Circuit and System Analysis
EHB 232E

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Outline I

1. Sinusoidal Steady-State Analysis
    - Phasor analysis
    - Properties of phasors
    - Representation of state-space equations
    - Transfer function
    - Kirchhoff's Laws in the Frequence Domain

Sinusoidal Steady-State Analysis

If a system is exponentially stable

$$\lim_{t \to \infty} \Phi(t) = 0$$

Forced response is the complete response.

$$x(t) = \Phi(t)x_0 - \Phi(t)x_p(t_0) + x_p(t)$$

Sinusoidal steady-state behavior

Sinusoidal steady-state behavior of linear time-invariant circuits when the circuits are driven by one or more sinusoidal sources at some frequency w and when, after all "transients" have died down, all currents and voltages are sinusoidal at frequency w.

Electric Circuits, James W. Nilsson and Susan A. Riedel, Ch.9 and 10

Phasor

The idea is to associate with each sine wave (of voltage or current) a complex number called the phasor.
$x(t)$ is a complex variable and in polar coordinate

$$x(t) = X_m e^{j(wt+\theta).}$$

where $j = \sqrt{-1}$, X_m ($|X|$) is called the magnitude of $x(t)$ and θ ($\angle$) is called the phase of $x(t)$.
Rectangular representation of complex $x(t)$ is

$$x(t) = X_m \cos(wt + \theta) + jX_m \sin(wt + \theta)$$

Real part

$$X_m \cos(wt + \theta) = \mathrm{Re}\{x(t)\}$$

Imaginary part

$$X_m \sin(wt + \theta) = \mathrm{Im}\{x(t)\}$$

from Euler's identity.

The quantity (**phasor**)

$$X = X_m e^{j\theta}$$

is a complex number that carries the amplitude and phase angle of the given sinusoidal function.This complex number is by definition the **phasor** representation of the given sinusoidal function.

Using phasor representation, a complex variable is given

$$x(t) = Xe^{wtj}$$

Examples: Electric Circuits, James W. Nilsson and Susan A. Riedel, pp. 334

Example

Phasor of the sinusoidal function

$$x(t) = 110\sqrt{2}\cos(wt + \frac{\pi}{2})$$

using

$$x(t) = \mathrm{Re}\{\underbrace{110\sqrt{2}e^{j\frac{\pi}{2}}}_{\text{phasor}} e^{jwt}\}$$

is obtain

$$X = 110\sqrt{2}e^{j\frac{\pi}{2}}$$

$X_m = 100\sqrt{2}$ (or $|X| = 100\sqrt{2}$), $\theta = \frac{\pi}{2}$ (or $\angle\frac{\pi}{2}$). $|X|_{\mathrm{rms}} = 100$
A complex number (phasor) in rectangular coordinate
$Z = a + bj = Z_m e^{j\theta}$, magnitude

$$Z_m = \sqrt{a^2 + b^2}$$

phase

$$\theta = \arctan(\frac{b}{a})$$

$$3+4j=5e^{j0.927} \quad 3=5\cos(0.927) \quad 4=5\sin(0.927)$$
$$5=\sqrt{3^2+4^2} \quad 0.927=\arctan\frac{4}{3}$$

$$1+j=\sqrt{2}e^{j0.785} \quad 1=\sqrt{2}\cos(0.785) \quad 1=\sqrt{2}\sin(0.785))$$
$$\sqrt{2}=\sqrt{1^2+1^2} \quad 0.785=\arctan\frac{1}{1}$$

$$5e^{j0.927}+\sqrt{2}e^{j0.785}=?$$
$$3+4j+1+j=4+5j=6.403e^{j0.896}$$

$$(3+4j)\times(1+j)=?$$

$$5e^{j0.927}\times\sqrt{2}e^{j0.785}=5\sqrt{2}e^{j1.71}$$

Properties of phasors

- $$y(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$
  $$Y = \alpha_1 X_1 + \alpha_2 X_2$$
- $$\frac{d}{dt}\{X_m e^{\theta j} e^{jwt}\} = \{\frac{d}{dt} X_m e^{\theta j} e^{jwt}\} = \{\underbrace{jwAX_m e^{\theta j}}_{\text{phasor}} e^{jwt}\}$$
- X is phasor of $x(t)$
  $$y(t) = \frac{d^n x(t)}{dt^n}$$
  then phasor of $y(t)$
  $$Y = (jw)^n X.$$
- A are B phasor
  - If $\mathrm{Re}\{Ae^{jwt}\} = \mathrm{Re}\{Be^{jwt}\}$ then $A = B$.
  - If $A = B$ then $\mathrm{Re}\{Ae^{jwt}\} = \mathrm{Re}\{Be^{jwt}\}$.

