_i$

- Can now give dynamics interpretation of left and right eigenvectors:

$$Av_i = \lambda_i v_i \quad , w_i A = \lambda_i w_i \quad , w_i^T v_j = \delta_{ij}$$

so if $x(0) = v_i$, then

$$\begin{aligned} x(t) &= \sum_{i=1}^{n} (w_i^T x(0)) e^{\lambda_i t} v_i \\ &= e^{\lambda_i t} v_i \end{aligned}$$

$\Rightarrow$ so **right** eigenvectors are initial conditions that result in relatively simple motions $x(t)$.

**With no external inputs, if initial condition only disturbs one mode, then the response consists of only that mode for all time.**

- If $A$ has complex conjugate eigenvalues, the process is similar but a little more complicated.
- Consider a 2x2 case with $A$ having eigenvalues $a \pm b\mathbf{i}$ and associated eigenvectors $e_1$, $e_2$, with $e_2 = \bar{e}_1$. Then

$$\begin{aligned} A &= \left[\, e_1 \,\middle|\, e_2 \,\right] \begin{bmatrix} a+b\mathbf{i} & 0 \\ 0 & a-b\mathbf{i} \end{bmatrix} \left[\, e_1 \,\middle|\, e_2 \,\right]^{-1} \\ &= \left[\, e_1 \,\middle|\, \bar{e}_1 \,\right] \begin{bmatrix} a+b\mathbf{i} & 0 \\ 0 & a-b\mathbf{i} \end{bmatrix} \left[\, e_1 \,\middle|\, \bar{e}_1 \,\right]^{-1} \equiv TDT^{-1} \end{aligned}$$

- Now use the transformation matrix

$$M = 0.5 \begin{bmatrix} 1 & -\mathbf{i} \\ 1 & \mathbf{i} \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1 & 1 \\ \mathbf{i} & -\mathbf{i} \end{bmatrix}$$

- Then it follows that

$$\begin{aligned} A &= TDT^{-1} = (TM)(M^{-1}DM)(M^{-1}T^{-1}) \\ &= (TM)(M^{-1}DM)(TM)^{-1} \end{aligned}$$

which has the nice structure:

$$A = \left[\, \mathtt{Re}(e_1) \,\middle|\, \mathtt{Im}(e_1) \,\right] \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \left[\, \mathtt{Re}(e_1) \,\middle|\, \mathtt{Im}(e_1) \,\right]^{-1}$$

where all the matrices are real.

- With complex roots, the diagonalization is to a block diagonal form.

- For this case we have that

$$e^{At} = \left[\ \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1)\ \right] e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix} \left[\ \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1)\ \right]^{-1}$$

- Note that $\left[\ \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1)\ \right]^{-1}$ is the matrix that inverts $\left[\ \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1)\ \right]$

$$\left[\ \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1)\ \right]^{-1} \left[\ \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1)\ \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- So for an initial condition to excite just this mode, can pick $x(0) = [\texttt{Re}(e_1)]$, or $x(0) = [\texttt{Im}(e_1)]$ or a linear combination.

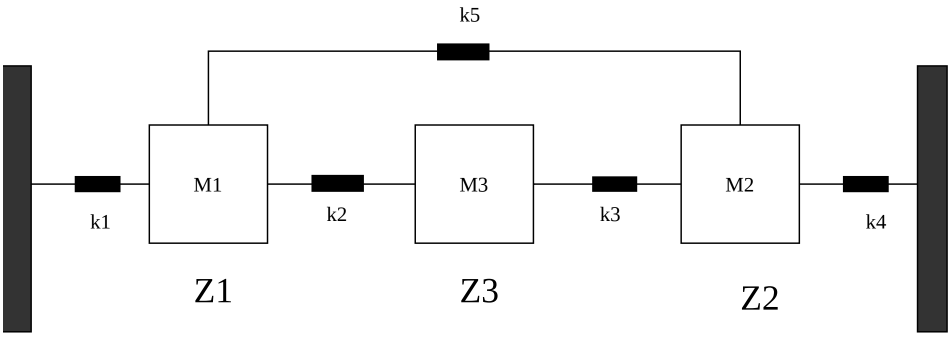
- Example $x(0) = [\texttt{Re}(e_1)]$

$$\begin{aligned} x(t) &= e^{At}x(0) = \left[\ \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1)\ \right] e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix} \cdot \\ &\qquad \left[\ \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1)\ \right]^{-1} [\texttt{Re}(e_1)] \\ &= \left[\ \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1)\ \right] e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= e^{at} \left[\ \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1)\ \right] \begin{bmatrix} \cos(bt) \\ -\sin(bt) \end{bmatrix} \\ &= e^{at} \left(\texttt{Re}(e_1)\cos(bt) - \texttt{Im}(e_1)\sin(bt)\right) \end{aligned}$$

which would ensure that only this mode is excited in the response

# Example: Spring Mass System

- Classic example: spring mass system consider simple case first: $m_i = 1$, and $k_i = 1$




$$
\begin{aligned}
x &= \begin{bmatrix} z_1 & z_2 & z_3 & \dot{z}_1 & \dot{z}_2 & \dot{z}_3 \end{bmatrix} \\
A &= \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix} \quad M = \mathrm{diag}(m_i) \\
K &= \begin{bmatrix} k_1 + k_2 + k_5 & -k_5 & -k_2 \\ -k_5 & k_3 + k_4 + k_5 & -k_3 \\ -k_2 & -k_3 & k_2 + k_3 \end{bmatrix}
\end{aligned}
$$

- Eigenvalues and eigenvectors of the undamped system

$$\lambda_1 = \pm 0.77\mathbf{i} \quad \lambda_2 = \pm 1.85\mathbf{i} \quad \lambda_3 = \pm 2.00\mathbf{i}$$

| $v_1$ | $v_2$ | $v_3$ |
|---|---|---|
| 1.00 | 1.00 | 1.00 |
| 1.00 | 1.00 | $-1.00$ |
| 1.41 | $-1.41$ | 0.00 |
| $\pm 0.77\mathbf{i}$ | $\pm 1.85\mathbf{i}$ | $\pm 2.00\mathbf{i}$ |
| $\pm 0.77\mathbf{i}$ | $\pm 1.85\mathbf{i}$ | $\mp 2.00\mathbf{i}$ |
| $\pm 1.08\mathbf{i}$ | $\mp 2.61\mathbf{i}$ | 0.00 |

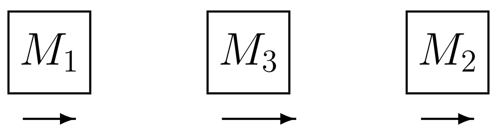
- Initial conditions to excite just the three modes:

$$x_i(0) = \alpha_1 \mathrm{Re}(v_i) + \alpha_2 \mathrm{Im}(v_1) \quad \forall \alpha_j \in \mathbb{R}$$

  – Simulation using $\alpha_1 = 1$, $\alpha_2 = 0$

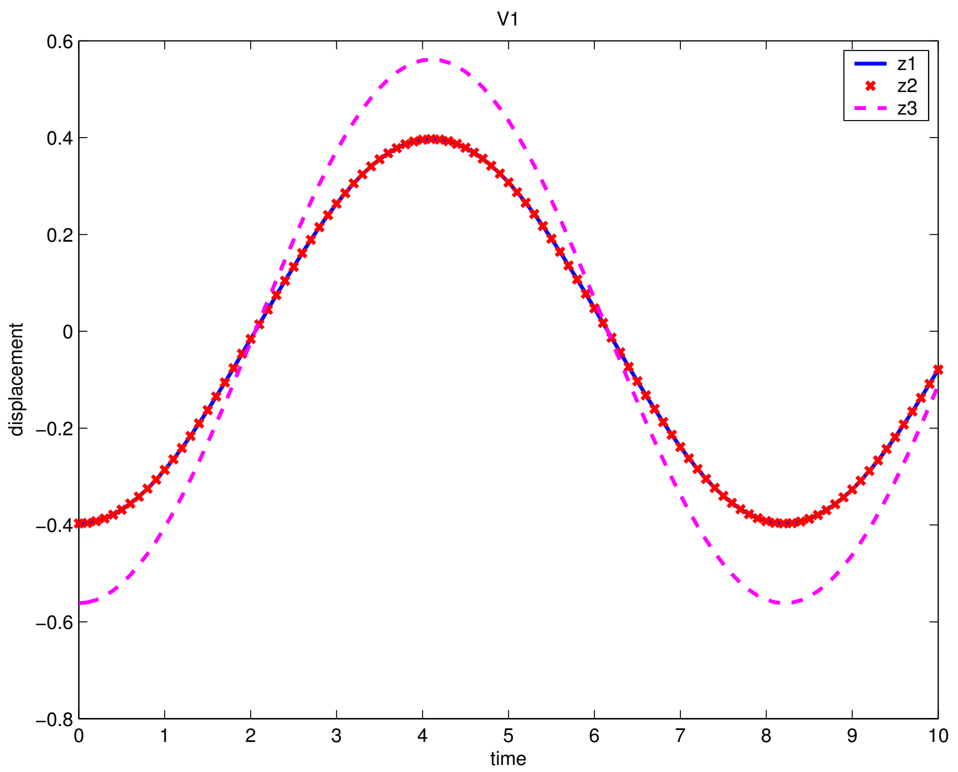
- Visualization important for correct physical interpretation

