Sayısal Filtreler ve Sistemler
EHB 433

Prof. Dr. Müştak E. Yalçın

Istanbul Technical University
Faculty of Electrical and Electronic Engineering

mustak.yalcin@itu.edu.tr

Outline I

Ayrık zamanlı sistemlerin durum denklemleri

$$\begin{aligned} v(k+1) &= f(v(k), x(k), k) \\ y(k) &= g(v(k), x(k), k) \end{aligned}$$

v durum değişken, x giriş ve y çıkış olarak tanımlanır. Lineer sistem :

$$\begin{aligned} v(k+1) &= Av(k) + Bx(k) \\ y(k) &= Cv(k) + Dx(k) \end{aligned}$$

Fark denkleminden

$$\begin{aligned} y(n)+ \quad & a_1 y(n-1) + a_2 y(n-2) + \cdots + a_N y(n-N) = b_0 x(n) \\ & + b_1 x(n-1) + \cdots + b_N x(n-N) \end{aligned}$$

durum denklemini elde edebilirmiyiz ?

$$
\begin{aligned}
y(n) &= -a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N) + b_0 x(n) \\
&\quad + b_1 x(n-1) + \ldots + b_N x(n-N) \\
v_N(n) &= -a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N) + b_1 x(n-1) \\
&\quad + \cdots + b_N x(n-N) \\
y(n) &= v_N(n) + b_0 x(n)
\end{aligned}
$$

$$
\begin{aligned}
v_N(n+1) &= -a_1 y(n) - a_2 y(n-1) - \cdots - a_N y(n-N+1) + b_1 x(n) \\
&\quad + \ldots + b_N x(n-N+1) \\
v_{N-1}(n) &= -a_2 y(n-1) - \cdots - a_N y(n-N+1) + b_2 x(n-1) \\
&\quad + \cdots + b_N x(n-N+1) \\
v_N(n+1) &= v_{N-1}(n) - a_1 y(n) + b_1 x(n) \\
&= v_{N-1}(n) - a_1 (v_N(n) + b_0 x(n)) + b_1 x(n)
\end{aligned}
$$

$$
\begin{aligned}
v_{N-1}(n+1) &= -a_2 y(n) - a_3 y(n-1) - \cdots - a_N y(n-N+2) + b_2 x(n) \\
&\quad + \cdots + b_N x(n-N+2) \\
v_{N-1}(n+1) &= v_{N-2}(n) - a_2 y(n) + b_2 x(n) \\
&= v_{N-2}(n) - a_2 (v_N(n) + b_0 x(n)) + b_1 x(n)
\end{aligned}
$$

Aşağıdaki şekilde yazılabilir ;

$$v(k+1) = Av(k) + Bx(k)$$

burda $v = [v_1 v_2 \cdots v_N]^T$ ve $A \in R^{N \times N}$ $B \in R^N$ olmak üzere

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & -a_N \\ 1 & 0 & 0 & \dots & 0 & 0 & -a_{N-1} \\ 0 & 1 & 0 & \dots & 0 & 0 & -a_{N-2} \\ . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & \dots & 1 & 0 & -a_2 \\ 0 & 0 & 0 & \dots & 0 & 1 & -a_1 \end{bmatrix}, B = \begin{bmatrix} b_N - a_N b_0 \\ b_{N-1} - a_{N-1} b_0 \\ . \\ . \\ . \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix}$$

$$y = Cv(k) + Dx(k)$$

$C = [0\ 0\ 0 \dots 0\ 1]$ ve $D = [b_0]$

Başlangıç koşulları yukarıda bulunan denklemler yardımıyla bulunur.

$$
\begin{aligned}
v_N(0) &= -a_1 y(-1) - a_2 y(-2) - \dots - a_N y(-N) + b_1 x(-1) \\
&\quad + \dots + b_N x(-N) \\
v_{N-1}(0) &= -a_2 y(-1) - \dots - a_N y(-N+1) + b_2 x(-1) + \dots + b_N x(-N + \\
. & \quad \dots \\
. & \quad \dots \\
. & \quad \dots
\end{aligned}
$$