Phasor & State-space equation

Lets find the sinusoidal particular solution (Xe^{jwt}) of linear time invariant state equation

$$\dot{x} = Ax + Be$$

for a sinusoidal input Ee^{jwt}. Substituting the solution and input

$$j(w)X = AX + BE$$

The sinusoidal solution is then

$$X = (jwI - A)^{-1}BE$$

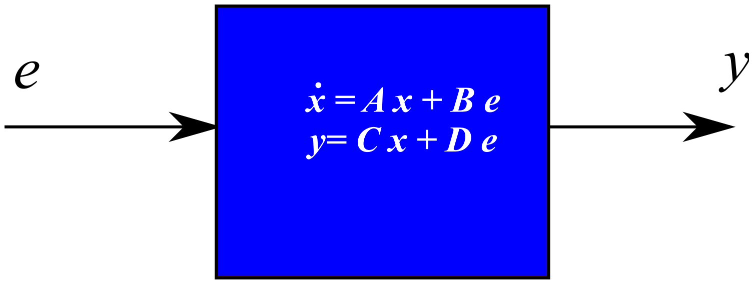
The solution is defined for $\det(jwI - A) \neq 0$ which means $jw \neq \lambda$.Input frequency is equal the natural frequency.

Transfer function

A linea time invariant single input single output system ($e, y \in R$) is defined by

$$\dot{x} = Ax + Be$$
$$y = Cx + De$$

what if $e \in R^m$, $y \in R^l$ ...



Using phasors, the output of the system

$$Y = \underbrace{\left(C(jwI - A)^{-1}B + D\right)}_{\text{transfer function}} E$$

Transfer function

$$H(jw) = \left(C(jwI - A)^{-1}B + D\right)$$

from input E to output Y

Example

$$\frac{dx}{dt} = \begin{bmatrix} -\sqrt{2} & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e(t); y = [0\ 1]x$$

Transfer function (which is a function of w !)

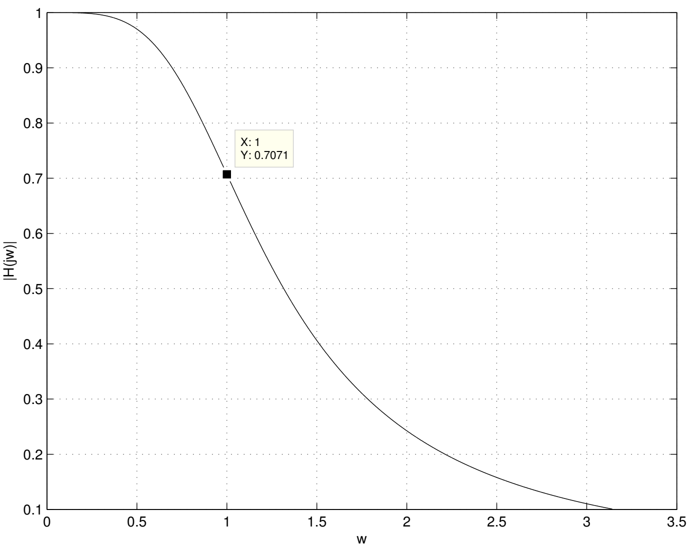
$$H(jw) = \frac{1}{1 - w^2 + j\sqrt{2}w}$$

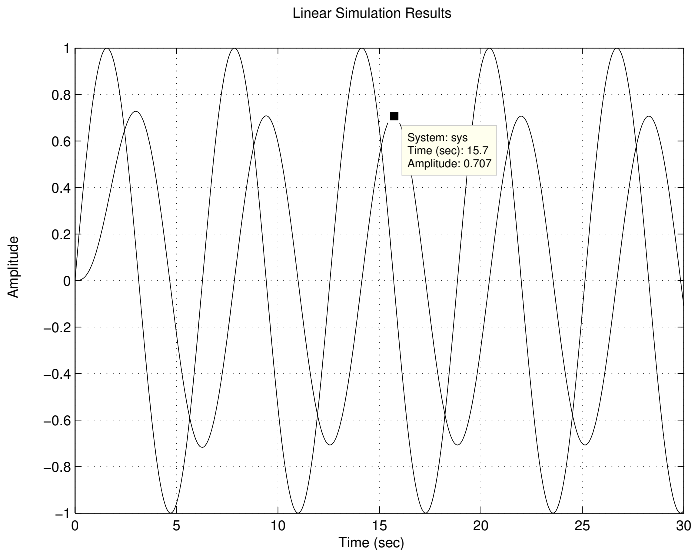
For the input $e(t) = \sin(wt)$(phasor of input signal $E = 1$) phasor of the output is