- Mode 1 $\lambda_1 = \pm 0.77\mathbf{i}$

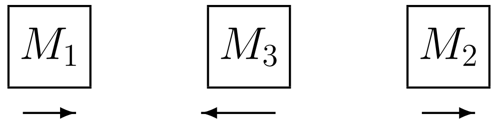



  – Lowest frequency mode, all masses move in same direction

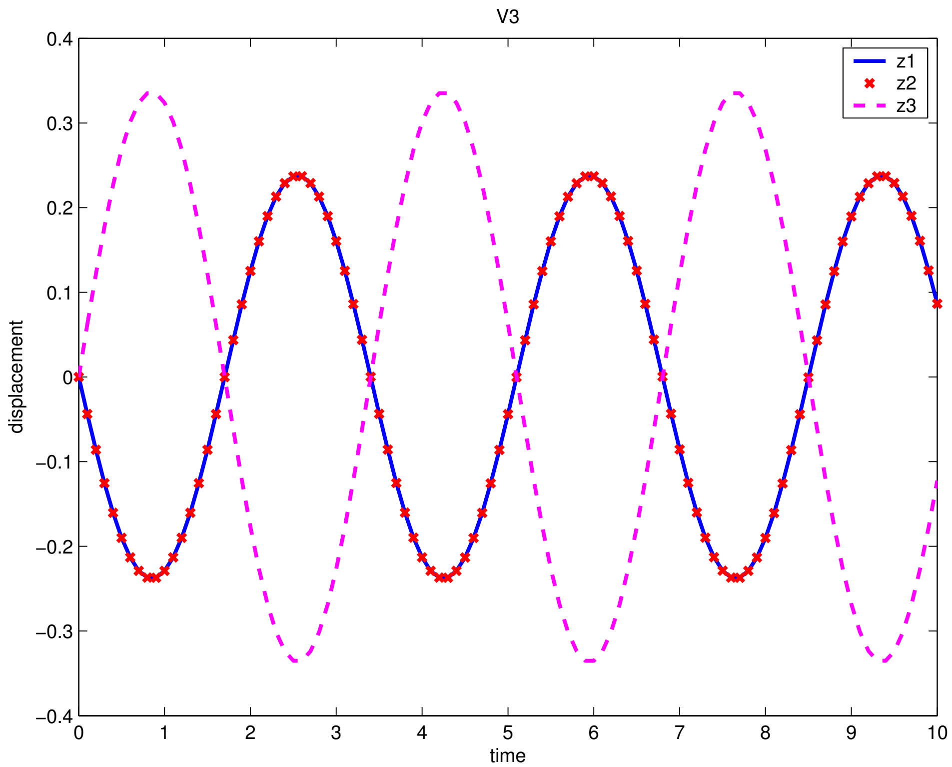
  – Middle mass has higher amplitude motions $z_3$, motions all in phase



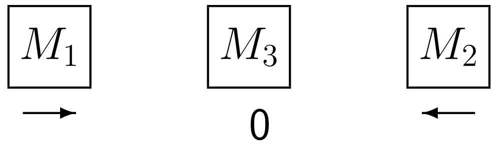
- Mode 2 $\lambda_2 = \pm 1.85\mathbf{i}$



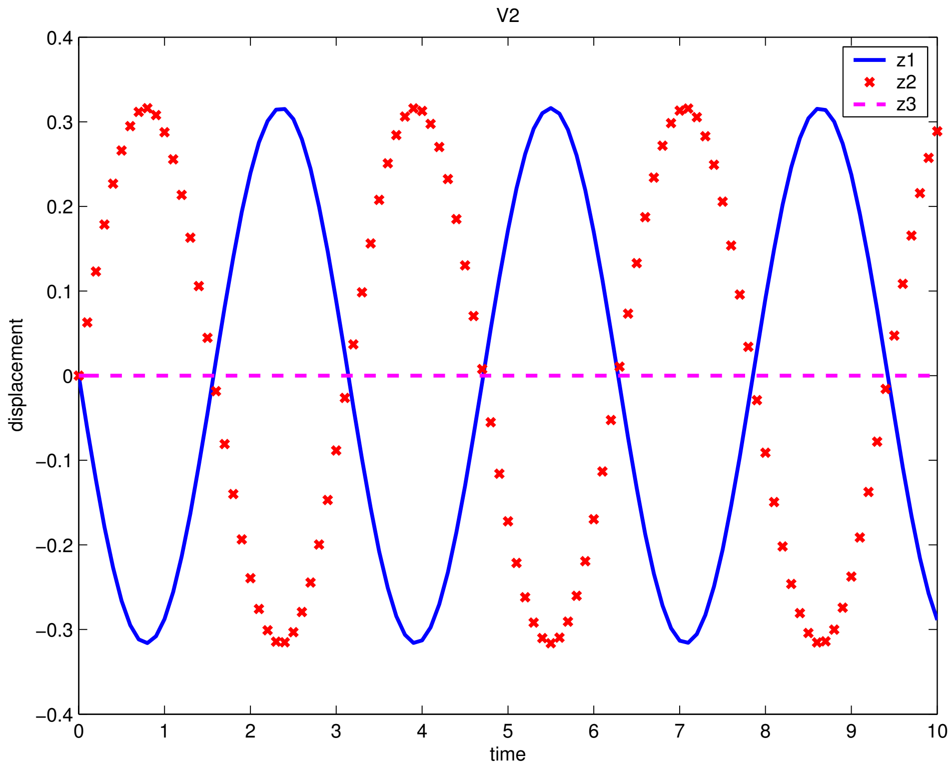

  – Middle frequency mode has middle mass moving in opposition to two end masses
  – Again middle mass has higher amplitude motions $z_3$



- Mode 3 $\lambda_3 = \pm 2.00\mathbf{i}$




  – Highest frequency mode, has middle mass stationary, and other two masses in opposition




- Eigenvectors with that correspond with more constrained motion of the system are associated with higher frequency eigenvalues

## 16.333: Lecture # 7

Approximate Longitudinal Dynamics Models

- A couple more stability derivatives
- Given mode shapes found identify simpler models that capture the main responses

# More Stability Derivatives

- Recall from 6–2 that the derivative stability derivative terms $Z_{\dot{w}}$ and $M_{\dot{w}}$ ended up on the LHS as modifications to the normal mass and inertia terms
  - These are the *apparent mass* effects – some of the surrounding displaced air is "entrained" and moves with the aircraft
  - Acceleration derivatives quantify this effect
  - Significant for blimps, less so for aircraft.

- Main effect: rate of change of the normal velocity $\dot{w}$ causes a transient in the downwash $\epsilon$ from the wing that creates a change in the angle of attack of the tail some time later – **Downwash Lag** effect

- If aircraft flying at $U_0$, will take approximately $\Delta t = l_t/U_0$ to reach the tail.
  - Instantaneous downwash at the tail $\epsilon(t)$ is due to the wing $\alpha$ at time $t - \Delta t$.

$$\epsilon(t) = \frac{\partial \epsilon}{\partial \alpha}\alpha(t - \Delta t)$$

  - Taylor series expansion

$$\alpha(t - \Delta t) \approx \alpha(t) - \dot{\alpha}\Delta t$$

  - Note that $\Delta\epsilon(t) = -\Delta\alpha_t$. Change in the tail AOA can be computed as

$$\Delta\epsilon(t) = -\frac{d\epsilon}{d\alpha}\dot{\alpha}\Delta t = -\frac{d\epsilon}{d\alpha}\dot{\alpha}\frac{l_t}{U_0} = -\Delta\alpha_t$$

- For the tail, we have that the lift increment due to the change in downwash is
$$\Delta C_{L_t} = C_{L_{\alpha_t}} \Delta\alpha_t = C_{L_{\alpha_t}} \dot{\alpha} \frac{d\epsilon}{d\alpha} \frac{l_t}{U_0}$$
The change in lift force is then
$$\Delta L_t = \frac{1}{2}\rho (U_0^2)_t S_t \Delta C_{L_t}$$

- In terms of the $Z$-force coefficient
$$\Delta C_Z = -\frac{\Delta L_t}{\frac{1}{2}\rho U_0^2 S} = -\eta \frac{S_t}{S} \Delta C_{L_t} = -\eta \frac{S_t}{S} C_{L_{\alpha_t}} \dot{\alpha} \frac{d\epsilon}{d\alpha} \frac{l_t}{U_0}$$

- We use $\bar{c}/(2U_0)$ to nondimensionalize time, so the appropriate stability coefficient form is (note use $C_z$ to be general, but we are looking at $\Delta C_z$ from before):
$$\begin{aligned} C_{Z_{\dot{\alpha}}} = \left( \frac{\partial C_Z}{\partial \left( \dot{\alpha}\bar{c}/2U_0 \right)} \right)_0 &= \frac{2U_0}{\bar{c}} \left( \frac{\partial C_Z}{\partial \dot{\alpha}} \right)_0 \\ &= -\eta \frac{2U_0}{\bar{c}} \frac{S_t}{S} \frac{l_t}{U_0} C_{L_{\alpha_t}} \frac{d\epsilon}{d\alpha} \\ &= -2\eta V_H C_{L_{\alpha_t}} \frac{d\epsilon}{d\alpha} \end{aligned}$$

- The pitching moment due to the lift increment is
$$\begin{aligned} \Delta M_{cg} &= -l_t \Delta L_t \\ \rightarrow \Delta C_{M_{cg}} &= -l_t \frac{\frac{1}{2}\rho (U_0^2)_t S_t \Delta C_{L_t}}{\frac{1}{2}\rho U_0^2 S \bar{c}} \\ &= -\eta V_H \Delta C_{L_t} = -\eta V_H C_{L_{\alpha_t}} \dot{\alpha} \frac{d\epsilon}{d\alpha} \frac{l_t}{U_0} \end{aligned}$$

- So that

$$
\begin{aligned}
C_{M_{\dot\alpha}} = \left(\frac{\partial C_M}{\partial\left(\dot\alpha\bar c/2U_0\right)}\right)_0 &= \frac{2U_0}{\bar c}\left(\frac{\partial C_M}{\partial\dot\alpha}\right)_0 \\
&= -\eta V_H C_{L_{\alpha_t}}\frac{d\epsilon}{d\alpha}\frac{l_t}{U_0}\frac{2U_0}{\bar c} \\
&= -2\eta V_H C_{L_{\alpha_t}}\frac{d\epsilon}{d\alpha}\frac{l_t}{\bar c} \\
&\equiv \frac{l_t}{\bar c}C_{Z_{\dot\alpha}}
\end{aligned}
$$

- Similarly, pitching motion of the aircraft changes the AOA of the tail. Nose pitch up at rate $q$, increases apparent downwards velocity of tail by $ql_t$, changing the AOA by

$$\Delta\alpha_t = \frac{ql_t}{U_0}$$

which changes the lift at the tail (and the moment about the cg).

