Circuit and System Analysis
EHB 232E

Prof. Dr. Müştak E. Yalçın

Istanbul Technical University
Faculty of Electrical and Electronic Engineering

mustak.yalcin@itu.edu.tr

Outline I

1. Laplace Transform in Circuit Analysis-Cont.
   - Network Functions
   - Characteristic polynomial
   - Stability and the Routh-Hurwitz Criterion
   - Routh-Hurwitz Criterion
   - Circuit elements in the s-domain

Network Functions

$$\mathrm{N}etwork\ Function = \frac{\mathcal{L}\{\mathrm{z}ero - state\ response\}}{\mathcal{L}\{\mathrm{i}nput\}}$$

$$\frac{Y(s)}{E(s)} = (C\Phi(s)B + D) = H(s)$$

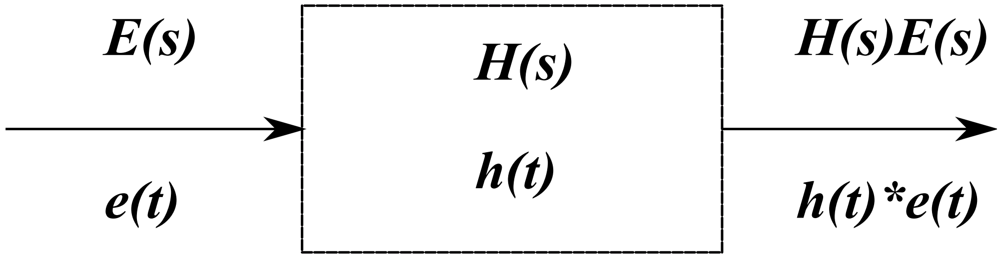
Transfer (Network) functions : voltage transfer functions, transfer admittances, current transfer function, transfer impedance.

$$y(t) = \mathcal{L}^{-1}\{H(s)E(s)\} = h(t) \star e(t) = \int_0^t h(t-\tau)e(\tau)d\tau$$

if we chose $e(t) = \delta(t)$

$$y(t) = \mathcal{L}^{-1}\{H(s)\ 1\} = \int_0^t h(t-\tau)\delta(\tau)d\tau = h(t)$$

NETWORK FUNCTION $= \mathcal{L}\{$IMPULS RESPONSE$\}$



$$\text{impuls response} = h(t) = \mathcal{L}^{-1}\{H(s)\}$$
$$\text{step response} = h(t)u(t) = \mathcal{L}^{-1}\{H(s)\tfrac{1}{s}\}$$
$$!\ \text{zero-input response} = 0\ !$$

what happen if zero-input response is not zero (or system is not stable)?

Characteristic polynomial

The characteristic polynomial of A is defined by

$$p(s) = \det\{sI - A\} = 0$$

The solutions of the characteristic equation are precisely the eigenvalues of the matrix A. The roots of the characteristic equation is $s_{pi} = \sigma_i + jw_i$

- if $\sigma_i > 0$ unstable
- if $\sigma_i < 0$ stable
- $\sigma_i = 0$ check the repeated eigenvalue !

Example : The transfer function for a linear time-invariant circuit is $H(s) = \frac{1}{s+1}$. If $E(t) = 3\cos t$ what is the steady-state expression of the output?

$$Y(s) = \frac{3s}{s^2+1}\frac{1}{s+1}$$

and in time domain

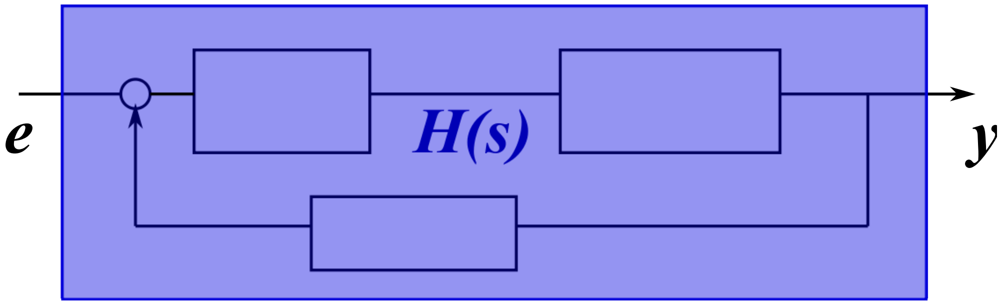
$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{3}{2}\frac{s+1}{s^2+1} - \frac{3}{2}\frac{1}{s+1}\right\} \\ &= \frac{3}{2}(\cos t + \sin t - e^{-t}) \end{aligned}$$

