# Circuit and System Analysis
EHB 232E

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# Outline I

1. Sinusoidal Steady-State Analysis
   - Mesh-Current Method in Frequency Domain
   - Node-voltage Method in Frequency Domain
   - Network functions
   - Superposition of Sinusoidal Steady States
   - Thevenin - Norton Equivalent Circuits

# Mesh-Current Method in Frequency Domain

The number of equations to be solved are equal to the number of independent loops $(n_e - n_d + 1)$. There exists a tree such that the meshes are Fundamental loops*.

$$B_1 \mathbf{R} B_1^T i_c + B_2 v_k = 0$$

where $v_k$ and $v_R$ voltages of independent voltage sources and resistors. Instead of $V_e = \mathbf{R} I_e$ using

$$V_e = \mathbf{Z} I_e$$

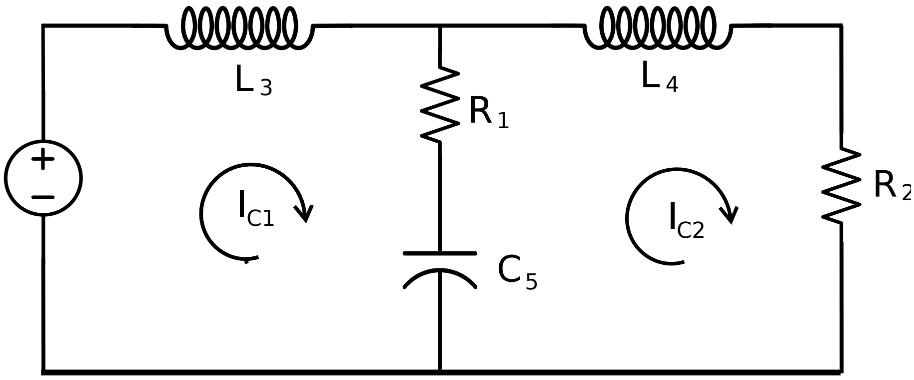
we have

$$B_1 \mathbf{Z} B_1^T I_c + B_2 V_k = 0$$

where $B_1 \mathbf{Z} B_1^T$ mesh impedance matrix.

See :EHB211 E

# Example




$$M1 \quad L_3 jwI_{c1} + (\frac{1}{C_5 jw} + R)(I_{c1} - I_{c2}) - V_G = 0$$
$$M2 \quad L_4 jwI_{c2} + R_2 I_{c2} + (\frac{1}{C_5 jw} + R)(I_{c2} - I_{c1}) = 0$$

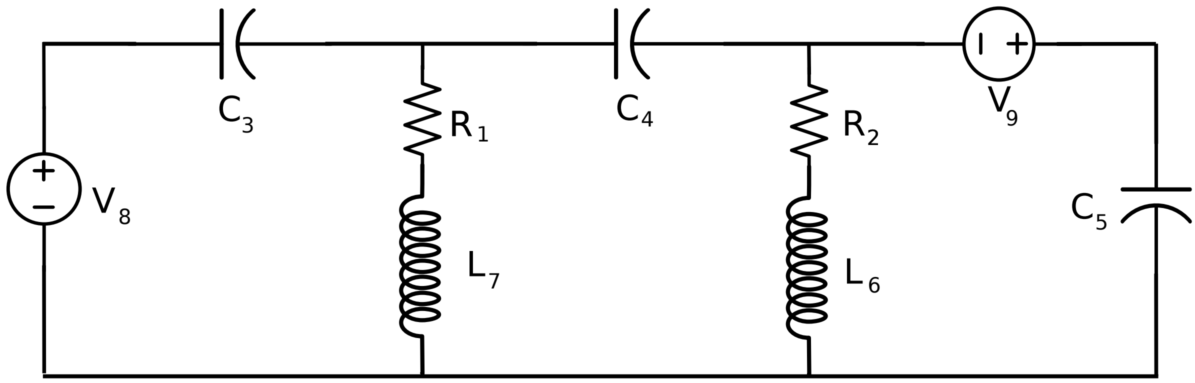
In matrix form

$$\begin{bmatrix} R_1 + \frac{1}{C_5 jw} + L_3 jw & -\frac{1}{C_5 jw} - R_1 \\ -\frac{1}{C_5 jw} - R_1 & L_4 jw + R_2 + R_1 + \frac{1}{C_5 jw} \end{bmatrix} \begin{bmatrix} I_{c1} \\ I_{c2} \end{bmatrix} = \begin{bmatrix} V_G \\ 0 \end{bmatrix}$$

$$V_{L_3} = L_3 jw[1\ \ 0]\begin{bmatrix} \cdot & -\frac{1}{Cjw} - R_1 \\ -\frac{1}{Cjw} - R_1 & \cdot \end{bmatrix}^{-1}\begin{bmatrix} V_G \\ 0 \end{bmatrix}$$

$v_G(t) = 4\cos(2\pi 60 t + \frac{\pi}{3}),\ L_3 = L_4 = 3mH,\ C = 4\mu F,\ R_1 = R_2 = 2k\Omega$

$$V_{L_3} = 310^{-3} j2\pi 60[1\ \ 0]\begin{bmatrix} \cdot & -\frac{10^6}{4j2\pi 60} - 2k \\ -\frac{10^6}{4j2\pi 60} - 2k & \cdot \end{bmatrix}^{-1}\begin{bmatrix} 4e^{\frac{\pi}{3}j} \\ 0 \end{bmatrix}$$