- Following same analysis as above: Lift increment

$$
\begin{aligned}
\Delta L_t &= C_{L_{\alpha_t}}\frac{ql_t}{U_0}\frac{1}{2}\rho(U_0^2)_t S_t \\
\Delta C_Z &= -\frac{\Delta L_t}{\frac{1}{2}\rho(U_0^2)S} = -\eta\frac{S_t}{S}C_{L_{\alpha_t}}\frac{ql_t}{U_0}
\end{aligned}
$$

$$
\begin{aligned}
C_{Z_q} \equiv \left(\frac{\partial C_Z}{\partial(q\bar c/2U_0)}\right)_0 = \frac{2U_0}{\bar c}\left(\frac{\partial C_Z}{\partial q}\right)_0 &= -\eta\frac{2U_0}{\bar c}\frac{l_t}{U_0}\frac{S_t}{S}C_{L_{\alpha_t}} \\
&= -2\eta V_H C_{L_{\alpha_t}}
\end{aligned}
$$

- Can also show that

$$C_{M_q} = C_{Z_q}\frac{l_t}{\bar c}$$

# Approximate Aircraft Dynamic Models

- It is often good to develop simpler models of the full set of aircraft dynamics.
  - Provides insights on the role of the aerodynamic parameters on the frequency and damping of the two modes.
  - Useful for the control design work as well
- Basic approach is to recognize that the modes have very separate sets of states that participate in the response.
  - **Short Period** – primarily $\theta$ and $w$ in the same phase. The $u$ and $q$ response is very small.
  - **Phugoid** – primarily $\theta$ and $u$, and $\theta$ lags by about 90°. The $w$ and $q$ response is very small.
- Full equations from before:

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \left[\begin{array}{c|c|c|c} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g\cos\Theta_0 \\ \frac{Z_u}{m-Z_{\dot{w}}} & \frac{Z_w}{m-Z_{\dot{w}}} & \frac{Z_q+mU_0}{m-Z_{\dot{w}}} & \frac{-mg\sin\Theta_0}{m-Z_{\dot{w}}} \\ \frac{[M_u+Z_u\Gamma]}{I_{yy}} & \frac{[M_w+Z_w\Gamma]}{I_{yy}} & \frac{[M_q+(Z_q+mU_0)\Gamma]}{I_{yy}} & -\frac{mg\sin\Theta_0\Gamma}{I_{yy}} \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta X^c \\ \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix}$$

- For the **Short Period** approximation,
  1. Since $u \approx 0$ in this mode, then $\dot{u} \approx 0$ and can eliminate the $X$-force equation.

$$\begin{bmatrix} \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \left[\begin{array}{c|c|c} \frac{Z_w}{m-Z_{\dot{w}}} & \frac{Z_q+mU_0}{m-Z_{\dot{w}}} & \frac{-mg\sin\Theta_0}{m-Z_{\dot{w}}} \\ \frac{[M_w+Z_w\Gamma]}{I_{yy}} & \frac{[M_q+(Z_q+mU_0)\Gamma]}{I_{yy}} & -\frac{mg\sin\Theta_0\Gamma}{I_{yy}} \\ 0 & 1 & 0 \end{array}\right] \begin{bmatrix} w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix}$$

  2. Typically find that $Z_{\dot{w}} \ll m$ and $Z_q \ll mU_0$. Check for 747:
     - $Z_{\dot{w}} = 1909 \ll m = 2.8866 \times 10^5$
     - $Z_q = 4.5 \times 10^5 \ll mU_0 = 6.8 \times 10^7$

$$\Gamma = \frac{M_{\dot{w}}}{m - Z_{\dot{w}}} \Rightarrow \Gamma \approx \frac{M_{\dot{w}}}{m}$$

$$\begin{bmatrix} \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \left[\begin{array}{c|c|c} \frac{Z_w}{m} & U_0 & -g\sin\Theta_0 \\ \frac{\left[M_w+Z_w\frac{M_{\dot{w}}}{m}\right]}{I_{yy}} & \frac{\left[M_q+(mU_0)\frac{M_{\dot{w}}}{m}\right]}{I_{yy}} & -\frac{mg\sin\Theta_0}{I_{yy}}\frac{M_{\dot{w}}}{m} \\ 0 & 1 & 0 \end{array}\right] \begin{bmatrix} w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix}$$

  3. Set $\Theta_0 = 0$ and remove $\theta$ from the model (it can be derived from $q$)

- With these approximations, the longitudinal dynamics reduce to

$$\dot{x}_{sp} = A_{sp}x_{sp} + B_{sp}\delta_e$$

where $\delta_e$ is the elevator input, and

$$x_{sp} = \begin{bmatrix} w \\ q \end{bmatrix} \quad , \quad A_{sp} = \begin{bmatrix} Z_w/m & U_0 \\ I_{yy}^{-1}\left(M_w + M_{\dot{w}}Z_w/m\right) & I_{yy}^{-1}\left(M_q + M_{\dot{w}}U_0\right) \end{bmatrix}$$

$$B_{sp} = \begin{bmatrix} Z_{\delta_e}/m \\ I_{yy}^{-1}\left(M_{\delta_e} + M_{\dot{w}}Z_{\delta_e}/m\right) \end{bmatrix}$$

- Characteristic equation for this system: $s^2 + 2\zeta_{sp}\omega_{sp}s + \omega_{sp}^2 = 0$, where the **full** approximation gives:

$$\begin{aligned} 2\zeta_{sp}\omega_{sp} &= -\left(\frac{Z_w}{m} + \frac{M_q}{I_{yy}} + \frac{M_{\dot{w}}}{I_{yy}}U_0\right) \\ \omega_{sp}^2 &= \frac{Z_w M_q}{m I_{yy}} - \frac{U_0 M_w}{I_{yy}} \end{aligned}$$

- Given approximate magnitude of the derivatives for a typical aircraft, can develop a **coarse** approximate:

$$\left.\begin{aligned} 2\zeta_{sp}\omega_{sp} \approx -\tfrac{M_q}{I_{yy}} \\ \omega_{sp}^2 \approx -\tfrac{U_0 M_w}{I_{yy}} \end{aligned}\right\} \rightarrow \quad \begin{aligned} \zeta_{sp} &\approx -\tfrac{M_q}{2}\sqrt{\tfrac{-1}{U_0 M_w I_{yy}}} \\ \omega_{sp} &\approx \sqrt{\tfrac{-U_0 M_w}{I_{yy}}} \end{aligned}$$

- Numerical values for 747

| | Frequency rad/sec | Damping |
|---|---|---|
| Full model | 0.962 | 0.387 |
| Full Approximate | 0.963 | 0.385 |
| Coarse Approximate | 0.906 | 0.187 |

Both approximations give the frequency well, but full approximation gives a much better damping estimate

- Approximations showed that short period mode frequency is determined by $M_w$ – measure of the *aerodynamic stiffness in pitch.*
  - Sign of $M_w$ negative if *cg* sufficient far forward – changes sign (mode goes unstable) when *cg* at the *stick fixed neutral point*. Follows from discussion of $C_{M_\alpha}$ (see 2–11)

- For the Phugoid approximation, start again with:

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \left[\begin{array}{c|c|c|c} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g\cos\Theta_0 \\ \frac{Z_u}{m-Z_{\dot{w}}} & \frac{Z_w}{m-Z_{\dot{w}}} & \frac{Z_q+mU_0}{m-Z_{\dot{w}}} & \frac{-mg\sin\Theta_0}{m-Z_{\dot{w}}} \\ \frac{[M_u+Z_u\Gamma]}{I_{yy}} & \frac{[M_w+Z_w\Gamma]}{I_{yy}} & \frac{\left[M_q+(Z_q+mU_0)\Gamma\right]}{I_{yy}} & -\frac{mg\sin\Theta_0\Gamma}{I_{yy}} \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta X^c \\ \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix}$$

1. Changes to $w$ and $q$ are very small compared to $u$, so we can

   – Set $\dot{w} \approx 0$ and $\dot{q} \approx 0$

   – Set $\Theta_0 = 0$

$$\begin{bmatrix} \dot{u} \\ 0 \\ 0 \\ \dot{\theta} \end{bmatrix} = \left[\begin{array}{c|c|c|c} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \\ \frac{Z_u}{m-Z_{\dot{w}}} & \frac{Z_w}{m-Z_{\dot{w}}} & \frac{Z_q+mU_0}{m-Z_{\dot{w}}} & 0 \\ \frac{[M_u+Z_u\Gamma]}{I_{yy}} & \frac{[M_w+Z_w\Gamma]}{I_{yy}} & \frac{\left[M_q+(Z_q+mU_0)\Gamma\right]}{I_{yy}} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta X^c \\ \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix}$$

2. Use what is left of the $Z$-equation to show that with these approximations (elevator inputs)

$$\begin{bmatrix} \frac{Z_w}{m-Z_{\dot{w}}} & \frac{Z_q+mU_0}{m-Z_{\dot{w}}} \\ \frac{[M_w+Z_w\Gamma]}{I_{yy}} & \frac{\left[M_q+(Z_q+mU_0)\Gamma\right]}{I_{yy}} \end{bmatrix} \begin{bmatrix} w \\ q \end{bmatrix} = -\begin{bmatrix} \frac{Z_u}{m-Z_{\dot{w}}} \\ \frac{[M_u+Z_u\Gamma]}{I_{yy}} \end{bmatrix} u - \begin{bmatrix} \frac{Z_{\delta_e}}{m-Z_{\dot{w}}} \\ \frac{\left[M_{\delta_e}+Z_{\delta_e}\Gamma\right]}{I_{yy}} \end{bmatrix} \delta_e$$

3. Use ($Z_{\dot{w}} \ll m$ so $\Gamma \approx \frac{M_{\dot{w}}}{m}$) and ($Z_q \ll mU_0$) so that:

$$\begin{bmatrix} Z_w & mU_0 \\ \left[M_w + Z_w\frac{M_{\dot{w}}}{m}\right] & [M_q + U_0 M_{\dot{w}}] \end{bmatrix} \begin{bmatrix} w \\ q \end{bmatrix}$$

$$= -\begin{bmatrix} Z_u \\ \left[M_u + Z_u\frac{M_{\dot{w}}}{m}\right] \end{bmatrix} u - \begin{bmatrix} Z_{\delta_e} \\ \left[M_{\delta_e} + Z_{\delta_e}\frac{M_{\dot{w}}}{m}\right] \end{bmatrix} \delta_e$$