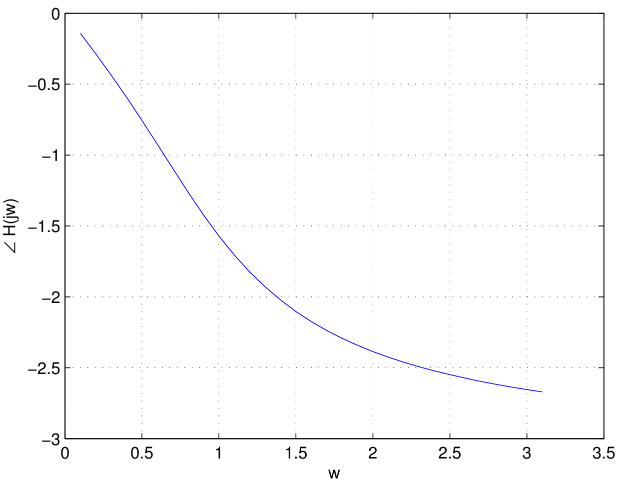
$$Y = H(j1)E = \frac{-j}{\sqrt{2}} = \frac{1}{\sqrt{2}} e^{-j\pi/2}$$

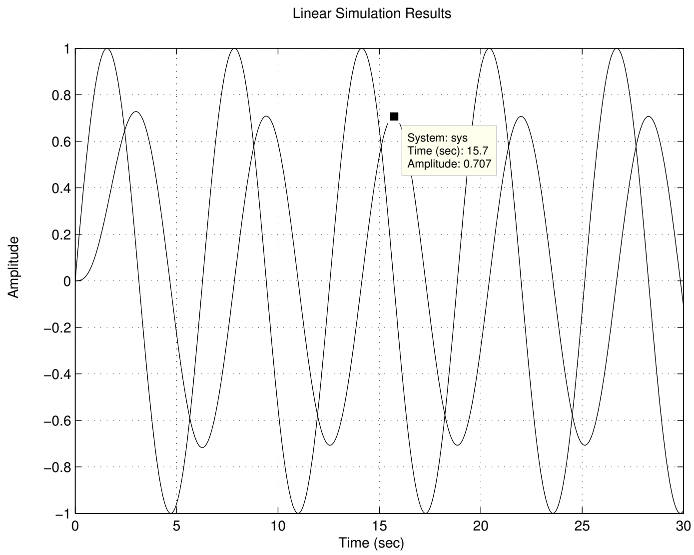
Hence output signal is

$$y(t) = \frac{1}{\sqrt{2}} \sin(t - \pi/2)$$


X: 1
Y: 0.7071
|H(jw)|
w


Linear Simulation Results
System: sys
Time (sec): 15.7
Amplitude: 0.707
Amplitude
Time (sec)


∠ H(jw)
w
0
−0.5
−1
−1.5
−2
−2.5
−3
0
0.5
1
1.5
2
2.5
3
3.5


Linear Simulation Results
Amplitude
Time (sec)
System: sys
Time (sec): 15.7
Amplitude: 0.707
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
5
10
15
20
25
30

Kirchhoff's Laws in the Frequence Domain

Lets assuming that $v_1, v_2 ... v_{n_e}$,represent voltages arourd a closed path in a circuit.
KVL requires that

$$\sum_{k=1}^{n_e} v_k(t) = 0$$

We assume that the circuit is operating in a sinusoidal steady state therefore

$$\sum_{k=1}^{n_e} \mathrm{Re}\left\{ V_k e^{jwt} \right\} = 0$$

Factoring the term e^{jwt} from each term yields

$$\sum_{k=1}^{n_e} V_k = 0.$$

A similar derivation applies to a set of sinusoidal currents (KCL). Thus if

$$\sum_{k=1}^{n_e} i_k(t) = 0$$

We assume that the circuit is operating in a sinusoidal steady state therefore

$$\sum_{k=1}^{n_e} \mathrm{Re}\{I_k e^{\theta_k)}\} = 0$$

Factoring the term e^{jwt} from each term yields

$$\sum_{k=1}^{n_e} I_k = 0.$$

Question: Four branches terminates at a common node. The reference direction of each branch current $(i_1, i_2, i_3, i_4,$ is toward the node if
$i_1 = 100\cos(wt + 25^\circ)$A $i_2 = 100\cos(wt + 145^\circ)$A
$i_3 = 100\cos(wt - 95^\circ)$A, find i_4