Circuit and System Analysis
EHB 232E

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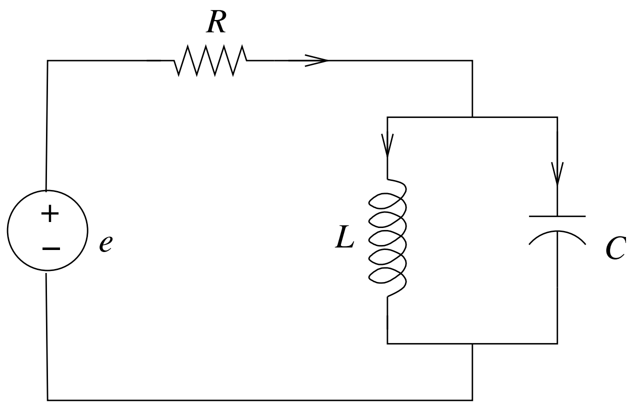
Outline I

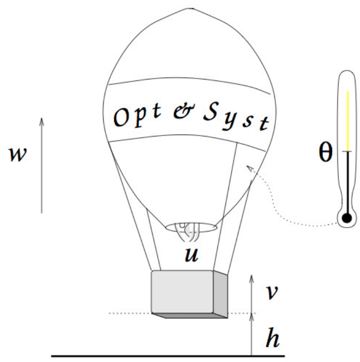
State Equations



State Equation of the circuit;

$$\frac{d}{dt}\begin{bmatrix} V_C \\ i_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} V_C \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix} e$$

State Equations

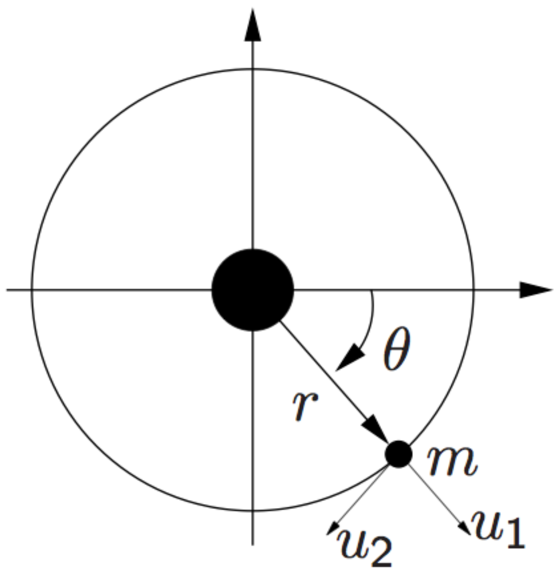


State Equation of The Hot Air Balloon;

$$\frac{d}{dt}\begin{bmatrix}\theta\\ v\\ h\end{bmatrix} = \begin{bmatrix}-\frac{1}{\alpha} & 0 & 0\\ \sigma & -\frac{1}{\beta} & \frac{1}{\beta}\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\theta\\ v\\ h\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}u$$

$\theta =$ temperature, $u =$ heating, $v =$ vertical velocity, $h =$ height, $w =$ vertical wind velocity

State Equations



Satellite Control;

$$\ddot{r}(t) = r(t)\dot{\theta}^2(t) - \frac{k}{r^2(t)} + u_1(t)$$
$$\ddot{\theta}(t) = \frac{\dot{\theta}(t)\dot{r}(t)}{r(t)} + \frac{1}{r(t)}u_2(t)$$

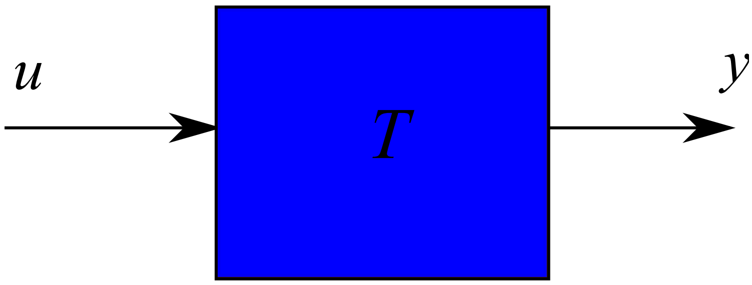
State Equations

A DC motor consists of an electromagnet made by winding wires around a core placed in a magnetic field made with permanent magnets or electromagnets. When current flows through the wires, the core spins.