4. Solve to show that

$$\begin{bmatrix} w \\ q \end{bmatrix} = \begin{bmatrix} \dfrac{mU_0M_u - Z_uM_q}{Z_wM_q - mU_0M_w} \\ \dfrac{Z_uM_w - Z_wM_u}{Z_wM_q - mU_0M_w} \end{bmatrix} u + \begin{bmatrix} \dfrac{mU_0M_{\delta_e} - Z_{\delta_e}M_q}{Z_wM_q - mU_0M_w} \\ \dfrac{Z_{\delta_e}M_w - Z_wM_{\delta_e}}{Z_wM_q - mU_0M_w} \end{bmatrix} \delta_e$$

5. Substitute into the reduced equations to get **full** approximation:

$$\begin{aligned} \begin{bmatrix} \dot{u} \\ \dot{\theta} \end{bmatrix} &= \begin{bmatrix} \frac{X_u}{m} + \frac{X_w}{m}\left(\frac{mU_0M_u - Z_uM_q}{Z_wM_q - mU_0M_w}\right) & -g \\ \left(\frac{Z_uM_w - Z_wM_u}{Z_wM_q - mU_0M_w}\right) & 0 \end{bmatrix} \begin{bmatrix} u \\ \theta \end{bmatrix} \\ &+ \begin{bmatrix} \frac{X_{\delta_e}}{m} + \frac{X_w}{m}\left(\frac{mU_0M_{\delta_e} - Z_{\delta_e}M_q}{Z_wM_q - mU_0M_w}\right) \\ \frac{Z_{\delta_e}M_w - Z_wM_{\delta_e}}{Z_wM_q - mU_0M_w} \end{bmatrix} \delta_e \end{aligned}$$

6. Still a bit complicated. Typically get that
   - $|M_uZ_w| \ll |M_wZ_u|$ (1.4:4)
   - $|M_wU_0m| \gg |M_qZ_w|$ (1:0.13)
   - $|M_uX_w/M_w| \ll X_u$ small

7. With these approximations, the longitudinal dynamics reduce to the **coarse** approximation

$$\dot{x}_{ph} = A_{ph}x_{ph} + B_{ph}\delta_e$$

where $\delta_e$ is the elevator input.

And

$$x_{ph} = \begin{bmatrix} u \\ \theta \end{bmatrix} \quad A_{ph} = \begin{bmatrix} \dfrac{X_u}{m} & -g \\ \dfrac{-Z_u}{mU_0} & 0 \end{bmatrix}$$

$$B_{ph} = \begin{bmatrix} \dfrac{\left(X_{\delta_e} - \left[\frac{X_w}{M_w}\right] M_{\delta_e}\right)}{m} \\ \dfrac{\left(-Z_{\delta_e} + \left[\frac{Z_w}{M_w}\right] M_{\delta_e}\right)}{mU_0} \end{bmatrix}$$

8. Which gives

$$\begin{aligned} 2\zeta_{ph}\omega_{ph} &= -X_u/m \\ \omega_{ph}^2 &= -\frac{gZ_u}{mU_0} \end{aligned}$$

Numerical values for 747

| | Frequency rad/sec | Damping |
|---|---|---|
| Full model | 0.0673 | 0.0489 |
| Full Approximate | 0.0670 | 0.0419 |
| Coarse Approximate | 0.0611 | 0.0561 |

- Further insights: recall that

$$\begin{aligned}\left(\frac{U_0}{QS}\right)\left(\frac{\partial Z}{\partial u}\right)_0 &= -\left(\frac{U_0}{QS}\right)\left(\frac{\partial L}{\partial u}\right)_0 \equiv -(C_{L_u}+2C_{L_0}) \\ &= -\frac{\mathbf{M^2}}{1-\mathbf{M^2}}C_{L_0} - 2C_{L_0} \approx -2C_{L_0}\end{aligned}$$

so

$$Z_u \equiv \left(\frac{\partial Z}{\partial u}\right)_0 = \left(\frac{\rho U_o S}{2}\right)(-2C_{L_0}) = -\frac{2mg}{U_0}$$

- Then

$$\begin{aligned}\omega_{ph} &= \sqrt{\frac{-gZ_u}{mU_0}} = \sqrt{\frac{mg^2}{mU_0^2}} \\ &= \sqrt{2}\frac{g}{U_0}\end{aligned}$$

which is **exactly** what Lanchester's approximation gave $\Omega \approx \sqrt{2}\frac{g}{U_0}$

- Note that

$$X_u \equiv \left(\frac{\partial X}{\partial u}\right)_0 = \left(\frac{\rho U_o S}{2}\right)(-2C_{D_0}) = -\rho U_o S C_{D_0}$$

