Basic of Electrical Circuits
EHB 211E

Prof. Dr. Müştak E. Yalçın

Istanbul Technical University
Faculty of Electrical and Electronic Engineering

mustak.yalcin@itu.edu.tr

Lecture 13

Contents I

State Equation

Capacitor current and or its voltage are given by

$$C\frac{dV_C}{dt} = i_C$$

and

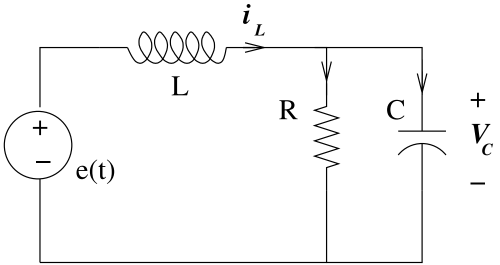
$$V_C(t) = \frac{1}{C}\int_{t_0}^{t} i_C(\tau)d\tau + V_C(0)$$

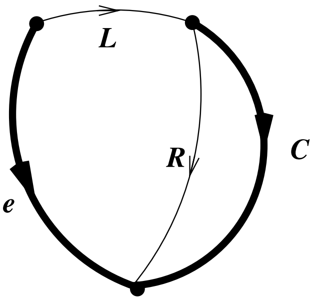
Inductor voltage and current are given by

$$L\frac{di_L}{dt} = V_L$$

and

$$i_L(t) = \frac{1}{L}\int_{t_0}^{t} V_L(\tau)d\tau + i_L(0)$$





Using Branch Voltages Method

$$i_C = i_L - i_R$$

Using the definition of L and C elements

$$\begin{aligned} C\frac{dV_C}{dt} &= \frac{1}{L}\int_{t_0}^{t} V_L(\tau)d\tau + i_L(0) - GV_R \\ &= \frac{1}{L}\left(\int_{t_0}^{t} e(\tau)d\tau - \int_{t_0}^{t} V_C(\tau)d\tau\right) + i_L(0) - GV_R \end{aligned}$$

we have an Integro-Differential Equation !.

We can represent the same circuit by differential equations of the form

$$\begin{aligned} C\frac{dV_C}{dt} &= i_L - i_R \\ L\frac{di_L}{dt} &= e - V_C \end{aligned}$$

We can write the state equations in matrix form:

$$\begin{bmatrix} \frac{dV_C}{dt} \\ \frac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} -G/C & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} V_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} e$$

where $i_R = GV_R = GV_C$. This equation can be recast into the standard form

$$\begin{aligned} \dot{X} &= AX + Bu \\ y &= CX + Du \end{aligned}$$

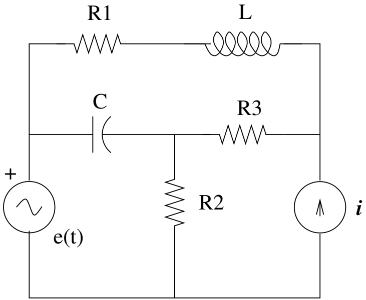
where X is state variable vector, y is output and u is input.

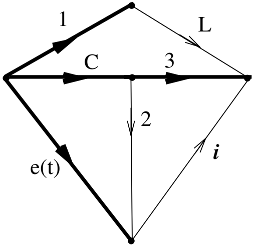
Obtaining State Equations

1. Pick a proper tree :
   - The voltage sources must be placed in the tree.
   - If the tree is not complete, the edges corresponding to as many capacitors as possible must be placed in the tree. If a capacitor in a loops which consisting entirely of capacitors and voltage sources. The capacitor must not be placed in the tree.
   - If the tree is not complete, the edges corresponding to the resistors must be chosen and as many resistors as possible must be included.
   - If still the tree is not completed, then, the edges corresponding to the inductors will be chosen until the tree is completed. If an inductors in a cut set which consisting entirely of inductors and current sources, the inductor must be placed in tree.
   - All the edges corresponding to the current sources must be placed in the co-tree.
2. After the selection of proper tree, the state variables are branch capacitor voltages and chord inductor currents.