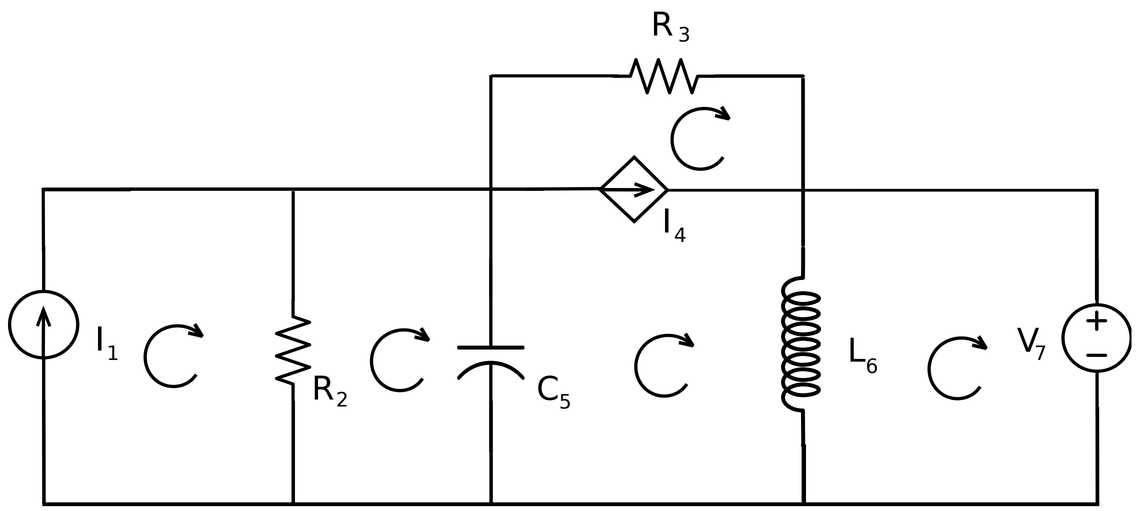

$$Z \begin{bmatrix} I_{c1} \\ I_{c2} \\ I_{c3} \end{bmatrix} = \begin{bmatrix} V_8 \\ 0 \\ V_9 \end{bmatrix}$$

where

$$Z =$$

$$\begin{bmatrix} \frac{1}{C_3 jw} + R_1 + L_7 jw & -R_1 - L_7 jw & 0 \\ -R_1 - L_7 jw & R_1 + (L_7 + L_6) jw + \frac{1}{C_4 jw} + R_2 & -R_2 - L_6 jw \\ 0 & -R_2 - L_6 jw & R_2 + L_6 jw + \frac{1}{C_5 jw} \end{bmatrix}$$

$I_4 = 2V_2$




$$\begin{bmatrix} R_2 & -R_2 & 0 & 0 & 0 \\ -R_2 & R_2 + \frac{1}{C_5 jw} & -\frac{1}{C_5 jw} & 0 & 0 \\ 0 & -\frac{1}{C_5 jw} & \frac{1}{C_5 jw} + L_6 jw & 0 & -L_6 jw \\ 0 & 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & -L_6 jw & L_6 jw \end{bmatrix} \begin{bmatrix} I_{c1} \\ I_{c2} \\ I_{c3} \\ I_{c4} \\ I_{c5} \end{bmatrix} = \begin{bmatrix} -V_1 \\ 0 \\ -V_4 \\ V_4 \\ -V_7 \end{bmatrix}$$

$$
\begin{aligned}
I_1 &= I_{c1} \\
I_4 &= 2V_2 \\
I_{c3} - I_{c4} &= -2V_1
\end{aligned}
$$

with above equ.s

$$
\begin{bmatrix}
R_2 & -R_2 & 0 & 0 & 0 & 1 & 0 \\
-R_2 & R_2 + \frac{1}{C_5 jw} & -\frac{1}{C_5 jw} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{C_5 jw} & \frac{1}{C_5 jw} + L_6 jw & 0 & -L_6 jw & 0 & 1 \\
0 & 0 & 0 & R_3 & 0 & 0 & -1 \\
0 & 0 & -L_6 jw & 0 & L_6 jw & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
I_{c1} \\ I_{c2} \\ I_{c3} \\ I_{c4} \\ I_{c5} \\ V_1 \\ V_4
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ -V \\ I_1 \\ 0
\end{bmatrix}
$$