$$
\begin{aligned}
L\dot{i}(t) &= v(t) - Ri(t) - k_b w(t) \\
I\dot{w}(t) &= k_T i(t) - \kappa w(t) - \tau(t)
\end{aligned}
$$

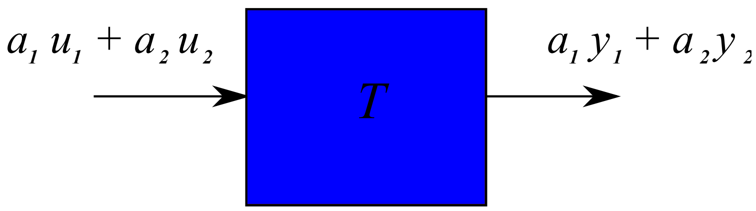
where w Angular Velocity, k_b back electromagnetic force constant, k_T motor torque constant, μ is the kinetic friction of the motor, and τ is the torque applied by the load.
See : http://leeseshia.org/download.html , page 203 (motor-RL-model.m)

Linear System



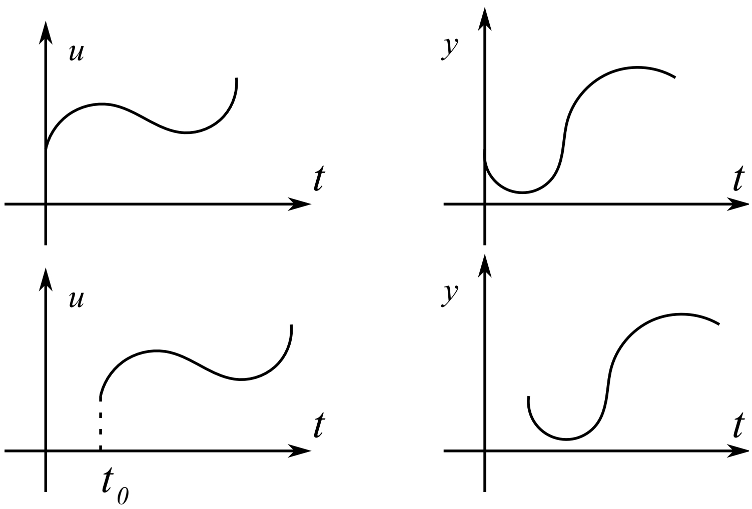
- T is the system
- u is the input.
- y is the output

Linear System



Here $a_1, a_2 \in R$ and Input u_1 gives response y_1, Input u_2 gives response y_2, Input $a_1 u_1 + a_2 u_2$ gives response $a_1 y_1 + a_2 y_2$.

Time-invariance



This implies that the dynamics do not change over time.

Internal descriptions: Linear State-space Equation

$$\dot{x} = Ax + Be, \quad x(t_0) = x_0$$

where the state variable $x \in R^n$, input $e \in R^m$ and $A \in R^{n \times n}$, $B \in R^{n \times m}$

Output $y \in R^l$

$$y = Cx + De$$

where $C \in R^{l \times n}$ and $D \in R^{l \times m}$.

Homogeneous Solution

The homogeneous solution is also called the natural response is the general solution of state equation when the input is set to zero;

$$\dot{x} = Ax, \quad x(t_0) = x_0.$$

The homogeneous solution is of the form

$$x_h(t) = ve^{\lambda t}$$

where $v \in R^n$. Substituting the proposed solution into the state equ.

$$\lambda e^{\lambda t} v = Ave^{\lambda t}$$

One obtain

$$(\lambda I - A)v = 0.$$

Since v is not zero, this means that the matrix $\lambda I - A$ is singular, which means that its determinant is 0 (non-invertible). Thus the roots of the function $\mathrm{det}(\lambda \mathrm{I} - \mathrm{A})$ are the eigenvalues of A, and it is clear that this determinant is a polynomial in λ such as

$$\mathrm{det}(\lambda \mathrm{I} - \mathrm{A}) = \lambda^{\mathrm{n}} + \mathrm{a}_{\mathrm{n}-1}\lambda^{\mathrm{n}-1} + ... + \mathrm{a}_0 = 0$$