Using convolution

$$h(t) = \mathcal{L}\{\frac{1}{s+1}\} = e^{-t}$$

$$\begin{aligned} y(t) &= \int_0^t e^{-t+\tau} 3\cos(\tau) d\tau \\ &= 3e^{-t}\left[\frac{e^{\tau}}{2}(\cos(\tau) + \sin(\tau))\Big|_0^t\right] \\ &= \frac{3}{2}(\cos t + \sin t - e^{-t}) \end{aligned}$$

We would like to determine the stability of a transfer function such as



The stability of

$$H(s) = \frac{P(s)}{Q(s)}$$

is determined by the roots of $Q(s) = 0$.

to find the roots of the nth-order polynomials !

There is an effective test for determining stability that does not require an explicit solution of the algebraic equation.

Routh-Hurwitz Criterion

Let

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots a_1 s + a_0$$

be a polynomial with real coefficients, with $a_n \neq 0$. Starting with the leading coefficient a_n, fill two rows of a table, as follows:

$$\begin{array}{rllll}
s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
s^{n-2} & b_1 & b_3 & b_5 & \ldots \\
s^{n-3} & c_1 & c_3 & c_5 & \ldots \\
. & . & . & . & \ldots \\
. & . & . & . & \ldots \\
s^1 & . & . & . & \ldots \\
s^0 & . & . & . & \ldots
\end{array}$$

Thus, the first row contains all coefficients with even subscript and the second those with odd subscripts.

Now a Routh-Hurwitz Table (RHT) with $(n+1)$ rows is formed by an algorithmic process, described below:

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_3 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 = \frac{b_1 a_{n-3} - b_3 a_{n-1}}{b_1}$$

$$c_3 = \frac{b_1 a_{n-5} - b_5 a_{n-1}}{b_1}$$

The polynomial $Q(s)$ is (Hurwitz) stable if and only if all the elements of the first column of its Routh-Hurwitz Table are non-zero and of the same sign.

Example

$M(s) = s^5 + 4s^4 + 2s^3 + 5s^2 + 3s + 6$

$$
\begin{array}{llll}
s^5 & 1 & 2 & 3 \\
s^4 & 4 & 5 & 6 \\
s^3 & 0.75 & 1.5 & \\
s^2 & -3 & 6 & \\
s^1 & 3 & & \\
s^0 & 6 & &
\end{array}
$$

we have -3 in first column. This polynomial is not Hurwitz! (The number of unstable roots of p(s) is equal to the number of changes of sign)

What is the stability condition of the Quadratic Polynomial ? (is stable if and only if all its coefficients are of the same sign)

What is the stability condition of the cubic polynomial ?

Example

Singular Cases : $m(s) = s^3 - 3s + 2$

$$\begin{array}{lll} s^3 & 1 & -3 \\ s^2 & 0(\epsilon) & 2 \\ s^1 & (-3-2)/\epsilon & \\ s^0 & 2 & \end{array}$$

We have a zero in the first column. We have two sign changes, confirming that the polynomial has two unstable roots. $(m(s) = (s-1)^2(s+2))$.

Replace the zero by a small ϵ (of arbitrary sign) and continue filling in the entries of the RHT.

Example

Pure imaginary pair of roots: $m(s) = s^3 + 2s^2 + s + 2$

$$\begin{array}{lll} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0(\epsilon) & \\ s^0 & 2 & \end{array}$$

we have a zero in the first column. Replace the zero by a small ϵ (of arbitrary sign) and continue filling in the entries of the RHT. $(m(s) = (s^2 + 1)(s + 2))$.

if $Q(s)$ has a simple pair of imaginary roots at and no other imaginary roots, then this can be detected in the RHT, in that it will have the whole of the s_1 row equal to zero and only two non-zero elements in the s_2 row, of the same sign.