3. Obtaining State Equations from the circuit: Express the voltage across each element corresponding to a branch and the current through each element corresponding to non-branch edge in terms of voltage sources, current sources, and state variables. * If not possible, assign a new voltage variable to a resistor corresponding to a branch and a new current variable to a resistor corresponding to a non-branch edge.
   - a Apply KVL to the fundamental loop determined by each non-branch inductor.
   - b Apply KCL to the fundamental cut-set determined by each branch capacitor.
   - c Apply KVL to the fundamental loop determined by each resistor with a new current variable assigned in *.
   - d Apply KCL to the node or super-node corresponding to the fundamental cut-set determined by each resistor with a new voltage variable assigned in *
   - e Solve the simultaneous equations obtained from steps c and d for the new variables in terms of the voltage sources, current sources, and the state variables.
   - f Substitute the expressions obtained in step e into the equations determined in steps a and b.

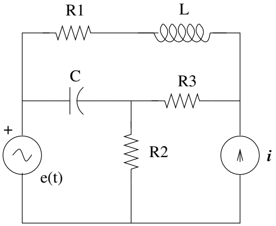
Example

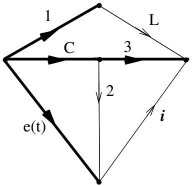




1. Graph is drawn and pick the proper tree.
2. V_C and i_L state variables.

$$\dot{V}_C = f(V_C, i_L, e(t), i(t)) \quad \dot{i}_L = f(V_C, i_L, e(t), i(t))$$





1. KVL for the fundamental loop determined by the inductor and KCL to the fundamental cut-set determined by the capacitor.

$$\begin{aligned} i_C + i_L - i_2 + i &= 0 \\ V_L - V_3 - V_C + V_1 &= 0 \end{aligned}$$

using the definition of the inductor and capacitor

$$\begin{aligned} C\frac{dV_C}{dt} &= -i_L + i_2 - i \\ L\frac{di_L}{dt} &= V_3 + V_C - V_1 \end{aligned}$$

KVL for the fundamental loop determined by R_2 and KCL to the fundamental cut-set determined by R_1 and R_3

$$\begin{aligned} R_2 i_2 &= e - V_C \\ G_1 V_1 &= i_L \\ G_3 V_3 &= -i_L - i \end{aligned}$$

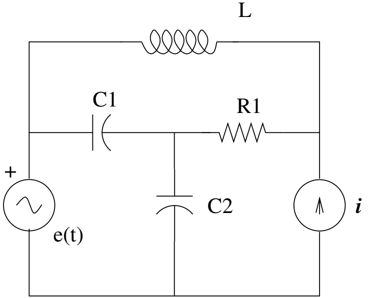
Substitute the expressions

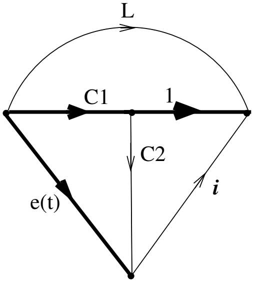
$$\frac{d}{dt}\begin{bmatrix} V_C \\ i_L \end{bmatrix} = \begin{bmatrix} \frac{-1}{R_2 C} & \frac{-1}{C} \\ \frac{1}{L} & \frac{-(R_3+R_1)}{L} \end{bmatrix} \begin{bmatrix} V_C \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{R_2 C} \\ 0 \end{bmatrix} e(t) + \begin{bmatrix} \frac{-1}{C} \\ -\frac{R_3}{L} \end{bmatrix} i$$

Degenerate Circuit

Circuit which contains any

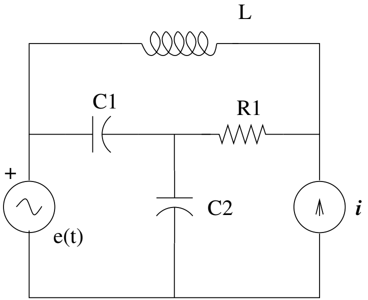
- Loops consisting entirely of capacitors and voltage sources.
- Cutsets consisting entirely of inductors and current sources.

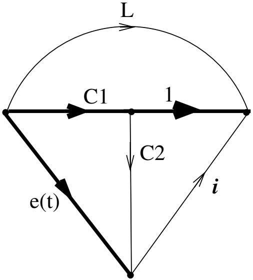




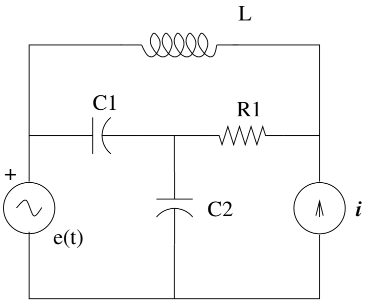
two capacitors and the voltage source make a loop.