Matris formunda yazıldığında $v(0) = Ey_0 + Fx_0$

$$
E = \begin{bmatrix} -a_N & 0 & 0 & \dots & 0 & 0 \\ -a_{N-1} & -a_N & 0 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ -a_2 & -a_3 & -a_4 & \dots & -a_{N-1} & 0 \\ -a_1 & -a_2 & -a_3 & \dots & -a_{N-1} & -a_N \end{bmatrix}, F = \begin{bmatrix} b_N & 0 & 0 & \dots & 0 & 0 \\ b_{N-1} & b_N & 0 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ b_2 & b_3 & b_4 & \dots & b_{N-1} & 0 \\ b_1 & b_2 & b_3 & \dots & b_{N-1} & b_N \end{bmatrix}
$$

$y_0 = [y(-1)\ y(-2) \dots y(-N+1)\ y(-N)]^T$ ve
$x_0 = [x(-1)\ x(-2) \dots x(-N+1)\ x(-N)]^T$.

Örnek

Örnek : Aşağıda transfer fonksiyonu verilen sistemin durum denklemlerini elde edin

$$H(z) = \frac{z^4 + 2z^3 + z + 1}{z^4 + 3z^3 + 2z^2 + z + 2}.$$

Çözüm :

$$\begin{aligned} y(n) + 3y(n-1) &+ 2y(n-2) + y(n-3) \\ &+ 2y(n-4) = x(n) + 2x(n-1) + x(n-3) + x(n-4) \end{aligned}$$

buradan

$$v_4(n) = ...$$

den başlayarak durum değişkenleri tanımlanır.

Yönetilebilir kanonik form

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

$$E = \frac{X}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

$$e(n) = x(n) - a_1 e(n-1) + a_2 e(n-2) + a_3 e(n-3)$$

$$\begin{aligned}
x_1(n) &= e(n-3) \\
x_2(n) &= e(n-2) \\
x_3(n) &= e(n-1) \\
x_1(n+1) &= e(n-2) = x_2(n) \\
x_2(n+1) &= e(n-1) = x_3(n) \\
x_3(n+1) &= e(n) = x(n) - a_1 e(n-1) + a_2 e(n-2) + a_3 e(n-3)
\end{aligned}$$

Yönetilebilir kanonik form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & 0 & -a_2 & -a_1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix}$$

$C = [b_M - a_N b_0 \;\; b_{M-1} - a_{N-1} b_0 \dots]$ ve $D = [b_0]$

Yönetilebilir kanonik form

2. derece direkt -II ve Transpose gerçekleme

14 April 2020 Tuesday 08:40

$$y(n) = a_1 y(n-1) + a_2 y(n-2) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

$$y(n) = v_2(n) + b_0 x(n)$$

$$v_2(n) = a_1 y(n-1) + a_2 y(n-2) + b_1 x(n-1) + b_2 x(n-2)$$

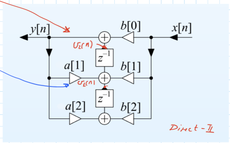
$$v_2(n+1) = a_1 y(n) + a_2 y(n-1) + b_1 x(n) + b_2 x(n-2)$$

$$v_2(n+1) = v_1(n) + a_1 y(n) + b_1 x(n) = v_1(n) + a_1 (v_2(n) + b_0 x(n)) + b_1 x(n)$$

$$v_1(n) = a_2 y(n-1) + b_2 x(n-2)$$

$$v_1(n+1) = a_2 y(n) + b_2 x(n) = a_2 (v_2(n) + b_0 x(n))$$

$$\begin{bmatrix} v_1(n+1) \\ v_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & a_2 \\ 1 & a_1 \end{bmatrix} \begin{bmatrix} v_1(n) \\ v_2(n) \end{bmatrix} + \begin{bmatrix} a_2 b_0 + b_2 \\ a_1 b_0 + b_1 \end{bmatrix} x(n)$$



