Fluid System Building Blocks

- Basic Modeling Elements
  - Resistance
  - Capacitance
  - Inertance
  - Pressure and Flow Sources
- Interconnection Relationships
  - Compatibility Law
  - Continuity Law

Thermal System Building Blocks

Fluid System Building Blocks

In fluid systems there are three basic building blocks:
Resistance, capacitance and inertance

Fluid systems can be considered to fall into two categories:

Hydraulic: the fluid is liquid (assume to be incompressible)

Pneumatic: it is a gas and can be compressed and shows a density change

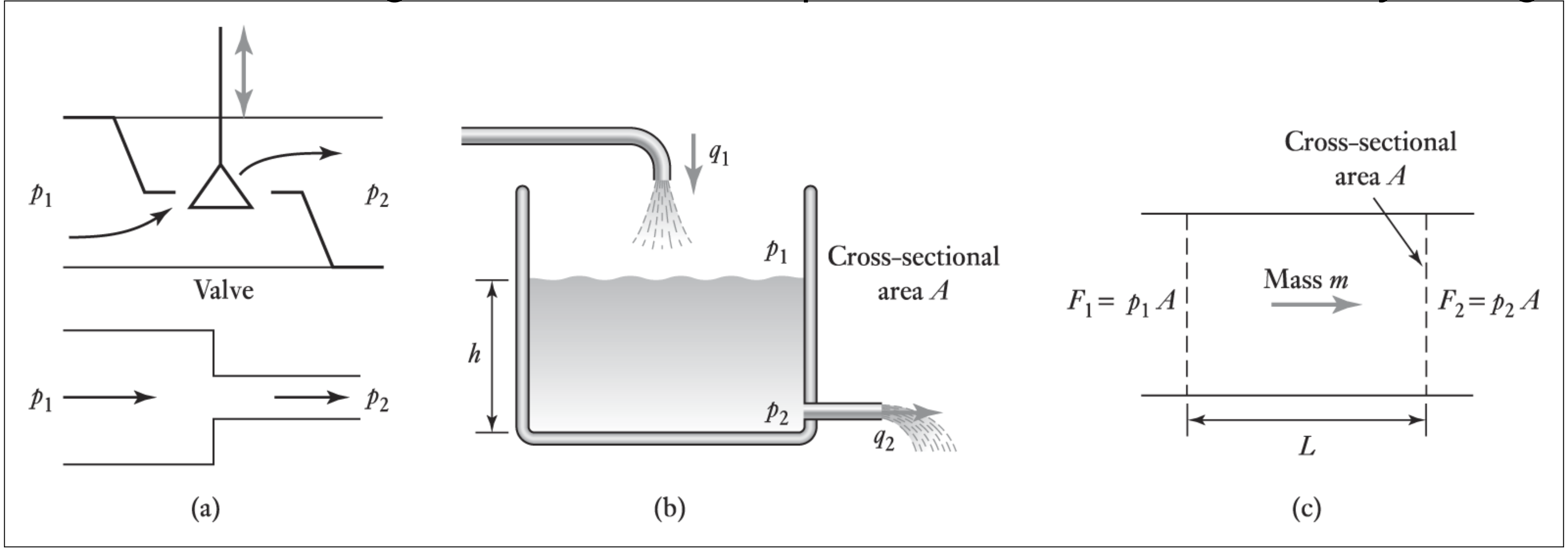
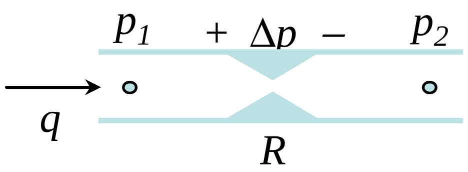


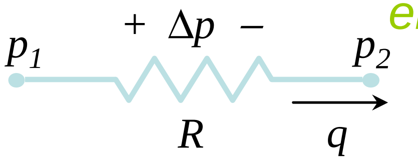
Figure 10.13 Hydraulic examples: (a) resistance, (b) capacitance, (c) inertance

Hydraulic Fluid system Building Blocks

- **Fluid Resistance**

  Describes any physical element with the characteristic that the pressure drop, Δp , across the element is proportional to the volume flow rate, *q*.





$$\Delta p = p_1 - p_2 = p_{12} = R \cdot q$$

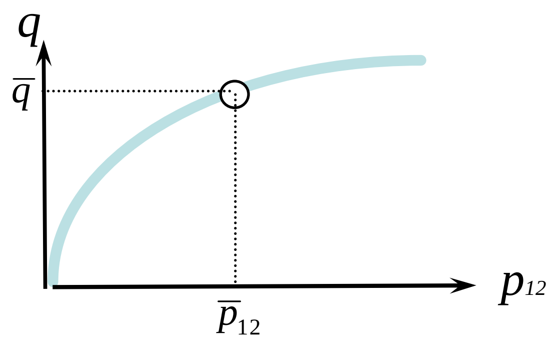
$$q = \frac{1}{R} \Delta p = \frac{1}{R} p_{12}$$

  – Orifices, valves, nozzles and friction in pipes can be modeled as fluid resistors.