# Node-voltage Method

The fundamental cut-set equations for the nodes(which do not correspond to node of a voltage sources)

$$Ai = 0$$

current sources $i_k$ and currents of one ports $i_e$

$$A_1 i_e + A_2 i_k = 0$$

in Sinusoidal steady-state $I_e = YV_e$

$$A_1 YV_e + A_2 I_k = 0$$

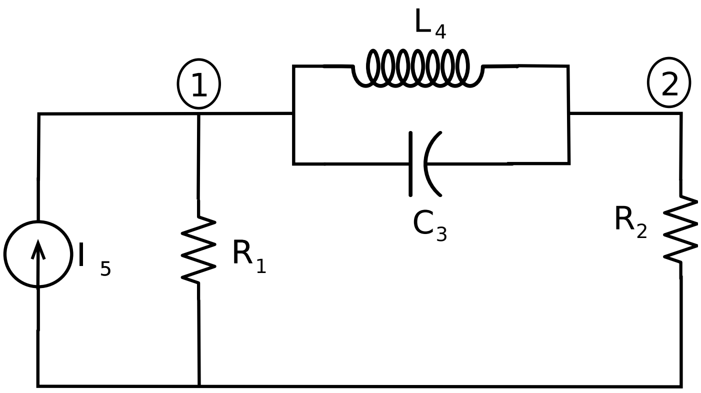
using $V_e = A_1^T V_d$ we have

$$A_1 YA_1^T V_d + A_2 i_k = 0$$

where $V_d$ is phasor of the node voltage.

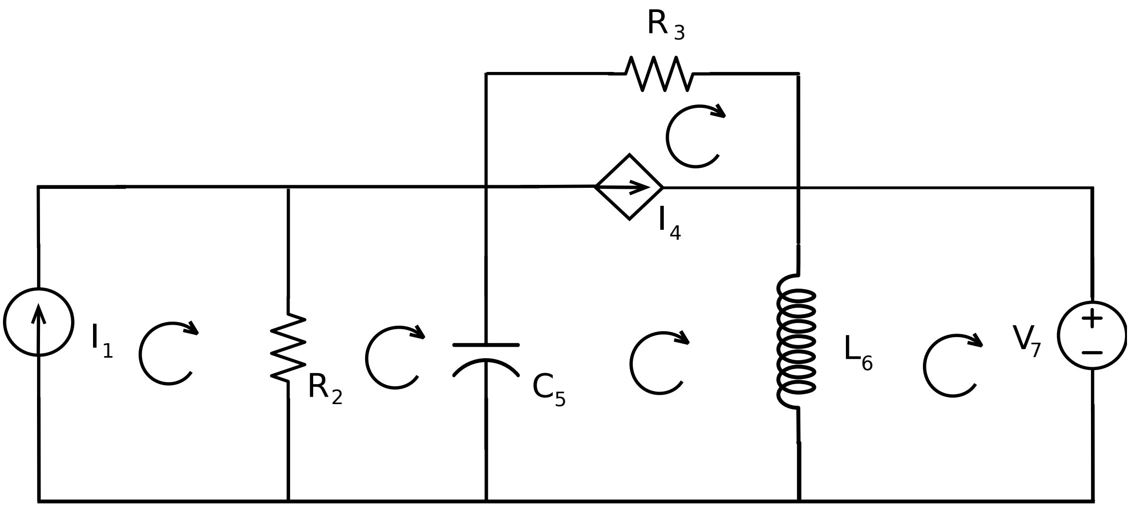
See :EHB211 E

# Example




$$\begin{bmatrix} G_1 + C_3jw + \frac{1}{L_3jw} & -C_3jw - \frac{1}{L_3jw} \\ -C_3jw - \frac{1}{L_3jw} & G_2 + C_3jw + \frac{1}{L_3jw} \end{bmatrix} \begin{bmatrix} V_{d1} \\ V_{d2} \end{bmatrix} = \begin{bmatrix} I_1 \\ 0 \end{bmatrix}$$

# Example




$$\begin{bmatrix} G_2 + G_3 + C_5jw & -G_3 \\ -G_3 & G_3 + \frac{1}{L_6jw} \end{bmatrix} \begin{bmatrix} V_{d1} \\ V_{d2} \end{bmatrix} = \begin{bmatrix} I_1 - I_4 \\ I_4 - I_7 \end{bmatrix}$$

# Example

$$V_7 = V_{d1}$$
$$I_4 = 2V_2 = 2V_{d1}$$

$$\begin{bmatrix} G_2 + G_3 + C_5 jw & -G_3 & 0 \\ -G_3 & G_3 + \frac{1}{L_6 jw} & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{d1} \\ V_{d2} \\ I_7 \end{bmatrix} = \begin{bmatrix} I_1 \\ 0 \\ V_7 \end{bmatrix}$$

$v_7(t) = 2\cos(2\pi 60t + \frac{2\pi}{3}),\ i_1(t) = 3\cos(2\pi 60t + \frac{7\pi}{3}),\ L_6 = 2H,\ C = 2F,$
$R_2 = R_3 = 1\Omega$

$$\begin{bmatrix} 1 + 1 + 2j2\pi 60 & -1 & 0 \\ -1 & 1 + \frac{1}{2j2\pi 60} & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{d1} \\ V_{d2} \\ I_7 \end{bmatrix} = \begin{bmatrix} 3e^{\frac{7\pi}{3}j} \\ 0 \\ 2e^{\frac{2\pi}{3}j} \end{bmatrix}$$

# Network functions

Consider a general linear time-invariant circuit $N$. Assume that $N$ is driven by one independent source, say, the sinusoidal current source $I_s$ represented by the phasor.