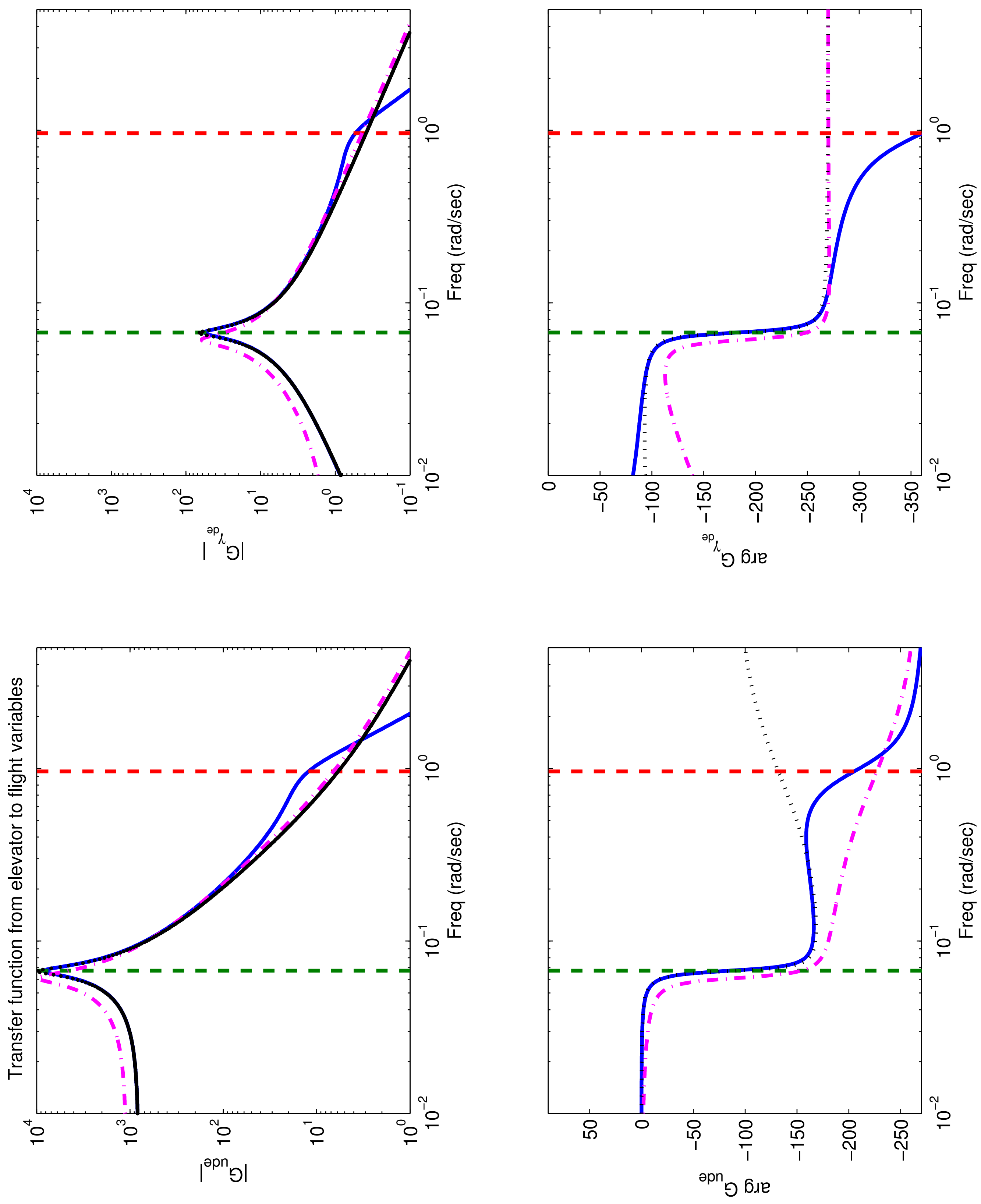
and

$$2mg = \rho U_o^2 S C_{L_0}$$

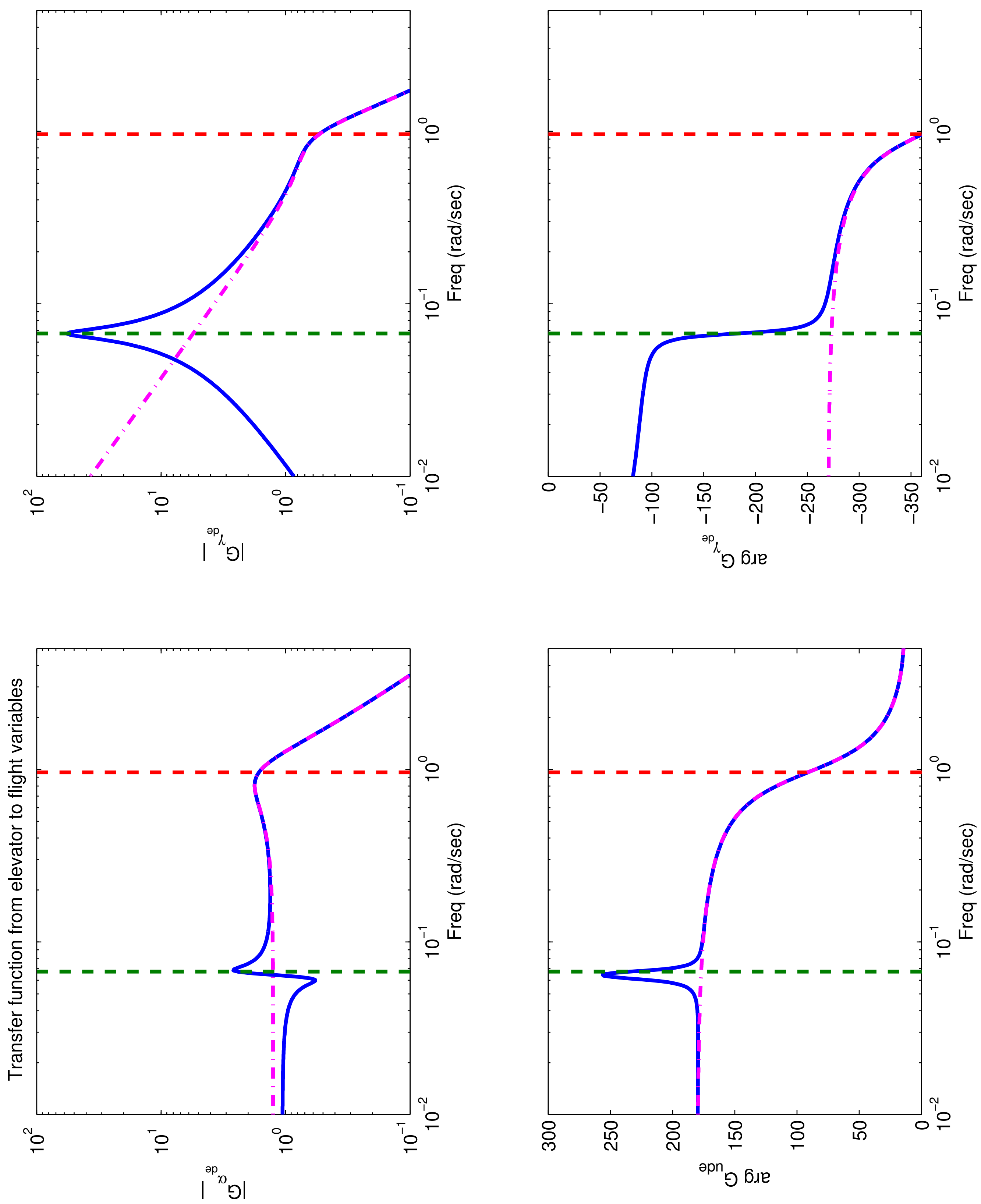
so

$$\begin{aligned}\zeta_{ph} &= -\frac{X_u}{2m\omega_{ph}} = -\frac{X_u U_0}{2\sqrt{2}mg} \\ &= \frac{1}{\sqrt{2}}\left(\frac{\rho U_o^2 S C_{D_0}}{\rho U_o^2 S C_{L_0}}\right) \\ &= \frac{1}{\sqrt{2}}\left(\frac{C_{D_0}}{C_{L_0}}\right)\end{aligned}$$

so the damping ratio of the approximate phugoid mode is **inversely proportional** to the **lift to drag** ratio.




Freq Comparison from elevator (Phugoid Model) – B747 at M=0.8. **Blue**– Full model, **Black**– Full approximate model, **Magenta**– Coarse approximate model




Freq Comparison from elevator (Short Period Model) – B747 at M=0.8. **Blue**– Full model, **Magenta**– Approximate model

# Summary

- Approximate longitudinal models are fairly accurate

- Indicate that the aircraft responses are mainly determined by these stability derivatives:

| Property | Stability derivative |
| --- | --- |
| Damping of the short period | $M_q$ |
| Frequency of the short period | $M_w$ |
| Damping of the Phugoid | $X_u$ |
| Frequency of the Phugoid | $Z_u$ |

- Given a change in $\alpha$, expect changes in $u$ as well. These will both impact the lift and drag of the aircraft, requiring that we re-trim throttle setting to maintain whatever aspects of the flight condition might have changed (other than the ones we wanted to change). We have:

$$\begin{bmatrix} \Delta L \\ \Delta D \end{bmatrix} = \begin{bmatrix} L_u & L_\alpha \\ D_u & D_\alpha \end{bmatrix} \begin{bmatrix} u \\ \Delta\alpha \end{bmatrix}$$

But to maintain $L = W$, want $\Delta L = 0$, so $u = -\frac{L_\alpha}{L_u}\Delta\alpha$

Giving $\Delta D = \left(-\frac{L_\alpha}{L_u} D_u + D_\alpha\right)\Delta\alpha$

$$\begin{aligned} C_{D_\alpha} = \frac{2C_{L_0}}{\pi e AR} C_{L_\alpha} \;\; &\rightarrow \;\; D_\alpha = QSC_{D_\alpha} \\ &\rightarrow \;\; L_\alpha = QSC_{L_\alpha} \\ &\qquad D_u = \frac{QS}{U_0}(2C_{D_0}) \qquad (4-16) \\ &\qquad L_u = \frac{QS}{U_0}(2C_{L_0}) \qquad (4-17) \end{aligned}$$

$$\begin{aligned} \Delta D &= QS\left(-\frac{C_{L_\alpha}}{2C_{L_0}/U_0}\left(\frac{2C_{D_0}}{U_0}\right) + C_{D_\alpha}\right)\Delta\alpha \\ &= \frac{QS}{C_{L_0}}\left(-C_{D_0} + \frac{2C_{L_0}^2}{\pi e AR}\right) C_{L_\alpha}\Delta\alpha \end{aligned}$$

$$\begin{aligned} \tan\Delta\gamma &= \frac{(T_0 + \Delta T) - (D_0 + \Delta D)}{L_0 + \Delta L} = \frac{-\Delta D}{L_0} \\ &= \left(\frac{C_{D_0}}{C_{L_0}} - \frac{2C_{L_0}}{\pi e AR}\right)\frac{C_{L_\alpha}\Delta\alpha}{C_{L_0}} \end{aligned}$$

For 747 (Reid 165 and Nelson 416), $AR = 7.14$, so $\pi e AR \approx 18$, $C_{L_0} = 0.654$ $C_{D_0} = 0.043$, $C_{L_\alpha} = 5.5$, for a $\Delta\alpha = -0.0185$rad (6–7) $\Delta\gamma = -0.0006$rad. This is the opposite sign to the linear simulation results, but they are both very small numbers.

# 16.333 Lecture # 8

Aircraft Lateral Dynamics

Spiral, Roll, and Dutch Roll Modes

# Aircraft Lateral Dynamics

- Using a procedure similar to the longitudinal case, we can develop the equations of motion for the **lateral dynamics**

$$\dot{x} = Ax + Bu \ , \ x = \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix} \ , \ u = \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

and $\dot{\psi} = r \sec\theta_0$

$$A = \begin{bmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \frac{Y_r}{m} - U_0 & g\cos\theta_0 \\ (\frac{L_v}{I'_{\rm xx}} + I'_{\rm zx}N_v) & (\frac{L_p}{I'_{\rm xx}} + I'_{\rm zx}N_p) & (\frac{L_r}{I'_{\rm xx}} + I'_{\rm zx}N_r) & 0 \\ (I'_{\rm zx}L_v + \frac{N_v}{I'_{\rm zz}}) & (I'_{\rm zx}L_p + \frac{N_p}{I'_{\rm zz}}) & (I'_{\rm zx}L_r + \frac{N_r}{I'_{\rm zz}}) & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{bmatrix}$$

where

$$\begin{aligned} I'_{\rm xx} &= (I_{\rm xx}I_{\rm zz} - I^2_{\rm zx})/I_{\rm zz} \\ I'_{\rm zz} &= (I_{\rm xx}I_{\rm zz} - I^2_{\rm zx})/I_{\rm xx} \\ I'_{\rm zx} &= I_{\rm zx}/(I_{\rm xx}I_{\rm zz} - I^2_{\rm zx}) \end{aligned}$$

and

$$B = \begin{bmatrix} (m)^{-1} & 0 & 0 \\ 0 & (I'_{\rm xx})^{-1} & I'_{\rm zx} \\ 0 & I'_{\rm zx} & (I'_{\rm zz})^{-1} \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \end{bmatrix}$$

# Lateral Stability Derivatives

- A key to understanding the lateral dynamics is **roll-yaw coupling**.
- $L_p$ rolling moment due to roll rate:
  - Roll rate $p$ causes right to move wing down, left wing to move up
    $\longrightarrow$ Vertical velocity distribution over the wing $W = py$
  - Leads to a spanwise change in the AOA: $\alpha_r(y) = py/U_0$
  - Creates lift distribution (chordwise strips)

$$\delta L_w(y) = \frac{1}{2}\rho U_0^2 C_{l_\alpha} \alpha_r(y) c_y dy$$

  - Net result is higher lift on right, lower on left
  - Rolling moment:

$$L = \int_{-b/2}^{b/2} \delta L_w(y) \cdot (-y) dy = -\frac{1}{2}\rho U_0^2 \int_{-b/2}^{b/2} C_{l_\alpha} \frac{py^2}{U_0} c_y dy \Rightarrow L_p < 0$$

  - **Key point: positive roll rate $\Rightarrow$ negative roll moment.**

- $L_r$ rolling moment due to yaw rate:
  - Positive $r$ has left wing advancing, right wing retreating
    $\longrightarrow$ Horizontal velocity distribution over wing $U = U_0 - ry$
  - Creates lift distribution over wing (chordwise strips)

$$L_w(y) \sim \frac{1}{2}\rho U^2 C_l c dy \approx \frac{1}{2}\rho (U_0^2 - 2U_0 r y) C_l c_y dy$$