Ex: The flow that goes through an orifice or a valve and the turbulent flow that goes through a pipe is related to the pressure drop by

$$q = k\sqrt{p_{12}}$$

Find the effective flow resistance of the element at certain operating point $(\bar{q}, \bar{p}_{12})$.



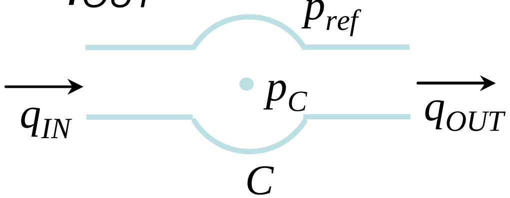
$$\frac{1}{R} = \left.\frac{dq}{dp_{12}}\right|_{(\bar{q}, \bar{p}_{12})} = \frac{k}{2\sqrt{\bar{p}_{12}}}$$

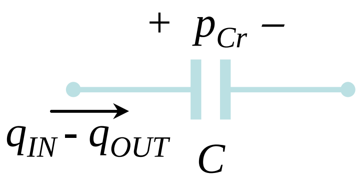
$$R = \frac{2\sqrt{\bar{p}_{12}}}{k} = \frac{2\bar{q}}{k^2}$$

Hydraulic Fluid system Building Blocks

- **Fluid Capacitance**

  Describes any physical element with the characteristic that the rate of change in pressure, p, in the element is proportional to the difference between the input flow rate, q_{IN} , and the output flow rate, q_{OUT} .

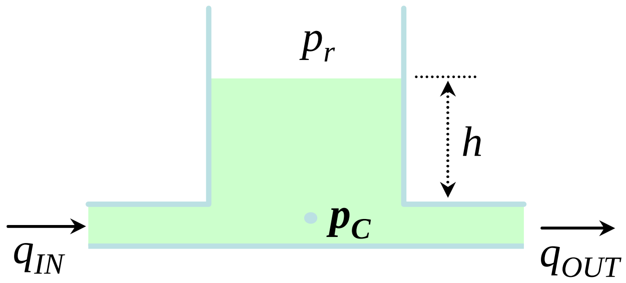




$$C\frac{d}{dt}\underbrace{\left(p_C - p_{ref}\right)}_{p_{Cr}} = C\cdot\dot{p}_{Cr} = q_{IN} - q_{OUT}$$

  – Hydraulic cylinder chambers, tanks, and accumulators are examples of fluid capacitors.

Ex: Consider an open tank with a constant cross-sectional area, A:



$$p_C = \rho g h + p_r \quad \Rightarrow \quad p_{Cr} = \rho g h$$

$$q_{IN} - q_{OUT} = \frac{d}{dt}(Volum) = \frac{d}{dt}(Ah) = A\dot{h}$$

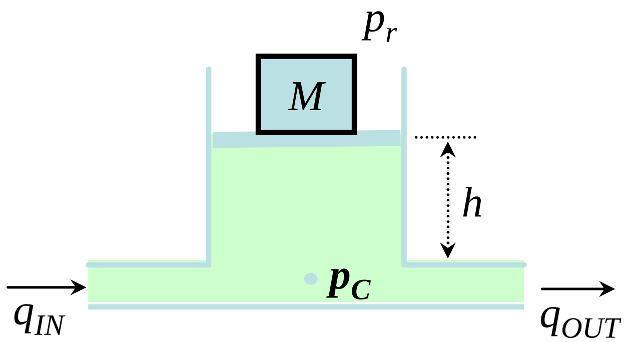
$$\dot{p}_{Cr} = \rho g \dot{h}$$

$$\Rightarrow \quad C = \frac{q_{IN} - q_{OUT}}{\dot{p}_{Cr}} = \frac{A\dot{h}}{\rho g \dot{h}} = \frac{A}{\rho g}$$