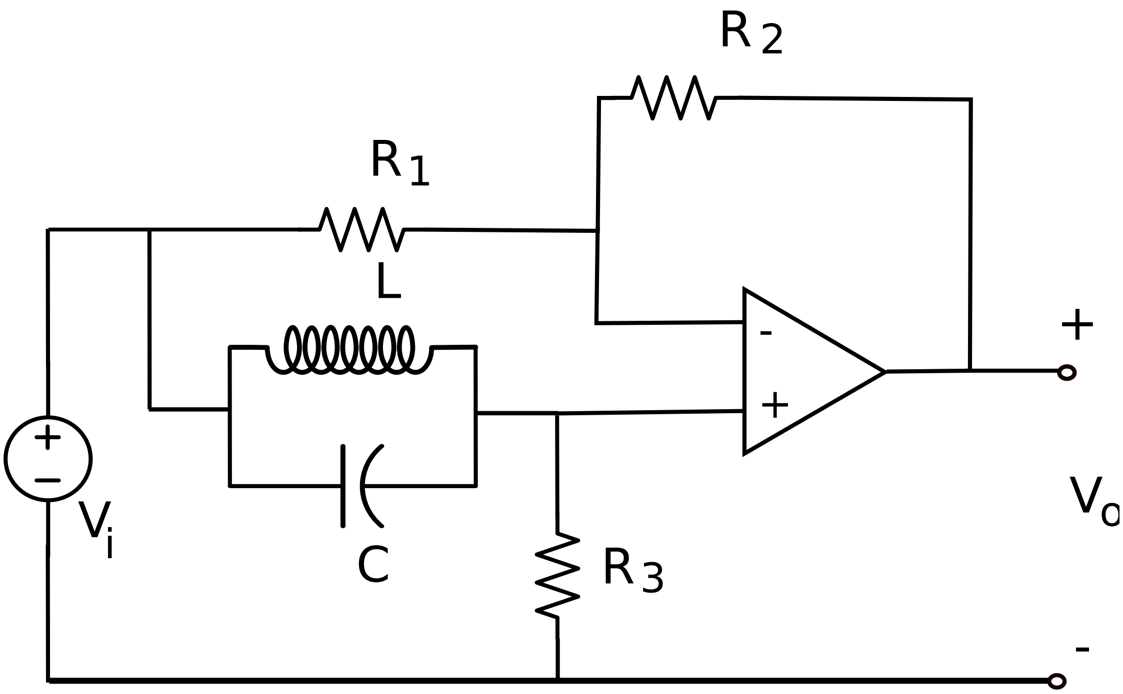
Suppose we want to calculate the node voltage $E$, and consider the dependence of the phasor $E$ on $w$.

$$\frac{E(jw)}{I_s}$$

is a function of $jw$ which depends only on the circuit $N$ and not on $I_s$. it is called the transfer impedance from $I$, to $E$.

Network functions : (a) Voltage transfer functions, (b) Transfer admittances, (c) Current transfer function, (d) Ttransfer impedance.

# Find voltage transfer functions from $V_i$ to $V_0$.

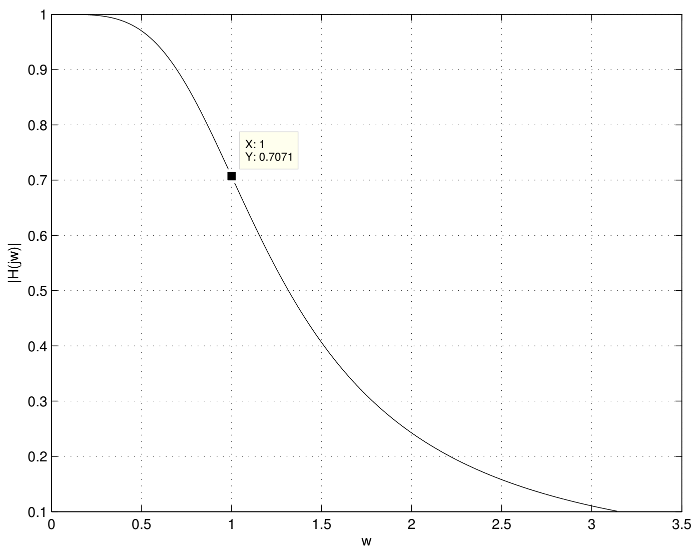


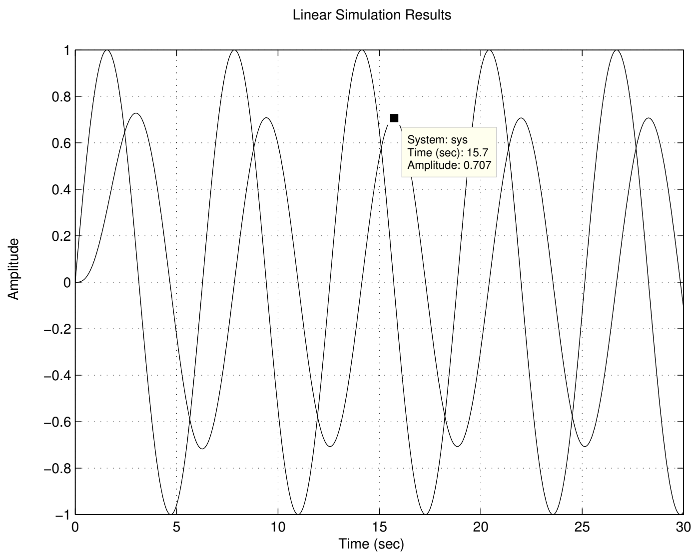

Example : Chua's book, Page 526, Examples 2 and 3.