  - Net result is higher lift on the left, lower on the right.
  - Rolling Moment: $L = \int_{-b/2}^{b/2} L_w(y) \cdot (-y) dy \approx \rho U_0 r \int_{-b/2}^{b/2} C_l c_y y^2 dy$
  - For large aspect ratio rectangular wing (crude)

$$L_r \approx (\frac{1}{6} \text{ to } \frac{1}{4}) C_L > 0$$

  - **Key point: positive yaw rate $\Rightarrow$ positive roll moment.**

- $N_p$ yawing moment due to roll rate:
  - Rolling wing induces a change in spanwise AOA, which changes the spanwise **lift** and **drag**.
  - Distributed drag change creates a yawing moment. Expect higher drag on right (lower on left) $\rightarrow$ positive yaw moment
  - There is both a change in the lift (larger on downward wing because of the increase in $\alpha$) and a rotation (leans forward on downward wing because of the larger $\alpha$). $\rightarrow$ negative yaw moment
  - In general hard to know which effect is larger. Nelson suggests that for a rectangular wing, crude estimate is that

$$N_p \approx \frac{1}{2}\rho U_0^2 Sb(-\frac{C_L}{8}) < 0$$

- $N_r$ yawing moment due to yaw rate:
  - Key in determining stability properties – mostly from fin.
  - Positive $r$ has fin moving to the left which increases the apparent angle of attack by

$$\Delta\alpha_f = \frac{rl_f}{(U_0)_f}$$

  - Creates increase in lift at the tail fin by

$$\Delta L_f = \frac{1}{2}\rho(U_0^2)_f S_f C_{L_{\alpha_f}} \Delta\alpha_f$$

  - Creates a change in the yaw moment of

$$N = -l_f \Delta L_f = -\frac{1}{2}\rho(U_0)_f S_f C_{L_{\alpha_f}} r l_f^2$$

  - So $N_r = -\frac{1}{2}\rho(U_0)_f S_f C_{L_{\alpha_f}} l_f^2 < 0$
  - **Key point: positive yaw rate $\Rightarrow$ negative yaw moment.**

| | $L$ | $N$ |
|---|---|---|
| $p$ | $< 0$ | ? |
| $r$ | $> 0$ | $< 0$ |

# Numerical Results

- The code gives the numerical values for all of the stability derivatives. Can solve for the eigenvalues of the matrix $A$ to find the modes of the system.

$$\begin{gathered}-0.0331 \pm 0.9470i\\ -0.5633\\ -0.0073\end{gathered}$$

  – Stable, but there is one very slow pole.

- There are 3 modes, but they are a **lot more complicated** than the longitudinal case.

| | | | |
|---|---|---|---|
| Slow mode | -0.0073 | $\Rightarrow$ | **Spiral Mode** |
| Fast real | -0.5633 | $\Rightarrow$ | Roll Damping |
| Oscillatory | $-0.0331 \pm 0.9470i$ | $\Rightarrow$ | **Dutch Roll** |

Can look at normalized eigenvectors:

| | Spiral | Roll | Dutch Roll | |
|---|---|---|---|---|
| $\beta = w/U_0$ | 0.0067 | -0.0197 | 0.3269 | -28° |
| $\hat{p} = p/(2U_0/b)$ | -0.0009 | -0.0712 | 0.1198 | 92° |
| $\hat{r} = r/(2U_0/b)$ | 0.0052 | 0.0040 | 0.0368 | -112° |
| $\phi$ | 1.0000 | 1.0000 | 1.0000 | 0° |

**Not as enlightening as the longitudinal case.**

# Lateral Modes

**Roll Damping** - well damped.

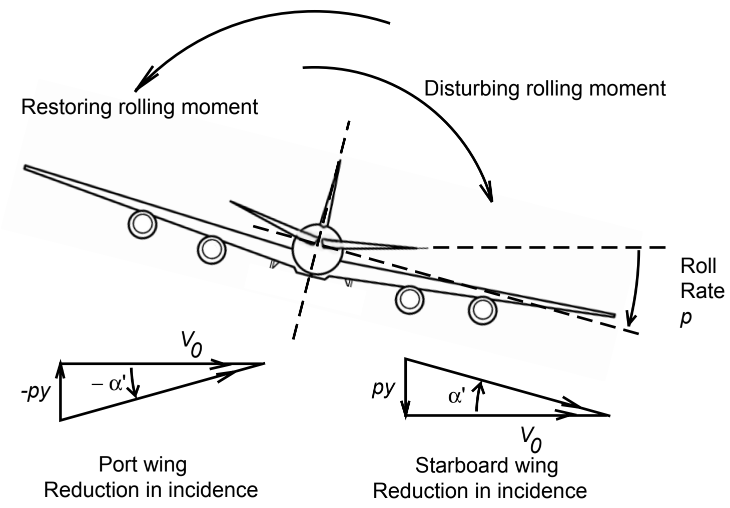
– As the plane rolls, the wing going down has an increased $\alpha$ (wind is effectively "coming up" more at the wing)

– Opposite effect for other wing.

– There is a difference in the lift generated by both wings $\longrightarrow$ more on side going down

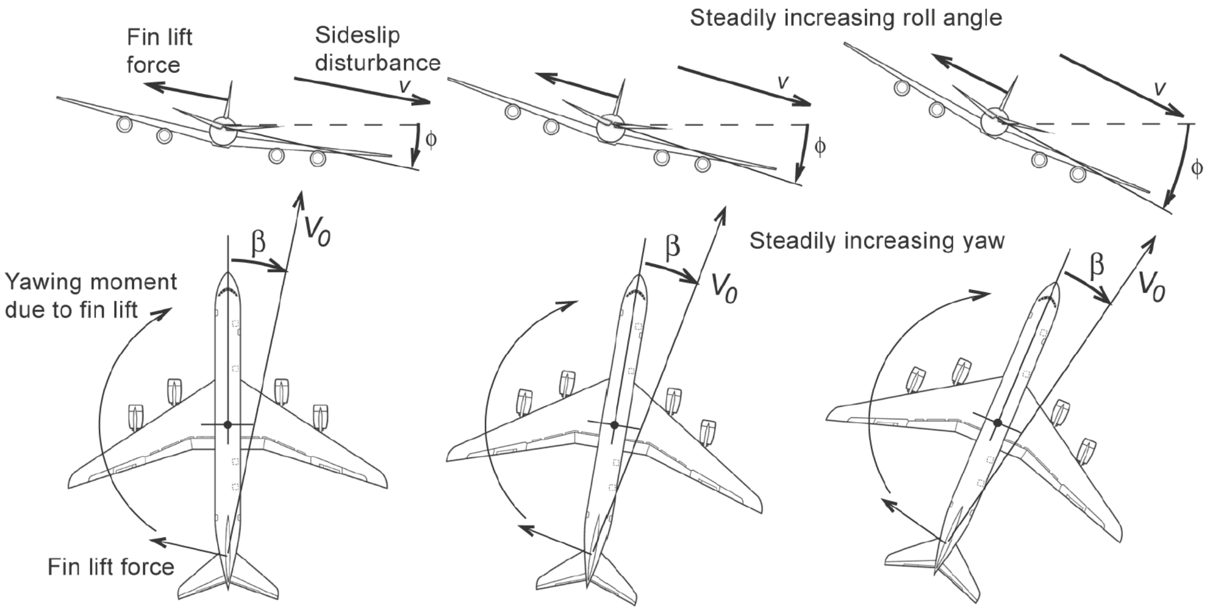
– The differential lift creates a **moment** that tends to **restore** the equilibrium. Recall that $L_p < 0$

– After a disturbance, the roll rate builds up exponentially until the restoring moment balances the disturbing moment, and a steady roll is established.



**Spiral Mode** - slow, often unstable.

- From level flight, consider a disturbance that creates a small roll angle $\phi > 0 \rightarrow$ This results in a small side-slip $v$ (vehicle *slides downhill*)
- Now the tail fin hits on the oncoming air at an incidence angle $\beta$ $\rightarrow$ extra tail lift $\rightarrow$ positive yawing moment
- Moment creates positive yaw rate that creates positive roll moment $(L_r > 0)$ that increases the roll angle and tends to increase the side-slip
  $\rightarrow$ makes things worse.
- If unstable and left unchecked, the aircraft would fly a slowly diverging path in roll, yaw, and altitude $\Rightarrow$ it would tend to *spiral* into the ground!!




- Can get a restoring torque from the wing **dihedral**

- Want a small tail to reduce the impact of the spiral mode.

**Dutch Roll** - damped oscillation in yaw, that couples into roll.

- Frequency similar to longitudinal short period mode, not as well damped (fin less effective than horizontal tail).
- Consider a disturbance from straight-level flight

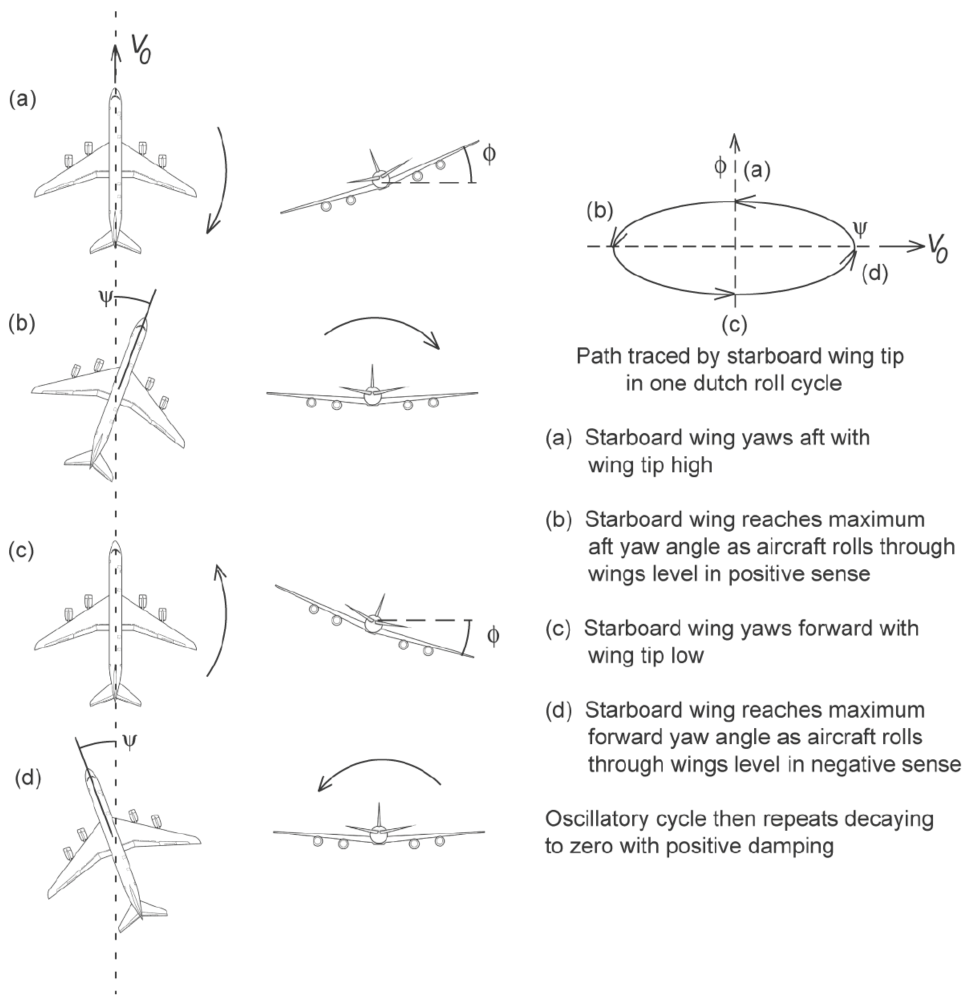
  $\rightarrow$ Oscillation in yaw $\psi$ (fin provides the *aerodynamic stiffness*)

