APPENDIX A

Some useful statistical equations

A and B are two independent random variables and their probability density functions are f(a) and f(b).

$$E[a] \quad = \int a.f(a).da = \mu_c$$

$$\sigma^2[a] \quad = E\{(a - E[a])^2\}$$
$$= E\{(a^2 - 2a\,E[a] + E^2[a])\}$$
$$= E[a^2] - 2\,E[a]E[a] + E^2[a] \quad = E[a^2] - E^2[a]$$

$$E\{E[a]\} \qquad = \int\left(\int_{-\infty}^{+\infty} a.f(a).da\right) f(a).da = \int a.f(a).da.\int f(a).da = E[a]$$

m and b are constant.

$$y = ma + b$$

$$E[y] \quad = \int (ma + b) f(a) da = mE[a] + b$$

$$\sigma^2[y] \quad = E\{(y - E[y])^2\}$$
$$= E\{(ma + b - mE[a] - b)^2\}$$
$$= m^2\sigma^2(a)$$

$$f(a.b) \quad = f(a).f(b)$$

$$E[a.b] \quad = \iint a.b.f(a).f(b).da.db = \int a.f(a).da \int b.f(b).db = E[a].E[b]$$

$$E[a + b] = \iint (a + b).f(a).f(b).da.db \quad = \iint a.f(a).f(b).da.db + \iint b.f(a).f(b).da.db$$
$$= \int a.f(a).da \int f(b).db \qquad + \int b.f(b).db \int f(a).da$$
$$= E[a] + E[b]$$

$$E[a - b] = E[a] - E[b]$$

$$
\begin{aligned}
\sigma^2[a.b] &= E\{[ab - E[a.b]]^2\} \\
&= E\{[ab - E[a].E[b]]^2\} \\
&= E\{[a^2b^2 - 2.a.b.E[a]E[b] + E^2[a]E^2[b]]\} \\
&= E[a^2b^2] - 2.E^2[a]E^2[b] + E^2[a]E^2[b] \\
&= (\sigma^2[a] + E^2[a]).(\sigma^2[b] + E^2[b]) - E^2[a]E^2[b] \\
&= \sigma^2[a].\sigma^2[b] + \sigma^2[a]E^2[b] + \sigma^2[b]E^2[a] + E^2[a]E^2[b] - E^2[a]E^2[b] \\
&= \sigma^2[a].\sigma^2[b] + E^2[b]\sigma^2[a] + E^2[a]\sigma^2[b]
\end{aligned}
$$

$$
\begin{aligned}
\sigma^2[a+b] \quad &= E\{[a+b-E[a+b]]^2\} \\
&= E\{[a+b-(E[a]+E[b])]^2\} \\
&= E\{[[a-E[a]]+[b-E[b]]]^2\} \\
&= \sigma_A^2 + \sigma_B^2 + 2E\{(ab - a.E[b] - b.E[a] + E[a]E[b])\} \\
&= \sigma_A^2 + \sigma_B^2 + 2\{E[ab] - E[a]E[b] - E[b]E[a] + E[a]E[b]\} \\
&= \sigma_A^2 + \sigma_B^2
\end{aligned}
$$

$$
\begin{aligned}
\sigma^2[a-b] \quad &= E\{[a-b-E[a-b]]^2\} \\
&= E\{[[a-E[a]]-[b-E[b]]]^2\} \\
&= \sigma_A^2 + \sigma_B^2
\end{aligned}
$$

$$
E\left[\frac{A}{B}\right] = \frac{E[A]}{E[B]}
$$

$$
\begin{aligned}
\sigma^2\left[\frac{A}{B}\right] &= E\left\{\left[\frac{A}{B} - E\left[\frac{A}{B}\right]\right]^2\right\} \\
&= E\left\{\frac{A^2}{B^2} - 2\frac{A}{B}\frac{E[A]}{E[B]} + \frac{E^2[A]}{E^2[B]}\right\} \\
&= \frac{E[A^2]}{E[B^2]} - \frac{E^2[A]}{E^2[B]} \\
&= \frac{\sigma^2[A] + E^2[A]}{\sigma^2[B] + E^2[B]} - \frac{E^2[A]}{E^2[B]} \\
&= \frac{\sigma^2[A]E^2[B] + E^2[A]E^2[B] - \sigma^2[B]E^2[A] - E^2[A]E^2[B]}{E^2[B][E^2[B] + \sigma^2[B]]} \\
&= \frac{E^2[B]\sigma^2[A] - E^2[A]\sigma^2[B]}{E^4[B]} \\
&= \frac{\sigma^2[A]}{E^2[B]} + \left(\frac{E[A]}{E[B]}\right)^2 \frac{\sigma^2[B]}{E^2[B]}
\end{aligned}
$$

If $g(x,y)$ is sufficiently smooth near the point (η_x, η_y) η_g and σ_g can be estimated as follows

$$\eta_g \cong g + \frac{1}{2}\left(\frac{\partial^2 g}{\partial x^2}\sigma_x{}^2 + 2\frac{\partial^2 g}{\partial x \partial y} r \sigma_x \sigma_y + \frac{\partial^2 g}{\partial y^2}\sigma_y{}^2\right)$$

$$\sigma_g^2 \cong \left(\frac{dg}{dx}\right)^2 \sigma_x^2 + 2\left(\frac{dg}{dx}\right)\left(\frac{dg}{dy}\right) r \sigma_y^2 \sigma_x^2 + \left(\frac{dg}{dy}\right)^2 \sigma_y^2$$

Assuming that random variable x has a mean value η_X and standard deviation σ_X^2.

$$y = f(x) \qquad \Rightarrow$$

$$\sigma_y^2 \cong \left| f(\eta_X)' \right|^2 \sigma_X^2$$