Distinct Eigenvalues

Homogeneous Solution

$$x_h(t) = \alpha_1 v_1 e^{\lambda_1 t} + \alpha_2 v_2 e^{\lambda_2 t} + ... + \alpha_n v_n e^{\lambda_n t}.$$

where $\lambda_1, ..., \lambda_n$ distinct eigenvalues and $v_1, ..., v_n$ corresponding eigenvectors. In matrix form

$$x_h(t) = \begin{bmatrix} v_1 e^{\lambda_1 t} & v_2 e^{\lambda_2 t} & ... & v_n e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ . \\ \alpha_n \end{bmatrix}$$

$\{v_1 e^{\lambda_1 t} ... v_n e^{\lambda_n t}\}$ is a set of n linearly-independent solutions which is called a fundamental set of solutions. A fundamental matrix $M(t)$ is formed by creating a matrix out of the n fundamental vectors.

$$M(t) = \begin{bmatrix} v_1 e^{\lambda_1 t} & v_2 e^{\lambda_2 t} & ... & v_n e^{\lambda_n t} \end{bmatrix}$$

A *fundamental matrix* $M(t)$ is formed by creating a matrix out of the n fundamental vectors.

$$M(t) = \left[\begin{array}{cccc} v_1 e^{\lambda_1 t} & v_2 e^{\lambda_2 t} & \dots & v_n e^{\lambda_n t} \end{array} \right]$$

Homogeneous Solution

$$x_h(t) = M(t)L$$

The fundamental matrix will satisfy the state equation:

$$\dot{M}(t) = AM(t)$$

The complete solution (including input $e(t)$)

$$x(t) = M(t)L + x_p(t)$$

	SOURCE($e(t)$)	PARTICULAR SOLUTION
Et^m	if 0 is not an eigenvalue of A	$X_0 + X_1 t + ... + X_m t^m$
$m = 0, 1, ...$	if 0 is a kth order repeated eigenvalue of A	$X_0 + X_1 t + ... + X_{k+m} t^{k+m}$
$Ee^{\sigma t}$	if σ is not an eigenvalue of A	$X_0 e^{\sigma t}$
	if σ is a kth order repeated eigenvalue of A	$(X_0 + X_1 t + ... + X_k t^k)e^{\sigma t}$

The particular solution must satisfy the state equation:

$$\dot{x_p} = Ax_p + Be$$

For $e(t) = cos(wt) = \frac{e^{jwt} + e^{-jwt}}{2}, \;\; \sigma = jw$!

Using the initial conditions

$$L = M^{-1}(t_0)(x_0 - x_p(t_0))$$

Substituting L into the complete solution

$$x(t) = M(t)M^{-1}(t_0)(x_0 - x_p(t_0)) + x_p(t)$$

The state transition matrix of the system

$$\Phi(t) = M(t)M^{-1}(t_0)$$

The complete solution

$$x(t) = \Phi(t)x_0 + x_p(t) - \Phi(t)x_p(t_0)$$

Zero-input response;

$$x_{\mathrm{zi}}(t) = \Phi(t)x_0$$

Zero-state response;

$$x_{zs}(t) = x_p(t) - \Phi(t)x_p(t_0)$$

Examples

Find fundamental matrix and state transition matrix of the system

$$\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e, \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for the initial condition .
Eigenvalues of the system

$$\det \left\{ \lambda I - \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix} \right\} = \lambda^2 + 3\lambda + 2$$

So we have $\lambda_1 = -1$ ve $\lambda_2 = -2$. Corresponding eigenvectors:
$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Homogeneous Solution

$$
\begin{aligned}
x_h(t) &= \alpha_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t} + \alpha_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} \\
&= \underbrace{\begin{bmatrix} -2e^{-t} & -e^{-2t} \\ e^{-t} & e^{-2t} \end{bmatrix}}_{\text{Fundamental matrix}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}
\end{aligned}
$$

Using the Fundamental matrix, state transition matrix:

$$
\begin{aligned}
\phi(t) &= M(t)M(t_0)^{-1} \\
&= \begin{bmatrix} -2e^{-t} & -e^{-2t} \\ e^{-t} & e^{-2t} \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t} \\ -e^{-t} + e^{-2t} & e^{-t} + 2e^{-2t} \end{bmatrix}.
\end{aligned}
$$

Zero-input solution

$$\begin{aligned} x_{zi} &= \phi(t)x(0) \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -e^{-t} + e^{-2t} \end{bmatrix}. \end{aligned}$$