AYNI FARK DENK.
İKİ FARKLI GERÇEKLENMESİ
+
iki durum da aynı transfer fonk. için
+
transpose

$$y(n) = a_1 y(n-1) + a_2 y(n-2) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

$$H(z) = (b_0 + b_1 z^{-1} + b_2) \frac{1}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

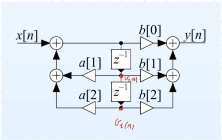
$$Y(z) = H(z) X(z) = (\quad) \underbrace{\frac{X(z)}{1 - a_1 z^{-1} - a_2 z^{-2}}}_{E(z)}$$

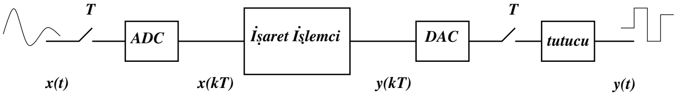
$$e(n) = a_1 e(n-1) + a_2 e(n-2) + x(n)$$

$$y(n) = b_0 e(n) + b_1 e(n-1) + b_2 e(n-2) = b_0 (a_1 v_2(n) + a_2 v_1(n) + x(n)) + b_1 v_2(n) + b_2 v_1(n)$$

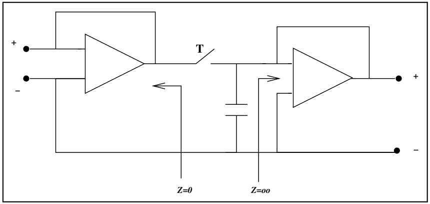
$$\left. \begin{matrix} v_2(n) = e(n-1) \\ v_1(n) = e(n-2) \end{matrix} \right\} \quad \begin{matrix} v_2(n+1) = e(n) = a_1 v_2(n) + a_2 v_1(n) + x(n) \\ v_1(n+1) = e(n-1) \end{matrix}$$

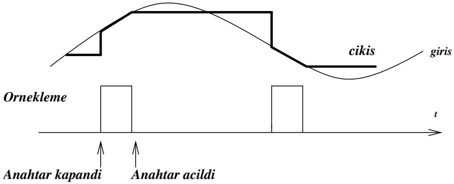
$$\begin{bmatrix} v_1(n+1) \\ v_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} \begin{bmatrix} v_1(n) \\ v_2(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(n)$$

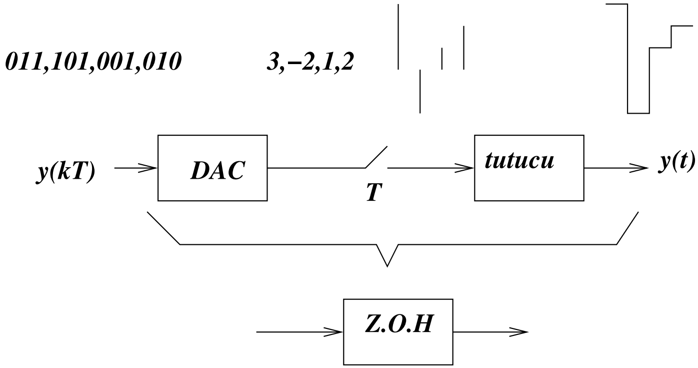




ADC: Örnekle ve Tut Devresi (Sample and Hold (S/H))

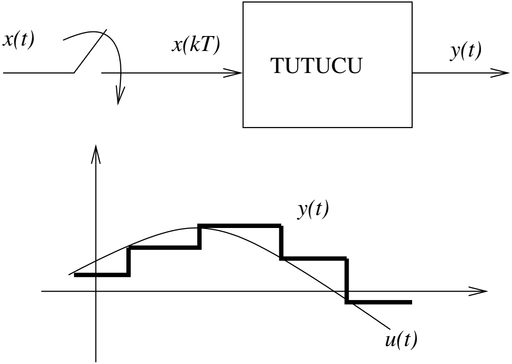






Yüksek dereceli tutucular işareti ZOH kıyasla daha yüksek doğrulukla şekillendirir.