It describe the energy storage with a liquid in the form of potential energy

Hydraulic Fluid Capacitance Examples

Ex: Will the effective capacitance change if in the previous open tank example, a load mass M is floating on top of the tank?



$$p_C = \rho g h + p_r + {}^{Mg}\!/\!_{A} \Rightarrow \quad p_{Cr} = \rho g h + {}^{Mg}\!/\!_{A}$$

$$q_{IN} - q_{OUT} = \frac{d}{dt}(Volum) = \frac{d}{dt}(Ah) = A\dot{h}$$

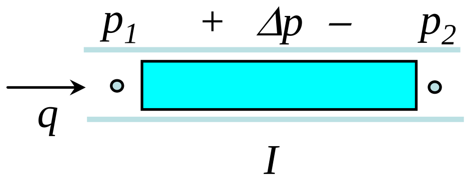
$$\dot{p}_{Cr} = \rho g \dot{h}$$

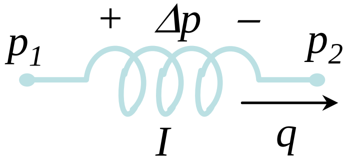
$$\Rightarrow \quad C = \frac{q_{IN} - q_{OUT}}{\dot{p}_{Cr}} = \frac{A\dot{h}}{\rho g \dot{h}} = \frac{A}{\rho g}$$

Hydraulic Fluid system Building Blocks

- **Fluid Inertance (Inductance)**

Describes any physical element with the characteristic that the pressure drop, Δp , across the element is proportional to the rate of change of the flow rate, q.



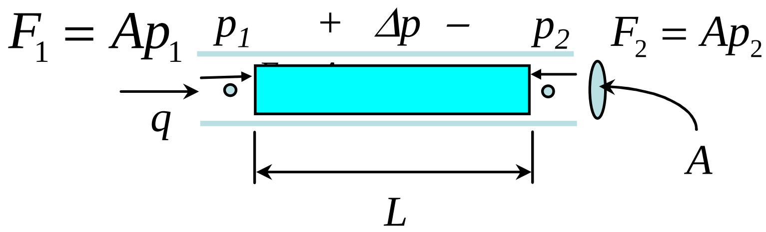


$$\Delta p = p_{12} = (p_1 - p_2) = I\frac{d}{dt}q = I \cdot \dot{q}$$

Long pipes are examples of fluid inertances.

It is concerned with fluid mass acceleration

Ex: Consider a section of pipe with cross-sectional area A and length L, filled with fluid whose density is ρ :



Start with force balance: F = ma

$$\sum F = F_1 - F_2 = A\left(p_1 - p_2\right) = Ap_{12}$$

$$m = \rho LA$$

$$\underbrace{Ap_{12}}_{\sum F} = \underbrace{\rho AL}_{m}\underbrace{\frac{dv}{dt}}_{a} = \rho AL\frac{d}{dt}\left(\frac{q}{A}\right)$$

$$p_{12} = \underbrace{\frac{\rho L}{A}}_{I}\frac{dq}{dt}$$

$$\Rightarrow \qquad I = \frac{\rho L}{A}$$

I is the hydraulic inertance

Pneumatic Fluid system Building Blocks

- **Pneumatic has the same three basic building blocks with hydraulic systems.**
- **Gases differ from liquids in being compressible i.e. change in pressure causes change in volume and hence density:**
- **The basic blocks are:**
- **Pneumatic Resistance,**
- **Pneumatic capacitance, &**
- **Pneumatic Inertance**

Pneumatic Fluid system Building Blocks

- Pneumatic Resistance:
- It is defined in terms of the mass rate of flow

$$P_1 - P_2 = R\frac{dm}{dt} = Rm^{\bullet}$$

m: mass of the gas; P1-P2: pressure difference; R: resistance

Pneumatic Fluid system Building Blocks

- **Pneumatic capacitance C**: is due to compressibility of the gas in some volume

Rate of change of mass inside the container is:

$$(\mathrm{d}m_1/\mathrm{d}t - \mathrm{d}m_2/\mathrm{d}t). = \frac{\mathrm{d}(\rho V)}{\mathrm{d}t} = \rho\frac{\mathrm{d}V}{\mathrm{d}t} + V\frac{\mathrm{d}\rho}{\mathrm{d}t}$$