Lets find zero-state response for $e = u(t)$ From table, particular solution

$$x_p = X_0$$

Substituted X_0 into the DE

$$\dot{X}_0 = AX_0 + B1$$

Particular solution will be $X_0 = -A^{-1}B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Complete solution

$$x(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -e^{-t} + e^{-2t} \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -e^{-t} + e^{-2t} \end{bmatrix}$$

Lets find complete solution for $e = 10\cos(t)$.
Input is $e(t) = \mathrm{Re}\{10e^{jt}\}$. Particular solution for this input (from table) is $x_p(t) = \mathrm{Re}\{X_0 e^{jt}\}$.
Substitute the particular solution into the DE.

$$(jI - A)X_0 = B10$$

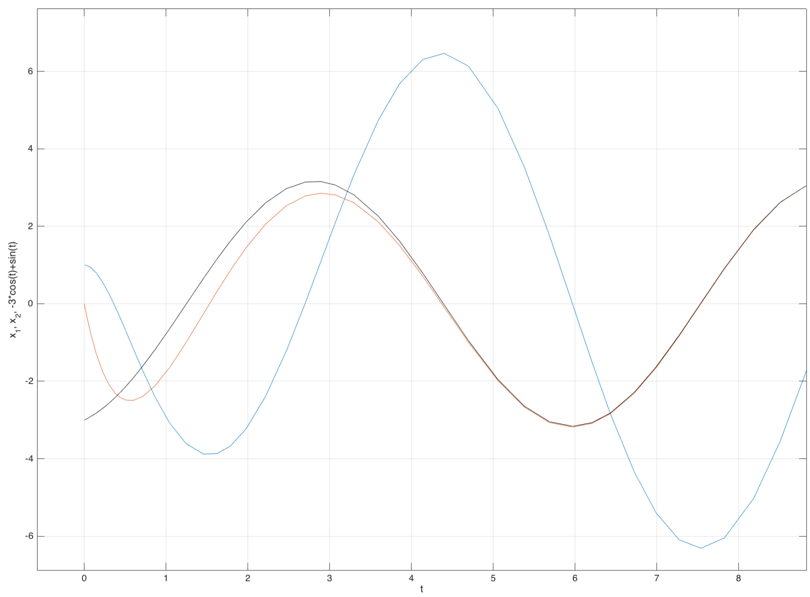
in order to find $X_0 = \begin{bmatrix} -2+6j \\ -3-j \end{bmatrix}$ The particular solution

$$\begin{aligned} x_p = \mathrm{Re}\{X_0 e^{jt}\} &= \mathrm{Re}\left\{\begin{bmatrix} -2+6j \\ -3-j \end{bmatrix}(\cos(t) + j\sin(t))\right\} \\ &= \begin{bmatrix} -2\cos(t) - 6\sin(t) \\ -3\cos(t) + \sin(t) \end{bmatrix} \end{aligned}$$

The complete solution: (ornek2.m)

$$x(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -e^{-t} + e^{-2t} \end{bmatrix} + \begin{bmatrix} -2\cos(t) - 6\sin(t) \\ -3\cos(t) + \sin(t) \end{bmatrix} - \phi(t)\begin{bmatrix} -2 \\ -3 \end{bmatrix}.$$

```
function xdot = s27(t,x)
xdot(1,1) = 0*x(1)+2*x(2);
xdot(2,1) = -x(1)-3*x(2)-10*cos(t);
>>[t,y]=ode23('s27',[0 60],[1 0]);
```



Question :

$$\lim_{t \to \infty} x_h(t) = ?$$

Repeated Eigenvalues

If we had n distinct eigenvalues, we have linearly independent n eigenvectors. It is possible to be less than n linearly independent eigenvectors if an eigenvalue is repeated. We could not find a independent eigenvector for a repeated eigenvalue.

Example :

$$\dot{x} = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} x$$

The eigenvalues are given by

$$\det(\lambda \mathrm{I} - \mathrm{A}) = (\lambda + 3)^2 = 0$$

$\lambda = -3$ is a repeated eigenvalue. The corresponding eigenvector is given by solving

$$\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

we get only one linearly independent eigenvector which is $v = [3 \;\; 1]^T$.