Example

$m(s) = s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50$

$$\begin{array}{llll} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & 0(8) & 0(96) & \\ s^2 & 24 & -50 & \\ s^1 & 112.7 & 0 & \\ s^0 & -50 & & \end{array}$$

$d(2s^4 + 48s^2 - 50)/ds = 8s^3 + 96$

the polynomial has unstable root.($s = 1$)

All the elements of a particular row are zero. The zero row is replaced by taking the coefficients of the derivative of the the auxiliary polynomial which is obtained from the values in the row above the zero row.

$m(s) = s^5 + 2s^3 + s$

s^5	1	2	1	
s^4	5	6	1	$d(s^5 + 2s^3 + s)/ds = 5s^4 + 6s^2 + 1$
s^3	$4/5$	$4/5$		
s^2	1	1		
s^1	0(2)			$d(s^2 + 1)/ds = 2s$
s^0	1			

Two times all the elements of a particular row are zero. Unstable! ($m(s) = s(s^2 + 1)^2$).

Circuit elements in the s-domain

s-domain equivalent circuit for each circuit element

A resistor in the s-domain

In time-domain

$$v = Ri$$

s-domain equivalent circuit for the resistor

$$V(s) = RI(s)$$

where $V(s) = \mathcal{L}\{v\}$ and $I(s) = \mathcal{L}\{i\}$.

A Capacitor in the s-domain:

Terminal current in time-domain

$$i = C\frac{dv}{dt}$$

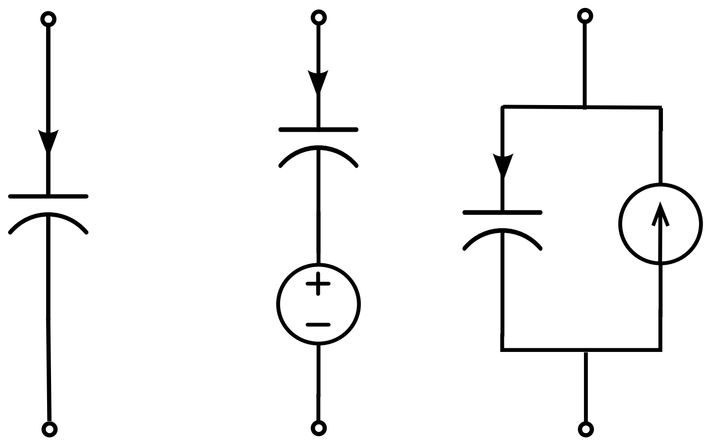
Laplace transform the above equ.

$$I(s) = CsV(s) - Cv(0)$$

or

$$V(s) = \frac{1}{Cs}I + \frac{1}{s}v(0)$$

Norton and Thevenin equivalents of a capacitor in the s-domain



An inductor in the s-domain:

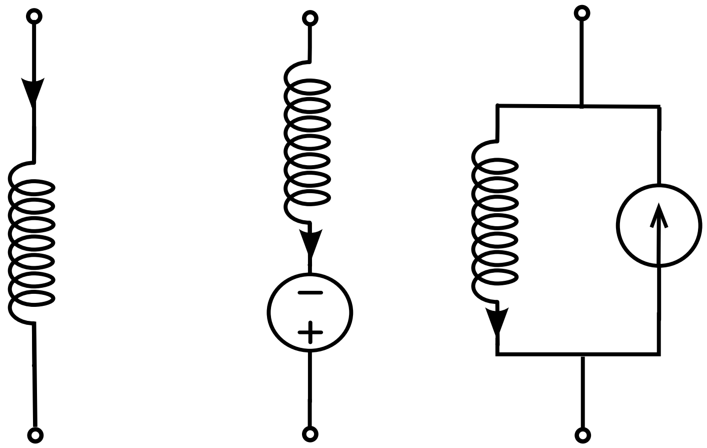
In time-domain

$$L\frac{di}{dt} = v$$

After Laplace transform

$$LsI(s) - Li(0) = V \quad \text{or} \quad I(s) = \frac{1}{Ls}V + \frac{1}{s}i(0)$$

Equivalent circuits for the inductor



Energy is stored in the inductor and capacitor