# Network functions and Sinusoidal Waveforms

A linear time-invariant circuit $N$ in the sinusoidal steady state of frequency $w$. $H(jw) = |H(jw)|e^{j\angle H(jw)}$ is the voltage transfer function from $V_s = |V_s|e^{j\angle V_s}$ to $V_k = |V_k|e^{j\angle V_k}$.

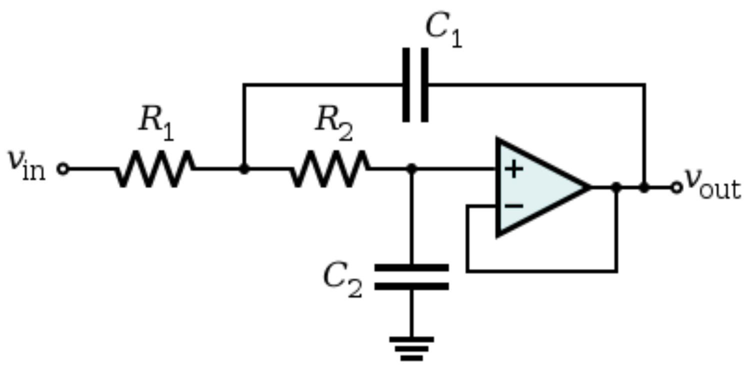
$$V_k = |V_k|e^{j\angle V_k} = |H(jw)|e^{j\angle H(jw)}|V_s|e^{j\angle V_s}$$
$$= |H(jw)||V_s|e^{j(\angle H(jw)+\angle V_s)}$$
$$v_k(t) = |H(jw)||V_s|\cos(wt + \angle H(jw) + \angle V_s)$$



# Example: Low Pass Filter

YouTube Video: RC Filter



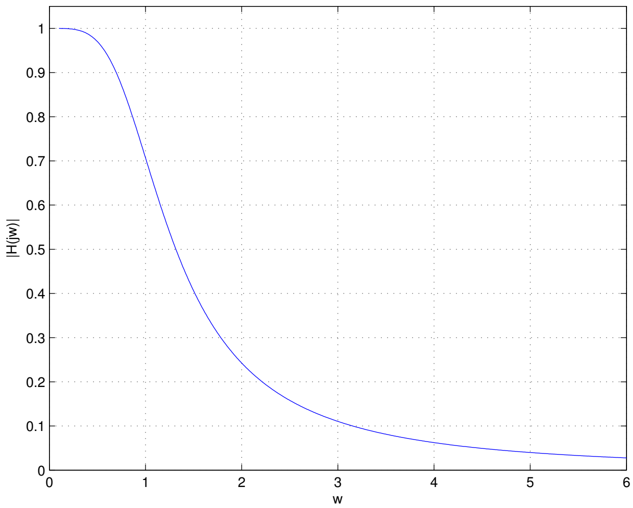

Verify

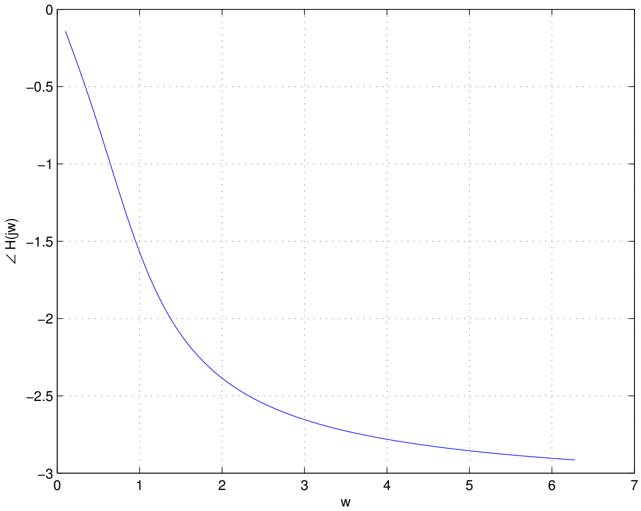
$$H(j\omega) = \frac{V_o}{V_i} = \frac{\omega_0^2}{\omega_0^2 - \omega + 2\alpha j\omega}$$

where $\omega_o^2 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$ and $2\alpha = \frac{(R_1+R_2)}{(R_1 R_2)} \frac{1}{C_1}$ (Note that $Q = \frac{\omega_0}{2\alpha}$)

# Example: Low Pass Filter

$\omega_o = 1$ and $2\alpha = \sqrt{2}$



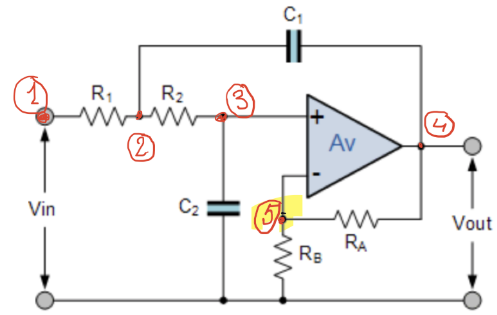




$H(j1) = \frac{1}{\sqrt{2}j} = \frac{1}{\sqrt{2}}e^{-\frac{\pi}{2}}$

$|H(j\omega)|\,|_{\text{in decibels}} = 20log(|H(j\omega)|)$

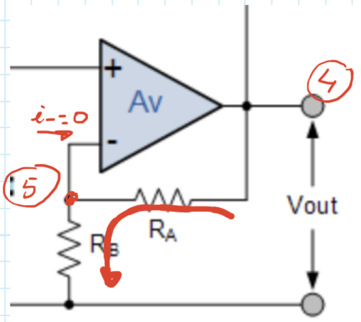
$|H(j0)| = 0dB, |H(j1)| = \frac{1}{\sqrt{2}} = -3dB$