Tutucuya ilişkin transfer fonksiyonu



$$x(kT) = \sum_{k=0}^{k=\infty} x(t)\delta(t-kT)$$

$$y(t) = \sum_{k=0}^{k=\infty} x(kT)(u(t-kT)-u(t-kT-T))$$

Tutucuya ilişkin transfer fonksiyonu

$$
\begin{array}{ccccc}
x(t) & \rightarrow & x(kT) & \rightarrow & y(t) \\
X(s) & \rightarrow G_S(s) \rightarrow & \hat{X}(s) & \rightarrow G_T(s) \rightarrow & Y(s)
\end{array}
$$

$$
\hat{X}(s) = \mathscr{L}\left\{\sum_{k=0}^{k=\infty} x(kT)\delta(t-kT)\right\} = \int_0^\infty \sum_{k=0}^{k=\infty} x(kT)\delta(t-kT)e^{-st}dt
$$

$$
\hat{X}(s) = \sum_{k=0}^{k=\infty} \int_0^\infty x(t)\delta(t-kT)e^{-st}dt = \sum_{k=0}^{k=\infty} x(kT)e^{-kTs}
$$

Tutucuya ilişkin transfer fonksiyonu

$$y(t) = \sum_{k=0}^{k=\infty} x(kT)(u(t-kT) - u(t-kT-T))$$

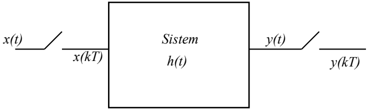
$$Y(s) = \sum_{k=0}^{k=\infty} x(kT)(\frac{1}{s}e^{-kTs} - \frac{1}{s}e^{-kTs-Ts})$$

$$Y(s) = \frac{1}{s}(1-e^{-Ts}) \sum_{k=0}^{k=\infty} x(kT)e^{-kTs}$$

$$Y(s) = \frac{1}{s}(1-e^{-Ts})\hat{X}(s)$$

$$G_{Tutucu}(s) = \frac{1}{s}(1-e^{-Ts})$$

Örnek Dizisi ile Sistem



$$\begin{array}{ccccccccc} x(t) & \rightarrow & x(kT) & \rightarrow h(t) \rightarrow & y(t) & \rightarrow y(nT) \\ X(s) & \rightarrow & \hat{X}(s) & \rightarrow H(s) \rightarrow & Y(s) & \rightarrow \hat{Y}(s) \end{array}$$

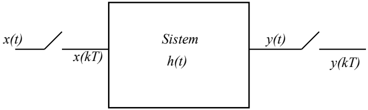
$$x(kT) = \sum_{k=0}^{k=\infty} x(t)\delta(t-kT), \qquad h(t) \star \delta(t-kT) = h(t-kT)$$

$$y(t) = \sum_{k=0}^{k=\infty} x(kT)h(t-kT)$$

$$y(nT) = \sum_{k=0}^{k=\infty} x(kT)h(nT-kT)$$

$$Y(z) = H(z)X(z)$$

Örnek Dizisi ile Sistem



$$y(nT) = \sum_{k=0}^{k=\infty} x(kT)h(nT - kT)$$

$$Y(z) = H(z)X(z)$$

$$H(z) = \mathscr{Z}\left\{\mathscr{L}^{-1}\left\{H(s)\right\}\right\}$$

Örnek:

$$H(z) = \mathscr{Z}\left\{\mathscr{L}^{-1}\left\{\frac{1}{s(s+a)}\right\}\right\}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s+a)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{1}{(s+a)}\right\} = (1 - e^{-at})u(t)$$

Örnek Dizisi ile Sistem

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s+a)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{1}{(s+a)}\right\} = (1-e^{-at})u(t)$$

$$H(z) = \sum_{k=0}^{\infty}(1-e^{-akT})z^{-k} = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}$$

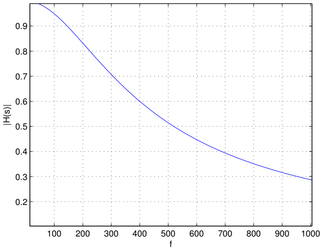
Örnek:

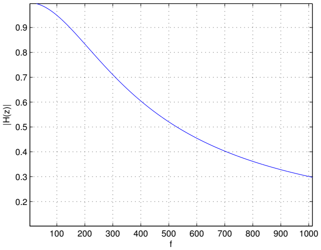
$$H(s) = \frac{2\pi 300}{s+2\pi 300}$$

$$H(z) = \mathscr{Z}\left\{\mathscr{L}^{-1}\left\{H(s)\right\}\right\} = \frac{1}{1-z^{-1}e^{-2\pi\frac{300}{6000}}}$$

```
[h,w]=freqs([0 2*pi*300],[1 2*pi*300]); plot(w/(2*pi),abs(h))
[hd,f]=freqz([1-exp(-(2*pi*300)*T) 0],[1
-exp(-(2*pi*300)*T)],10000,1/6000); plot(f,abs(hd))
```

Not: w=0'da genlik 1 olması için kazanç terimi eklenmiştir!

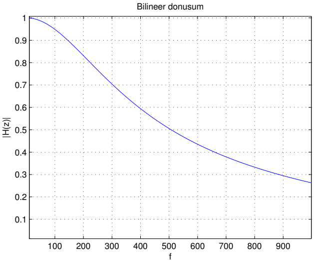

|H(s)|
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
100 200 300 400 500 600 700 800 900 1000
f


|H(z)|
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
100 200 300 400 500 600 700 800 900 1000
f

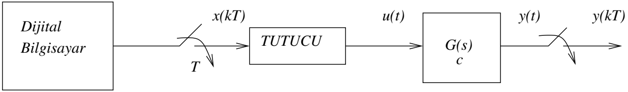
Bilineer dönüşüm kullanarak $H(z)$'yi elde etmeye çalışalım ($a = 2\pi 300$ $T = 1/60000$).

$$H(s) \rightarrow s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \rightarrow H(z) = \frac{aT(1+z^{-1})}{(2+aT)+(aT-2)z^{-1}}$$

```
[hd1,wd1]=freqz([a*T a*T],[(2+a*T) (T*a-2)],10000,1/T);
```



Tutucu kullanırak sayısallaştırılmış sisteme ilişkin tf.



$$\frac{Y(s)}{\hat{X}(s)} = G(s) = G_{Tutucu}(s)G_c(s) = \frac{(1-e^{-Ts})}{s}G_c(s)$$

$$\frac{Y(z)}{X(z)} = \mathscr{Z}\{\mathscr{L}^{-1}\{G(s)\}\}$$

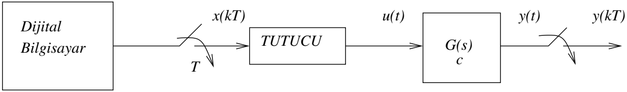
Örnek: $G_c(s) = \frac{a}{(s+a)}$

$$H(s) = \frac{(1-e^{-Ts})}{s}\frac{1}{(s+a)} = \frac{(1-e^{-Ts})}{s(s+a)}$$

$$h(t) = (1-e^{-Ts})(\frac{1}{s} - \frac{1}{(s+a)}) = (1-e^{-aT})u(t) - (1-e^{-a(t-T)})u(t-T)$$

Katsuhiko Ogata, Discrete-Time Control Systems, Pearson Education (1994) Chapter 3

Tutucu kullanırak sayısallaştırılmış sisteme ilişkin tf.



$$G(s) = (1 - e^{-Ts})\frac{G_c(s)}{s}$$

$$\mathscr{Z}\{\mathscr{L}^{-1}\{(1 - e^{-Ts})\frac{G_c(s)}{s}\}\}$$

$$\mathscr{Z}\{\mathscr{L}^{-1}\{\frac{G_c(s)}{s}\} - \mathscr{L}^{-1}\{e^{-Ts}\frac{G_c(s)}{s}\}\} = (1 - z^{-1})\mathscr{Z}\{\mathscr{L}^{-1}\{\frac{G_c(s)}{s}\}\}$$