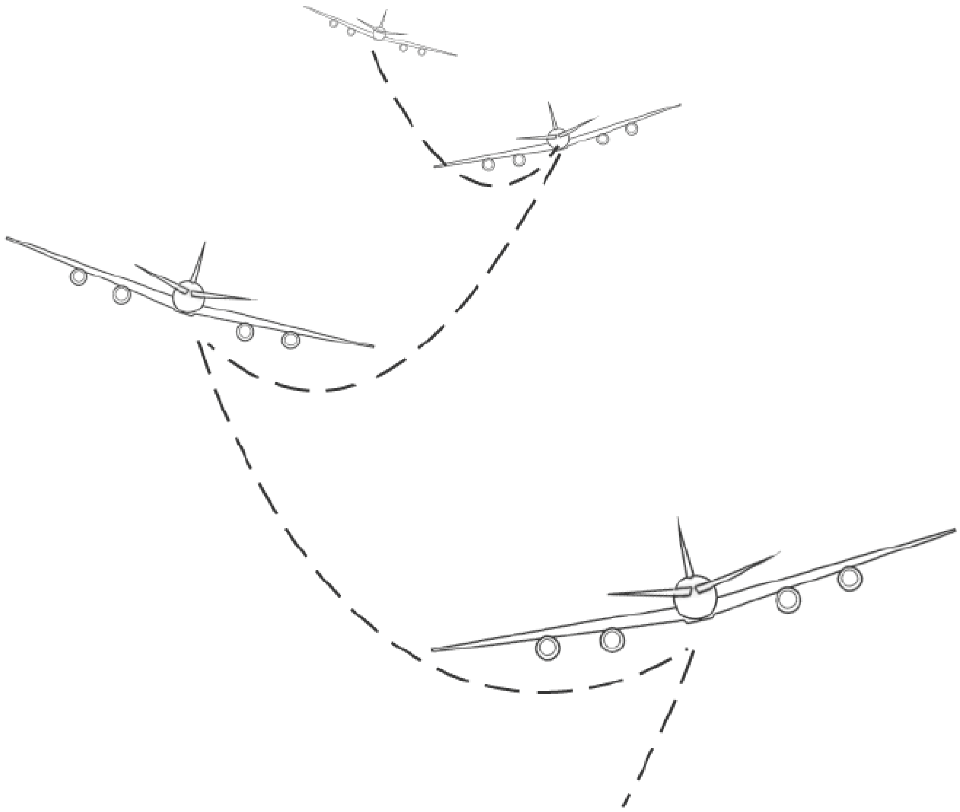
  $\rightarrow$ Wings moving back and forth due to yaw motion result in oscillatory differential lift/drag (wing moving forward generates more lift) $L_r > 0$

  $\rightarrow$ Oscillation in roll $\phi$ that lags $\psi$ by approximately 90°

  $\Rightarrow$ *Forward going wing is low*

  Oscillating roll $\Rightarrow$ sideslip in direction of low wing.





- Do you know the origins on the name of the mode?
- Damp the Dutch roll mode with a large tail fin.

# Aircraft Actuator Influence

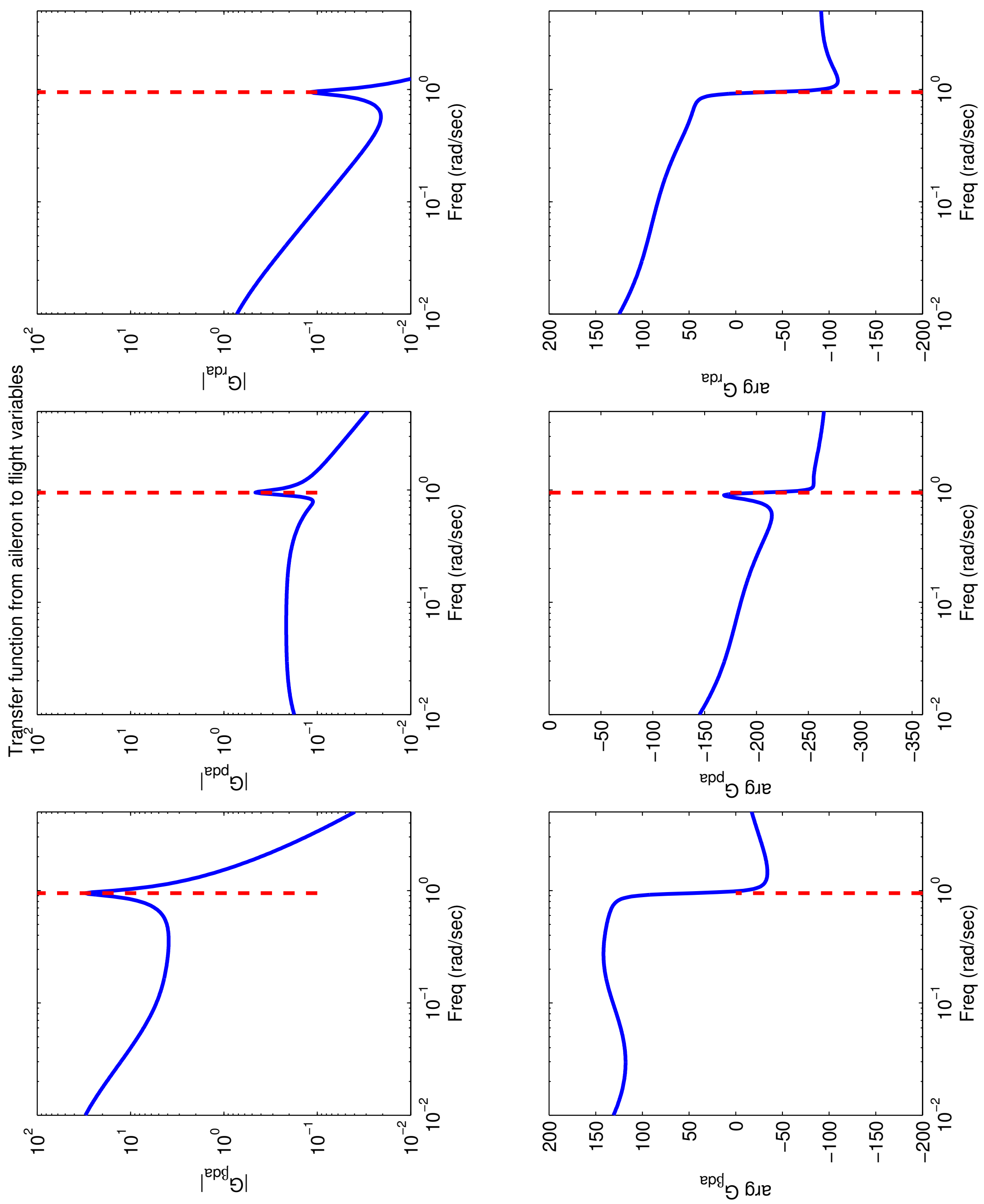



Figure 1: Aileron impulse to flight variables. Response primarily in $\phi$.

- Transfer functions dominated by lightly damped Dutch-roll mode.
- Note the rudder is physically quite high, so it also influences the A/C roll.
- Ailerons influence the Yaw because of the differential drag

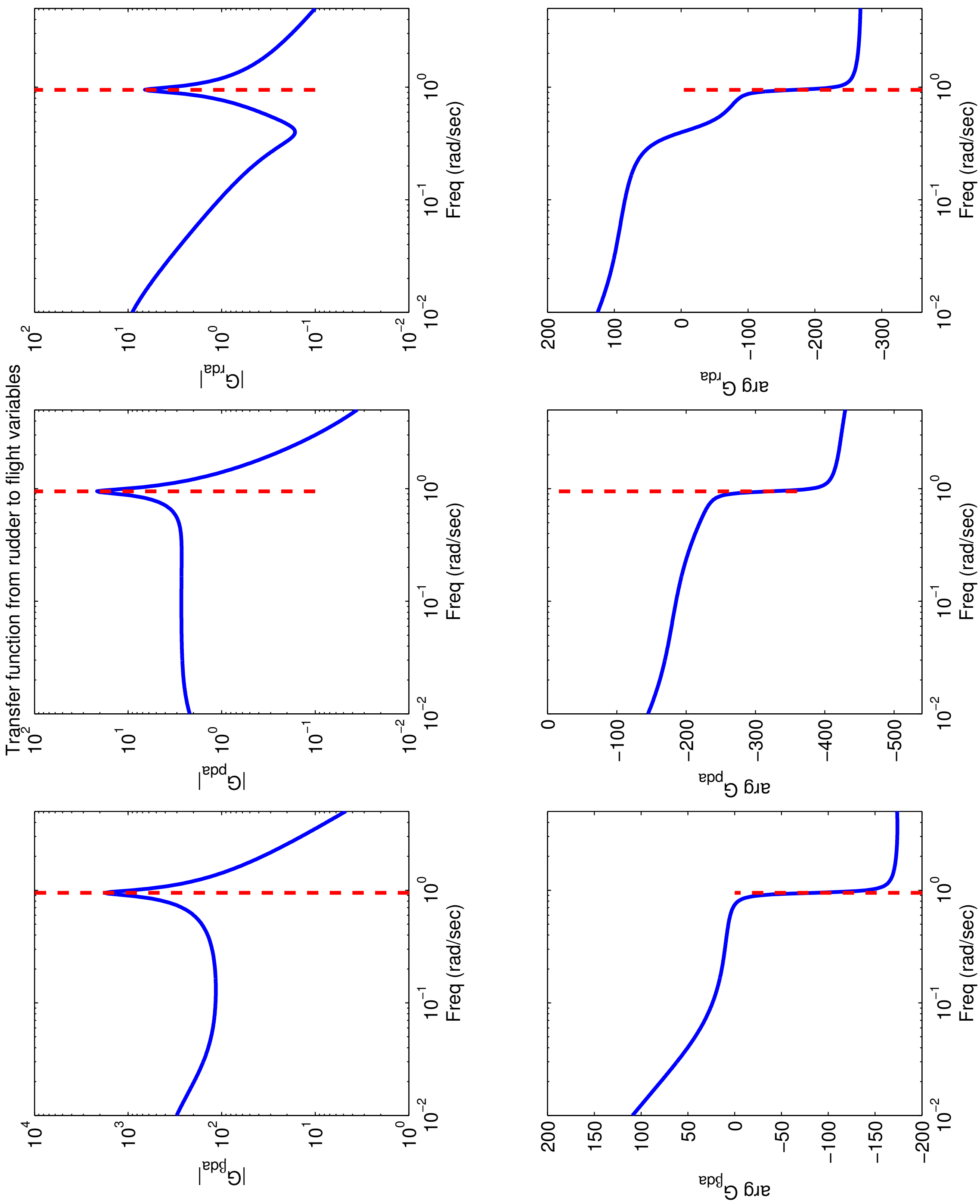



Figure 2: Aileron impulse to flight variables. Response primarily in $\phi$.