## Second Order Low Pass Filter




Transfer function: $\frac{V_o}{V_i} = ?$

NODE 5

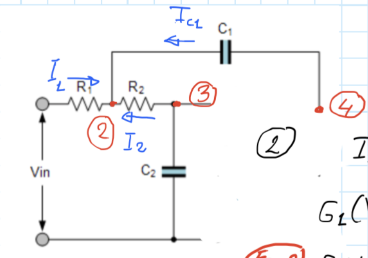



for OPAMP $V_{n3} = V_{n5}$

$V_{n5} = \frac{R_B}{R_A + R_B} \cdot V_{n4}$ Note $V_{n4} = V_o$

$= k V_{n4}$

NODE 2

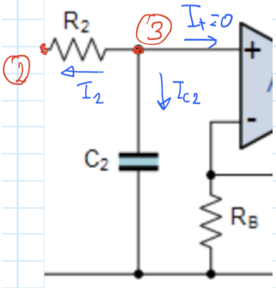



(2) $I_1 + I_2 + I_{c1} = 0$

$G_1(V_{n1} - V_{n2}) + G_2(V_{n3} - V_{n2}) + C_1 jw(V_{n4} - V_{n2}) = 0$

(Equ 1) $G_1 V_{n1} - (G_1 + G_2 + C_1 jw)V_{n2} + G_2 V_{n3} + C_1 jw V_{n4} = 0$ Note $V_{n1} = V_i$

$V_{n3} = V_{n5} = k V_{n4}$




$V_{n2} = ?$

$I_2 + I_{C2} = 0$ KCL

$G_2(V_{n3} - V_{n2}) + V_{n3} C_2 j\omega = 0$

$(G_2 + C_2 j\omega) V_{n3} = G_2 V_{n2} \Rightarrow V_{n2} = \frac{(G_2 + C_2 j\omega)}{G_2} \cdot V_{n3} = \frac{(G_2 + C_2 j\omega)}{G_2} \cdot k V_{n4}$

Equ. 2

$G_1 V_{n1} - (G_1 + G_2 + C_1 j\omega) \cdot \frac{(G_2 + C_2 j\omega)}{G_2} \cdot k V_{n4} + G_2 \cdot k V_{n4} + C_1 j\omega V_{n4} = 0$

$V_{n1} \to V_i$, $V_{n4} \to V_o$

$G_1 V_{n1} + \left( C_1 j\omega + G_2 k - (G_1 + G_2 + C_1 j\omega) \frac{(G_2 + C_2 j\omega)}{G_2} \cdot k \right) V_{n4} = 0$

$$\frac{V_o}{V_i} = \frac{G_1}{(G_1 + G_2 + C_1 j\omega)\frac{(G_2 + C_2 j\omega)}{G_2} \cdot k - C_1 j\omega - G_2 k}$$

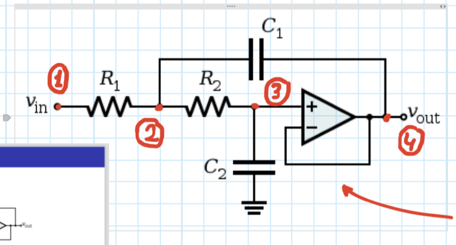
$$= \frac{G_1 \cdot G_2}{-k C_1 C_2 \omega^2 + k((G_1+G_2)C_2 + G_2 C_1) j\omega + (G_1+G_2) k G_2 - G_2 C_1 j\omega - G_2^2 k}$$

$$= \frac{G_1 \cdot G_2}{-k C_1 C_2 \omega^2 + k(G_1 + G_2) C_2 j\omega + G_1 G_2 k}$$

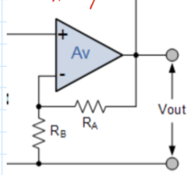
$G = \frac{1}{R}$

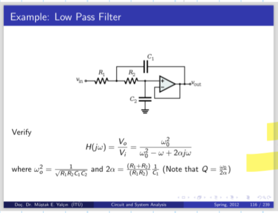
$$= \left(\frac{1}{k}\right) \frac{1}{-C_1 C_2 R_1 R_2 \omega^2 + (R_1 + R_2) C_2 j\omega + 1}$$

$k = \frac{R_B}{R_B + R_A} \Rightarrow \frac{1}{k} = 1 + \frac{R_A}{R_B} \geq 1$




This part is removed !!