For the other eigenvector, we search for a solution of the form $vte^{\lambda t} + we^{\lambda t}$. Lets plug it into the DE

$$ve^{\lambda t} + v\lambda te^{\lambda t} + w\lambda e^{\lambda t} = A(vte^{\lambda t} + we^{\lambda t})$$

We can equate the coefficients

$$Av = \lambda v$$

and

$$Aw = v + \lambda w$$

Example :

$$\dot{x} = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} x$$

$$Aw = v + \lambda w$$

$$\begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Thus eigenvector is $w = [1/2 \;\; 0]^T$

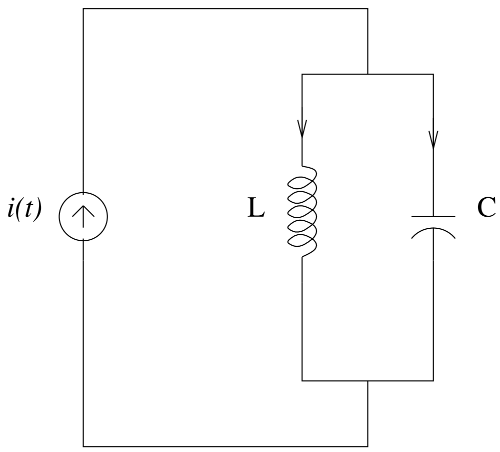
Homogeneous Solution

$$x_h(t) = \alpha_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-3t} + \alpha_2 \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{-3t} \right)$$

$$\lim_{t \to \infty} x_h(t) = ?$$

Example

Find the complete solution for $C = 1F$, $L = 1H$, $V_C(0) = 1$ ve $i_L(0) = 0$ and $i(t) = \cos(t)$.



$$
\begin{aligned}
L\dot{i}_L &= V_C \\
C\dot{V}_C &= -i_L + i(t).
\end{aligned}
$$

In standart form

$$
\frac{d}{dt}\begin{bmatrix} i_L \\ V_C \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} i_L \\ V_C \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i(t)
$$

Solve

$$\det\left\{\lambda I - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\} = \lambda^2 + 1$$

thus eigenvalues are $\lambda_1 = j$, $\lambda_2 = -j$ and corresponding eigenvectors are $[j \ \ -1]^T$, $[-j \ \ -1]^T$. The Fundamental matrix:

$$M = \begin{bmatrix} je^{jt} & -je^{-jt} \\ -e^{jt} & -e^{-jt} \end{bmatrix}$$

and the state transition matrix:

$$\phi(t) = \begin{bmatrix} je^{jt} & -je^{-jt} \\ -e^{jt} & -e^{-jt} \end{bmatrix} \begin{bmatrix} j & -j \\ -1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

Zero-input response:

$$x_{zi}(t) = \left[\begin{array}{cc} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{array} \right] \left[\begin{array}{c} 0 \\ 1 \end{array} \right] = \left[\begin{array}{c} \sin(t) \\ \cos(t) \end{array} \right]$$

Particular solution for $e = \cos t = \mathrm{Re}\{e^{jt}\}$ is chosen from the table such as $x_p = (X_0 + X_1 t)e^{jt}$. Substituting x_p into the DE

$$\begin{array}{rcl} (jI - A)X_1 & = & 0 \\ (jI - A)X_0 & = & B - X_1 \end{array}$$

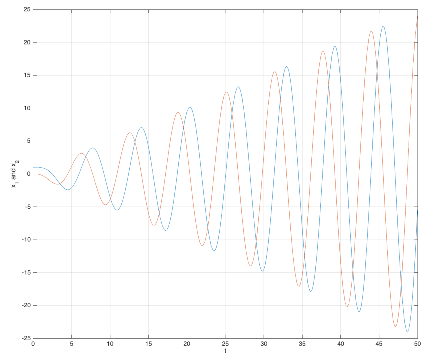
we obtain $X_0 = \left[\begin{array}{c} 0.5 \\ 0 \end{array} \right]$ and $X_1 = \left[\begin{array}{c} -0.5j \\ 0.5 \end{array} \right]$ The particular solution

$$x_p = \mathrm{Re}\left\{ \left(\left[\begin{array}{c} 0.5 \\ 0 \end{array} \right] + \left[\begin{array}{c} -0.5j \\ 0.5 \end{array} \right] t \right) (\cos(t) + j\sin(t)) \right\}$$