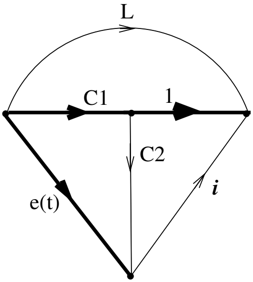
Degenerate Circuit





1. $C2$ must be placed to co-tree.





2. The state variable are V_{C1} i_L.
3. KCL and KVL

$$\begin{aligned} i_{C1} + i_L - i_{C2} + i &= 0 \\ V_L - V_1 - V_{C1} &= 0 \end{aligned}$$

Using the definition of C and L elements, the state equations;

$$\begin{aligned} C_1 \frac{dV_{C1}}{dt} &= -i_L - i + i_{C2} \\ L \frac{di_L}{dt} &= V_1 + V_{C1} \end{aligned}$$

Degenerate Circuit

Apply KVL to the fundamental loop determined by $C2$ and KCL to the fundamental cut-set determined by $R1$

$$\begin{aligned} V_{C2} &= e - V_{C1} \\ G_1 V_1 &= -i_L - i \end{aligned}$$

In order to obtain i_{C2} in terms of the voltage sources, current sources, and the state variables, we will use the definition of capacitor ($i_{C2} = C_2 \frac{dV_{C2}}{dt}$).

$$C_2 \frac{dV_{C2}}{dt} = C_2 \frac{de}{dt} - C_2 \frac{dV_{C1}}{dt}$$

The state equation in standard form

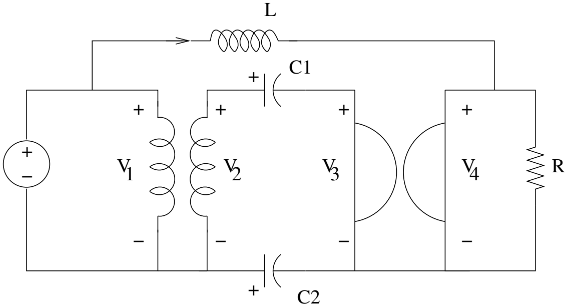
$$\begin{aligned} C_1 \frac{dV_{C1}}{dt} &= -i_L - i + C_2 \frac{de}{dt} - C_2 \frac{dV_{C1}}{dt} \\ L \frac{di_L}{dt} &= -R_1(i_L - i) + V_{C1} \end{aligned}$$

$$\frac{d}{dt}\begin{bmatrix} V_{C1} \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{C_2+C_1} \\ \frac{1}{L} & \frac{-R}{L} \end{bmatrix}\begin{bmatrix} V_{C1} \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{C_2}{C_1+C_2} \\ 0 \end{bmatrix}\frac{de}{dt} + \begin{bmatrix} \frac{-1}{C_1+C_2} \\ \frac{-R}{L} \end{bmatrix} i$$

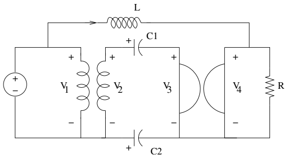
RLC and multi-terminal elements

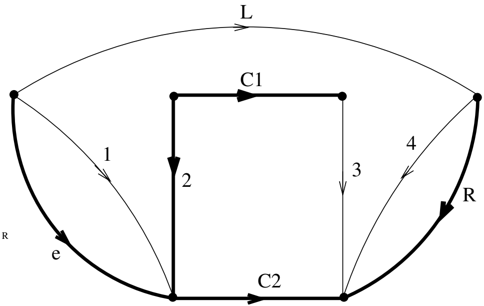
All the edge corresponding to the dependent voltage source must be placed in tree. All the edge corresponding to the dependent current source must be placed in co-tree.

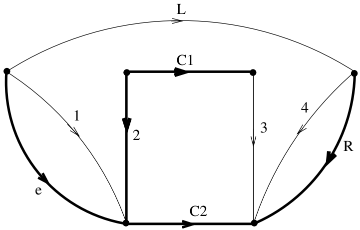
Example



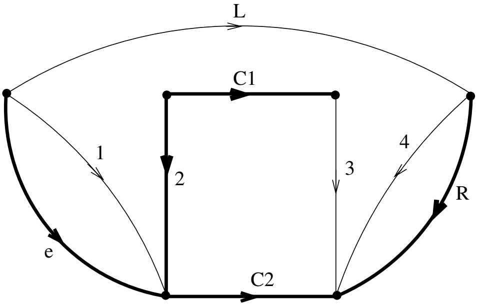
Transformer $V_2 = nV_1$, $i_1 = -ni_2$ and Gyrator $i_3 = -\alpha V_4$, $i_4 = \alpha V_3$


L
C1
+
+
+
+
+
−
V1
V2
V3
V4
R
−
−
−
−
+
C2


L
C1
1
2
3
4
R
e
C2



1. Graph is drawn. The voltage sources e, capacitors $C1$ and $C2$ are placed to tree. The tree is not complete, edge 2 is a dependent voltage source which is placed to tree. The edges 3 and 4 are placed to co-tree.
2. V_{C1}, V_{C2} and i_L are state variable.