$$= \frac{1}{-C_1C_2R_1R_2\omega^2 + (R_1+R_2)C_2 j\omega + 1}$$

without k

$$H(j\omega) = \frac{V_o}{V_i} = \frac{\frac{1}{R_1R_2C_1C_2}}{-\omega^2 + \frac{(R_1+R_2)C_2}{R_1R_2C_1C_2} j\omega + \frac{1}{R_1R_2C_1C_2}}$$

$\omega = 0 \qquad H(j0) = 1$

$$\omega^2 = \frac{1}{R_1R_2C_1C_2} \qquad H(j\omega) = \frac{\frac{1}{R_1R_2C_1C_2}}{-\frac{1}{R_1R_2C_1C_2} + \frac{(R_1+R_2)C_2}{R_1R_2C_1C_2}\cdot\frac{1}{\sqrt{R_1R_2C_1C_2}}\, j + \frac{1}{R_1R_2C_1C_2}}$$

$$= -j\,\frac{\sqrt{R_1R_2C_1C_2}}{(R_1+R_2)C_2}$$

Let's $R_1 = R_2 = R$ and $C_1 = 2C_2$

$$= -j\,\frac{R}{2R}\sqrt{\frac{C_1C_2}{C_2^2}}$$

$$= -j\,\frac{1}{2}\sqrt{2}$$

$$= -j\,\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}j}$$

$$\omega = \hat{\omega} = \frac{1}{\sqrt{R_1R_2C_1C_2}} \qquad H(j\hat{\omega}) = \frac{1}{\sqrt{2}}e^{-j\pi/4}$$

$\omega = \infty \qquad H(j\omega) = 0$

SPICE PROG.

OPAMP →

OPAMP MODEL

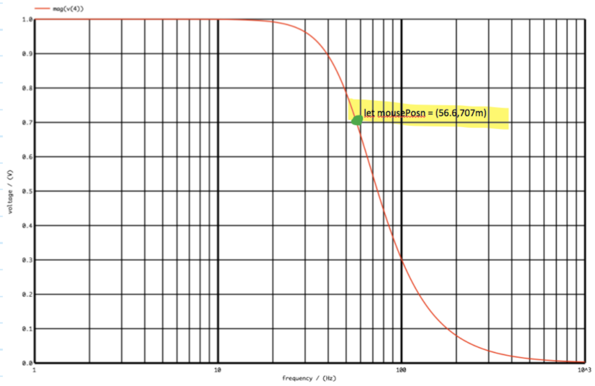
```
R1 1 2 1k
R2 2 3 1k
C2 3 0 2U
C1 2 4 4U
XOP 3 4 4 OPAMP1
.SUBCKT OPAMP1     1   2   6
* INPUT IMPEDANCE
RIN       1          2          10MEG
* DC GAIN (100K) AND POLE 1 (10HZ)
EGAIN   3 0        1 2        100K
RP1       3          4          1K
CP1       4          0          15.915UF
* OUTPUT BUFFER AND RESISTANCE*
EBUFFER          5 0        4 0        1
ROUT    5          6          10
.ENDS
Vin 1 0 DC 0.0 AC 1.0 0.0
.ac dec 100 1Hz 1000Hz
.END
```

⟹ plot mag(v(4))

$$w = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$$

$$w = 353{,}55$$

$$f = 56{,}2\,Hz$$



# Superposition of Sinusoidal Steady States

Let $N$ be a linear time-invariant circuit which is driven by two sinusoidal independent sources operating at two different frequencies.

Voltage source is specified by phasor $E$, and operates at frequency $w_1$. The current source is specified by phasor $I$, and operates at frequency $w_2$.

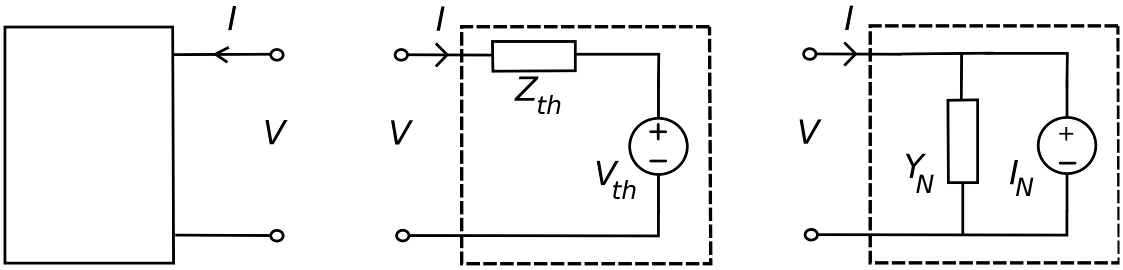
$H_1(jw)$ be the voltage transfer function of from $E$ to $V_k$ and $H_2(jw)$ be the transfer impedance from $I$ to $V_k$.