The complete solution

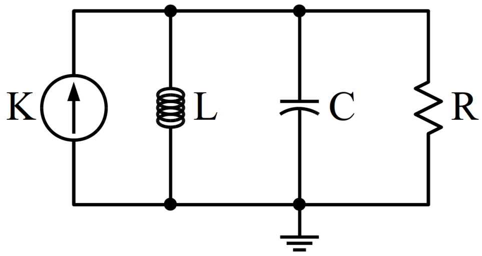
$$
\begin{aligned}
x(t) &= \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + \begin{bmatrix} 0.5\cos(t) + 0.5t\sin(t) \\ 0.5t\cos(t) \end{bmatrix} \\
&\quad - \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \sin(t) + 0.5t\sin(t) \\ \cos(t) + 0.5t\cos(t) + 0.5\sin(t) \end{bmatrix}
\end{aligned}
$$

$\lim_{t\to\infty} x(t) = ?$



Example

Find the complete solution for $R = 1/3\Omega$, $C = 1F$, $L = 1/2H$, $V_C(0) = 1V$ ve $i_L(0) = 1A$ and $i(t) = \cos(wt)$.



In standart form

$$\frac{d}{dt}\begin{bmatrix} i_L \\ V_C \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix}\begin{bmatrix} i_L \\ V_C \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix} i_K(t)$$

The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = -1$ and corresponding eigenvectors are $[1 \; -1]^T$, $[2 \; -1]^T$.

The Fundamental matrix:

$$M = \begin{bmatrix} e^{-2t} & 2e^{-t} \\ -e^{-2t} & -e^{-t} \end{bmatrix}$$

and the state transition matrix:

$$\phi(t) = \begin{bmatrix} e^{-2t} & 2e^{-t} \\ -e^{-2t} & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t} \\ e^{-2t} - e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

Zero-input response:

$$x_{zi}(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t} \\ e^{-2t} - e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$$

Particular solution for $e = \cos \omega t = \mathrm{Re}\{e^{j\omega t}\}$ is chosen from the table such as $x_p = X_0 e^{j\omega t}$. Substituting x_p into the DE

$$(j\omega I - A)X_0 \quad = \quad B$$

we obtain

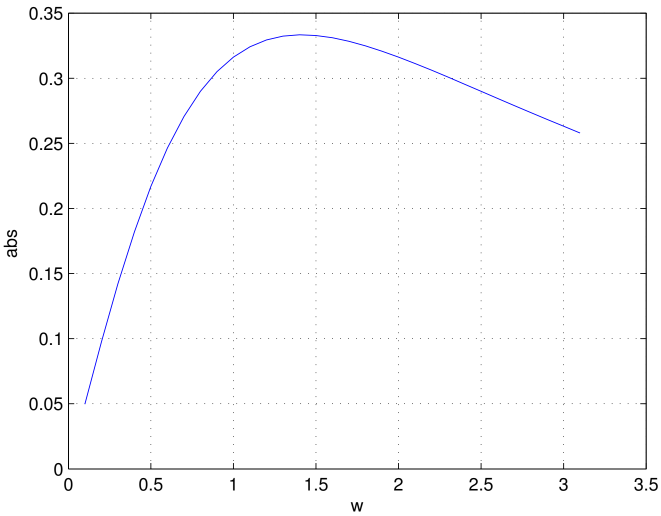
$$X_0 = \begin{bmatrix} j\omega & -2 \\ 1 & j\omega + 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{-\omega^2 + j3\omega + 2} \begin{bmatrix} 2 \\ j\omega \end{bmatrix}$$

The particular solution

$$x_p(t) = Re\left\{ \tfrac{1}{2-\omega^2+j3\omega} \begin{bmatrix} 2 \\ j\omega \end{bmatrix} e^{j\omega t} \right\}$$
$$= \tfrac{1}{(2-\omega^2)^2+9\omega^2} \begin{bmatrix} 2(2-\omega^2)\cos(\omega t) + 6\omega \sin(\omega t) \\ 3\omega^2 \cos(\omega t) - \omega(2-\omega^2)\sin(\omega t) \end{bmatrix}$$

When the magnitude of $v_C(t)$ become maximum ?