3. From the fundamental cut-sets and loop, we have

$$\begin{aligned} i_{C1} + i_3 &= 0 \\ i_{C2} + i_L + i_3 &= 0 \\ V_L + V_R - V_{C2} - e &= 0 \end{aligned}$$

The state equations;

$$\begin{aligned} C_1 \frac{dV_{C1}}{dt} &= -i_3 \\ C_2 \frac{dV_{C2}}{dt} &= -i_L - i_3 \\ L \frac{di_L}{dt} &= -V_R + V_{C2} + e \end{aligned}$$

Express the i_3 and V_R as function of state variable and independent sources

$$\begin{aligned} i_R &= -i_4 + i_L = i_L - \alpha V_3 = i_L - \alpha(-V_{C1} + V_2 + V_{C2}) \\ &= i_L - \alpha(-V_{C1} + ne + V_{C2}) \\ i_3 &= \alpha V_4 = \alpha V_R = \alpha R i_R \end{aligned}$$

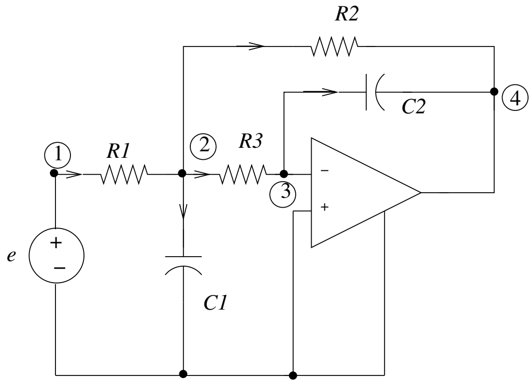
$$\frac{d}{dt}\begin{bmatrix} V_{C1} \\ V_{C2} \\ i_L \end{bmatrix} = \begin{bmatrix} -- & -- & -- \\ -- & -- & -- \\ -- & -- & -- \end{bmatrix}\begin{bmatrix} V_{C1} \\ V_{C2} \\ i_L \end{bmatrix} + \begin{bmatrix} -- \\ -- \\ -- \end{bmatrix} e$$

Obtaining State Equations directly from the circuit

Consider a dynamic circuit that does not contain any

- Loops consisting entirely of capacitors and voltage sources.
- Cutsets consisting entirely of inductors and current sources.

The objective of the analysis is the express the currents of capacitors and the voltages of the inductors as a function of the voltages of the capacitors, the currents of the inductors and the independent sources.



$$
\begin{aligned}
C_1 \frac{dV_{C1}}{dt} &= G1(V_{d1} - V_{d2}) - G_3(V_{d2} - V_{d3}) - G_2(V_{d2} - V_{d4}) \\
C_2 \frac{dV_{C2}}{dt} &= G_3(V_{d2} - V_{d3}
\end{aligned}
$$

$$\begin{aligned} V_{d1} &= e \\ V_{d3} &= 0 \\ V_{d2} &= V_{C1} \\ V_{d4} &= -V_{C2} \end{aligned}$$

Using the above equations, the state equations;

$$\begin{aligned} C_1 \frac{dV_{C1}}{dt} &= G1(e - V_{C1}) - G_3(V_{C1}) - G_2(V_{C1} + V_{C2}) \\ C_2 \frac{dV_{C2}}{dt} &= G_3 V_{C1} \end{aligned}$$

In standard matris form:

$$\frac{d}{dt} \begin{bmatrix} V_{C1} \\ V_{C2} \end{bmatrix} = \begin{bmatrix} -\frac{G_1+G_2+G_3}{C_1} & -\frac{G_2}{C_1} \\ -\frac{G3}{C_2} & 0 \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \end{bmatrix} + \begin{bmatrix} \frac{G_1}{C_1} 0 \end{bmatrix} e$$