By the superposition theorem, the resulting steady state is the superposition of two sinusoids

$$v_k(t) = |H_1(jw)||E|\cos(w_1 t + \angle H_1(jw) + \angle E) + |H_2(jw)||I| \cos w_2 t + \dots$$

$w_1 = rw_2$ if $r$ is a rational number then periodic if $r$ is a irrational number then almost periodic

# Thevenin - Norton Equivalent Circuits

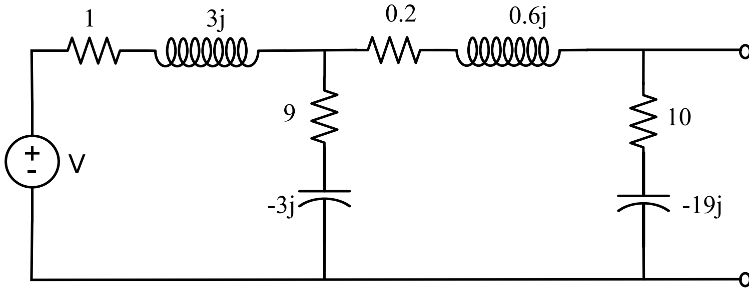


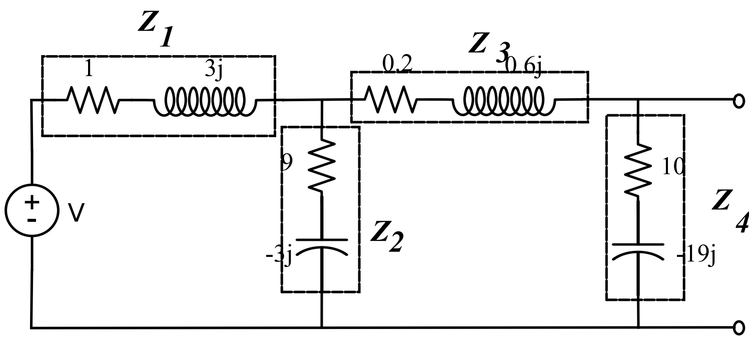

The techniques for finding the Thevenin equivalent voltage ($V_{\mathrm{th}}$) and impedance ($Z_{\mathrm{th}}(jw)$) are identical to those used for resistive circuits, except that the frequency domain equivalent circuit involves the manipulation of complex quantities.

Driving-point characteristic of Thevenin equivalent circuit is defined by

$$V = Z_{\mathrm{th}}(jw)I + V_{\mathrm{th}}$$

More detail EHB211 E: Slayt 192


1
3j
0.2
0.6j
9
10
V
-3j
-19j

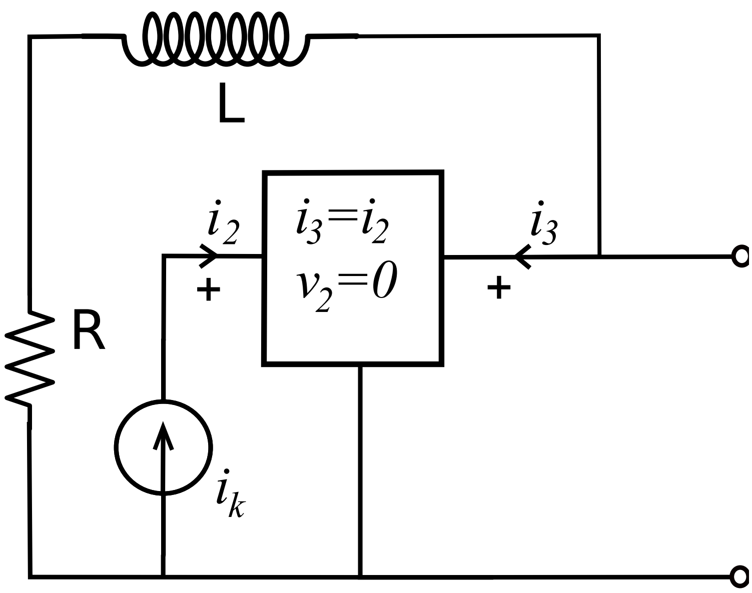



$$Z_{\mathrm{th}} = (((Z_1 // Z_2) + Z_3) // Z_4)$$

$$V_1 = \frac{(Z_3 + Z_4)//Z_2}{(Z_3 + Z_4)//Z_2 + Z_1} V$$

$$V_{\mathrm{th}} = \frac{Z_4}{Z_3 + Z_4}\ V_1$$

# Example



# Example

$Y = j$, $i_1 = 2v_2$ and $i_2 = -2v_1$ In steady state $I_{c1} = 1 - j$. Find complex power of two-port.

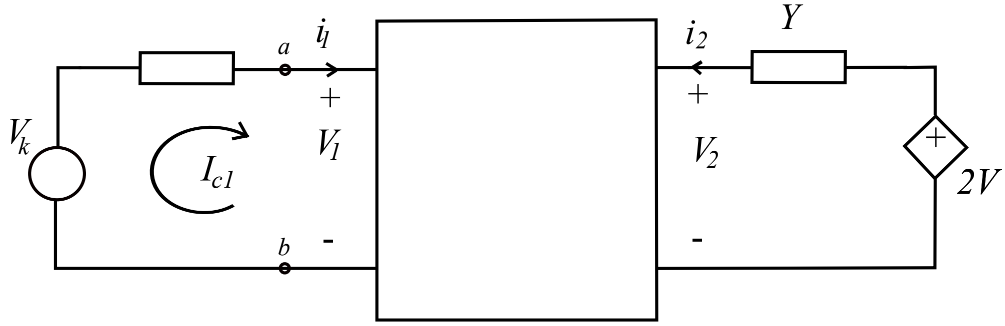



Figure: $(i_1 = 2v_2,\ i_2 = -2v_1$ ve $Y = j$ and $I_{c1} = 1 - j\ )$.