$$v_C(t) = Re\left\{\frac{j\omega}{2-\omega^2+j3\omega}e^{j\omega t}\right\}$$



For $\omega = \sqrt{2}$

$$x(t) = \begin{bmatrix} \dfrac{\sqrt{2}\sin(\sqrt{2}t)}{3} \\ \dfrac{\cos(\sqrt{2}t)}{3} \end{bmatrix} + \begin{bmatrix} 2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t} \\ e^{-2t} - e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}\left(\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}0\\1/3\end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} -7e^{-2t} + 10e^{-t} + \sqrt{2}\sin(\sqrt{2}t) \\ 7e^{-2t} - 5e^{-t} + \cos(\sqrt{2}t) \end{bmatrix}$$

Series expansion of Homogeneous Solution

A *Mac-Laurin* series expansion of

$$\dot{x} = Ax$$

about $x(0)$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ . \\ x_n(t) \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \\ . \\ . \\ . \\ x_n(0) \end{bmatrix} + \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ . \\ . \\ . \\ \dot{x}_n(0) \end{bmatrix} t + \ldots + \begin{bmatrix} x_1^{(k)}(0) \\ x_2^{(k)}(0) \\ . \\ . \\ . \\ x_n^{(k)}(0) \end{bmatrix} \frac{t^k}{k!} + \ldots$$

Using $\dot{x}(0) = Ax(0)$, $\ddot{x}(0) = A^2x(0)$, $\ldots x^{(k)}(0) = A^{(k)}x(0)$ we obtain

$$x(t) = (I + At + \frac{1}{2!}A^2t^2 \ldots + \frac{1}{k!}A^kt^k \ldots)x(0)$$

State transition matrix

Zero-input response

$$x_{\mathrm{zi}}(t) = \Phi(t)x_0$$

$$\Phi(t) = I + At + \frac{1}{2!}A^2t^2 \ldots + \frac{1}{k!}A^kt^k \ldots$$

Matrix exponential

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 \ldots + \frac{1}{k!}A^kt^k \ldots.$$

$$\dot{x} = Ax \quad x(t_0) = x_0 \rightarrow x(t) = e^{A(t-t_0)}x_0$$

$$x(t_1) = \Phi(t_1, t_0)x(t_0) = e^{A(t_1-t_0)}x(t_0)$$

Properties of state transition matrix

- $$\Phi(0) = e^{A0} = I$$
- $$\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0)$$
- $$\Phi(t_1, t_0) = \Phi(t_0, t_1)^{-1}$$
- $$\frac{d\Phi(t)}{dt} = A\Phi(t)$$

Non-homogeneous state equations

$$\dot{x} = Ax + Be(t) \quad x(0) = x_0$$

Lets suppose it has a solution such as

$$x(t) = \Phi(t)s(t)$$

Substituting the solution into the DE

$$\frac{d\Phi(t)s(t)}{dt} = \frac{d\Phi(t)}{dt}s(t) + \frac{ds(t)}{dt}\Phi(t) = A\Phi(t)s(t) + Be(t)$$

Using the property $\frac{d\Phi(t)}{dt} = A\Phi(t)$ we obtain

$$\Phi(t)\frac{ds(t)}{dt} = Be(t)$$

$$\frac{ds(t)}{dt} = \Phi(-t)Be(t)$$

Lets integrate

$$s(t) = s_0 + \int_0^t \Phi(-\tau)Be(\tau)d\tau$$

and substitute $s(t)$ into the solution

$$x(t) = \Phi(t)s_0 + \Phi(t)\int_0^t \Phi(-\tau)Be(\tau)d\tau = \Phi(t)s_0 + \int_0^t \Phi(t-\tau)Be(\tau)d\tau$$

From the initial condition $s(0) = x(0)$ thus

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Be(\tau)d\tau$$

$$x(t) = \underbrace{\Phi(t)x(0)}_{\text{zero-input response}} + \underbrace{\int_0^t \Phi(t-\tau)Be(\tau)d\tau}_{\text{zero-state response}}$$

$$x(t) = \underbrace{\Phi(t)x(0)}_{\text{zero-input response}} + \underbrace{x_p(t) - \Phi(t)x_p(t_0)}_{\text{zero-state response}}$$

$$x(t) = \underbrace{\Phi(t)x(0) - \Phi(t)x_p(0)}_{\text{Natural Response}} + \underbrace{x_p(t)}_{\text{Forced reponse}}$$

$$x_{\text{zs}}(t) = x_p(t) - \Phi(t)x_p(0) = \int_0^t \Phi(t-\tau)Be(\tau)d\tau$$