Since $(\mathrm{d}V/\mathrm{d}t) = (\mathrm{d}V/\mathrm{d}p)(\mathrm{d}p/\mathrm{d}t)$ and, for an ideal gas, $pV = mRT$ with consequently $p = (m/V)RT = \rho RT$ and $\mathrm{d}\rho/\mathrm{d}t = (1/RT)(\mathrm{d}p/\mathrm{d}t)$, then

$$\text{rate of change of mass in container} = \rho\frac{\mathrm{d}V}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{V}{RT}\frac{\mathrm{d}p}{\mathrm{d}t}$$

where R is the gas constant and T the temperature, assumed to be constant, on the Kelvin scale. Thus

$$\frac{\mathrm{d}m_1}{\mathrm{d}t} - \frac{\mathrm{d}m_2}{\mathrm{d}t} = \left(\rho\frac{\mathrm{d}V}{\mathrm{d}p} + \frac{V}{RT}\right)\frac{\mathrm{d}p}{\mathrm{d}t}$$

Pneumatic Fluid system Building Blocks

$$\frac{dm_1}{dt} - \frac{dm_2}{dt} = \left(\rho \frac{dV}{dp} + \frac{V}{RT}\right)\frac{dp}{dt}$$

The pneumatic capacitance due to the change in volume of the container C_1 is defined as

$$C_1 = \rho \frac{dV}{dp}$$

and the pneumatic capacitance due to the compressibility of the gas C_2 as

$$C_2 = \frac{V}{RT}$$

Hence

$$\frac{dm_1}{dt} - \frac{dm_2}{dt} = (C_1 + C_2)\frac{dp}{dt}$$

or

$$p_1 - p_2 = \frac{1}{C_1 + C_2}\int (\dot{m}_1 - \dot{m}_2)\,dt$$

Pneumatic Fluid system Building Blocks

- **Pneumatic inertance:** is due to the pressure drop necessary to accelerate a block of gas

$$(p_1 - p_2)A = \frac{d(mv)}{dt}$$

$$mv = \rho LA\frac{q}{A} = \rho Lq$$

and so

$$(p_1 - p_2)A = L\frac{d(\rho q)}{dt}$$

But $\dot{m} = \rho q$ and so

$$p_1 - p_2 = \frac{L}{A}\frac{d\dot{m}}{dt}$$

$$p_1 - p_2 = I\frac{d\dot{m}}{dt}$$

with the pneumatic inertance I being $I = L/A$.

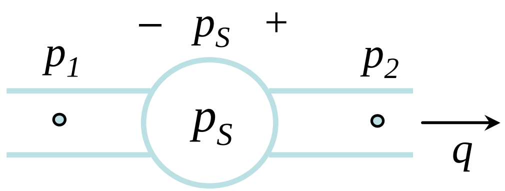
Building block	Describing equation	Energy stored or power dissipated
Hydraulic		
Inertance	$q = \frac{1}{I}\int (p_1 - p_2)\,\mathrm{d}t$	$E = \frac{1}{2}Iq^2$
	$p = L\frac{\mathrm{d}q}{\mathrm{d}t}$	
Capacitance	$q = C\frac{\mathrm{d}(p_1 - p_2)}{\mathrm{d}t}$	$E = \frac{1}{2}C(p_1 - p_2)^2$
Resistance	$q = \frac{p_1 - p_2}{R}$	$P = \frac{1}{R}(p_1 - p_2)^2$
Pneumatic		
Inertance	$\dot{m} = \frac{1}{I}\int (p_1 - p_2)\,\mathrm{d}t$	$E = \frac{1}{2}I\dot{m}^2$
Capacitance	$\dot{m} = C\frac{\mathrm{d}(p_1 - p_2)}{\mathrm{d}t}$	$E = \frac{1}{2}C(p_1 - p_2)^2$
Resistance	$\dot{m} = \frac{p_1 - p_2}{R}$	$P = \frac{1}{R}(p_1 - p_2)^2$

Table 10.3 Hydraulic and pneumatic building blocks

Fluid system Building Blocks

- **Pressure Source (Pump)**
  - An ideal pressure source of a hydraulic system is capable of maintaining the desired pressure, regardless of the flow required for what it is driving.

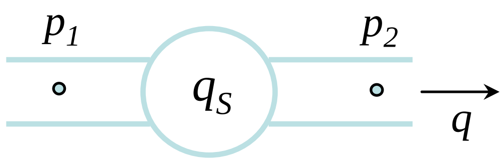
Voltage Source



$$p_{21} = p_2 - p_1 = p_S$$

- **Flow Source (Pump)**
  - An ideal flow source is capable of delivering the desired flow rate, regardless of the pressure required to drive the load.