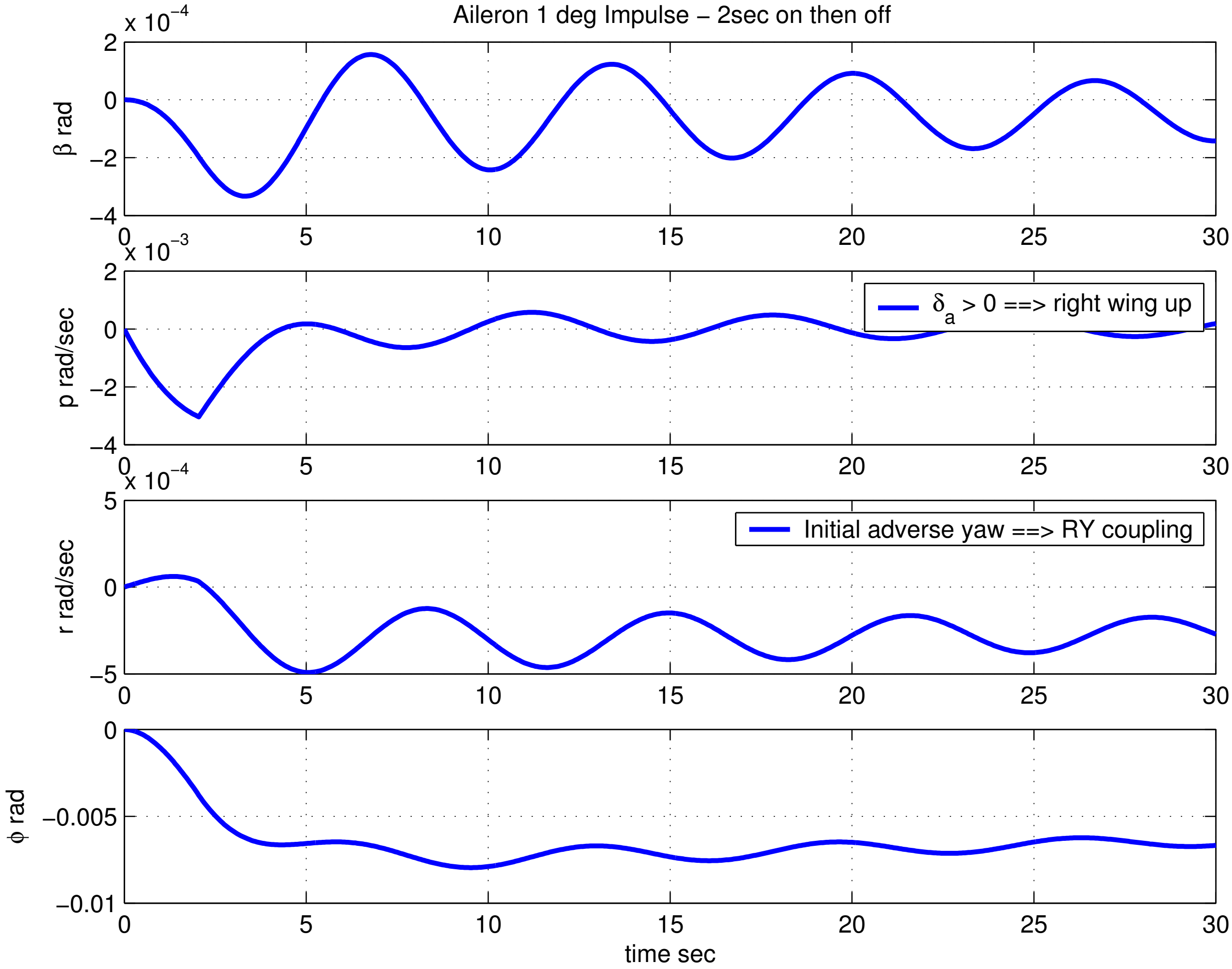



Figure 3: Aileron impulse to flight variables

- **Aileron** $\delta_a$ =1deg impulse for 2 sec.
  - Since $\delta_a > 0$ then right aileron goes down, and right wing goes up → Reid's notation, and it is **not** consistent with the picture on 6–4 (from Nelson).
  - Influence of the roll mode seen in the response of $p$ to application and release of the aileron input.
  - See effect of *adverse yaw* in the yaw rate response caused by the differential drag due to aileron deflection.
  - Spiral mode harder to see.
  - Dutch mode response in other variables clear (1 rad/sec $\sim$ 6 sec period).

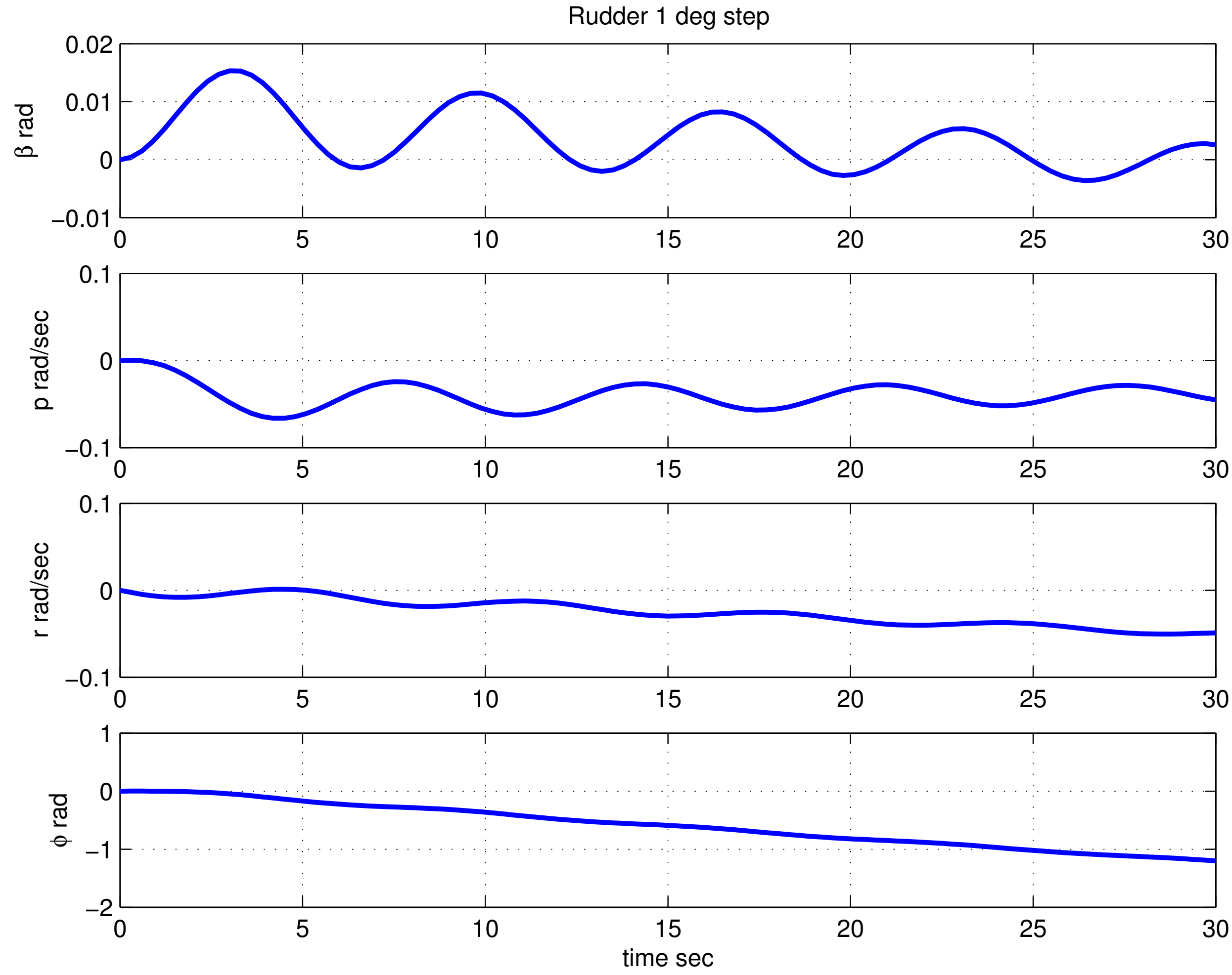



Figure 4: Rudder step to flight variables

- **Rudder step input** 1deg step.
  - Dutch roll response very clear. Other 2 modes are much less pronounced.
  - $\beta$ shows a very lightly damped decay.
  - $p$ clearly excited as well. Doesn't show it, but often see evidence of adverse roll in $p$ response where initial $p$ is opposite sign to steady state value. Reason is that the forces act on the fin which is well above the *cg* $\rightarrow$ and the aircraft responds rapidly (initially) in roll.
  - $\phi$ ultimately oscillates around $2.5^{\circ}$