Current Source



$$q = q_S$$

Building up a model for a fluid system

Example1

For the shown simple hydraulic system derive an expression for the height of the fluid in the container. Consider the system consist of a capacitor, the liquid in the container, with a resistor and a valve

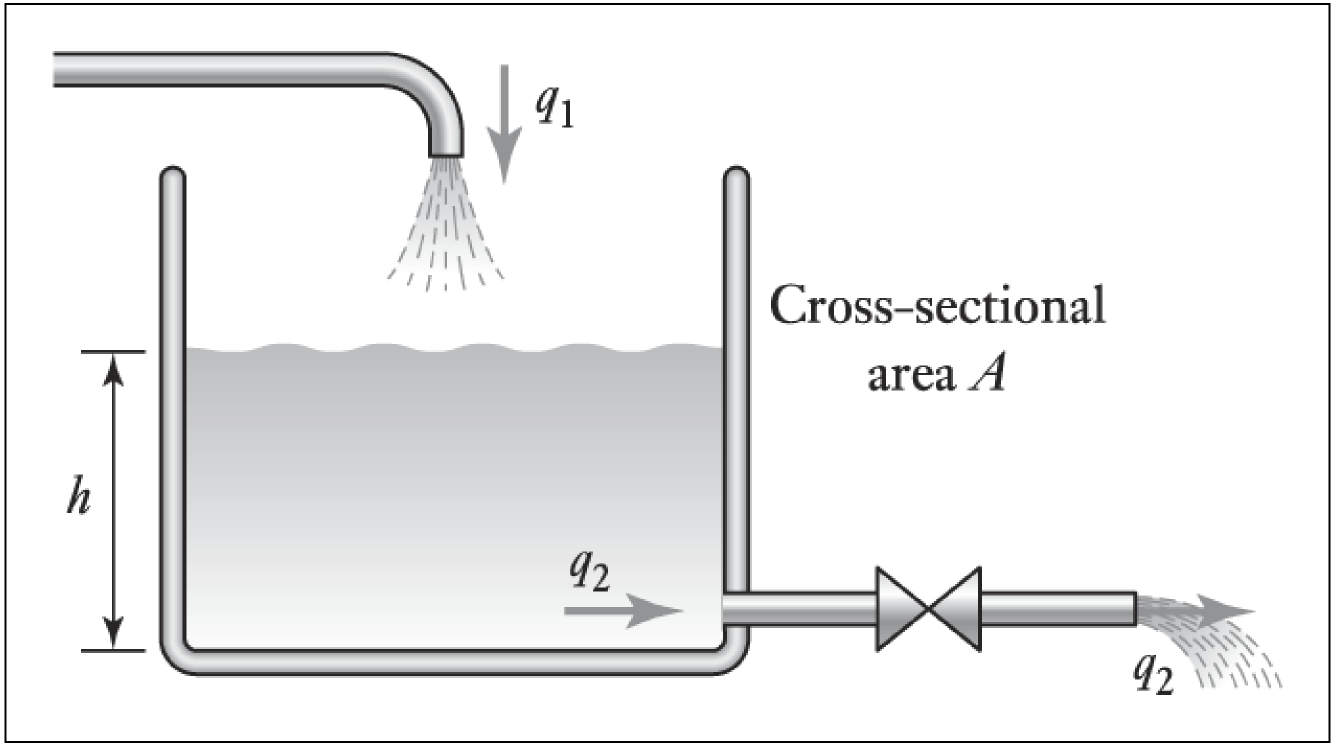


Figure 10.14 A fluid system

Solution:

Inertance can be neglected since flow rates change only very slowly. For the capacitor we can write

$$q_1 - q_2 = C\frac{dp}{dt}$$

The rate at which liquid leaves the container q_2 equals the rate at which it leaves the valve. Thus for the resistor

$$p_1 - p_2 = Rq_2$$

The pressure difference $(p_1 - p_2)$ is the pressure due to the height of liquid in the container and is thus $h\rho g$. Thus $q_2 = h\rho g/R$ and so substituting for q_2 in the first equation gives

$$q_1 - \frac{h\rho g}{R} = C\frac{d(h\rho g)}{dt}$$

and, since $C = A/\rho g$,

$$q_1 = A\frac{dh}{dt} + \frac{\rho g h}{R}$$

This equation describes how the height of liquid in the container depends on the rate of input of liquid into the container.

Building up a model for a fluid system

Example2:

For the shown hydraulic system derive expression for the fluid level in the two containers

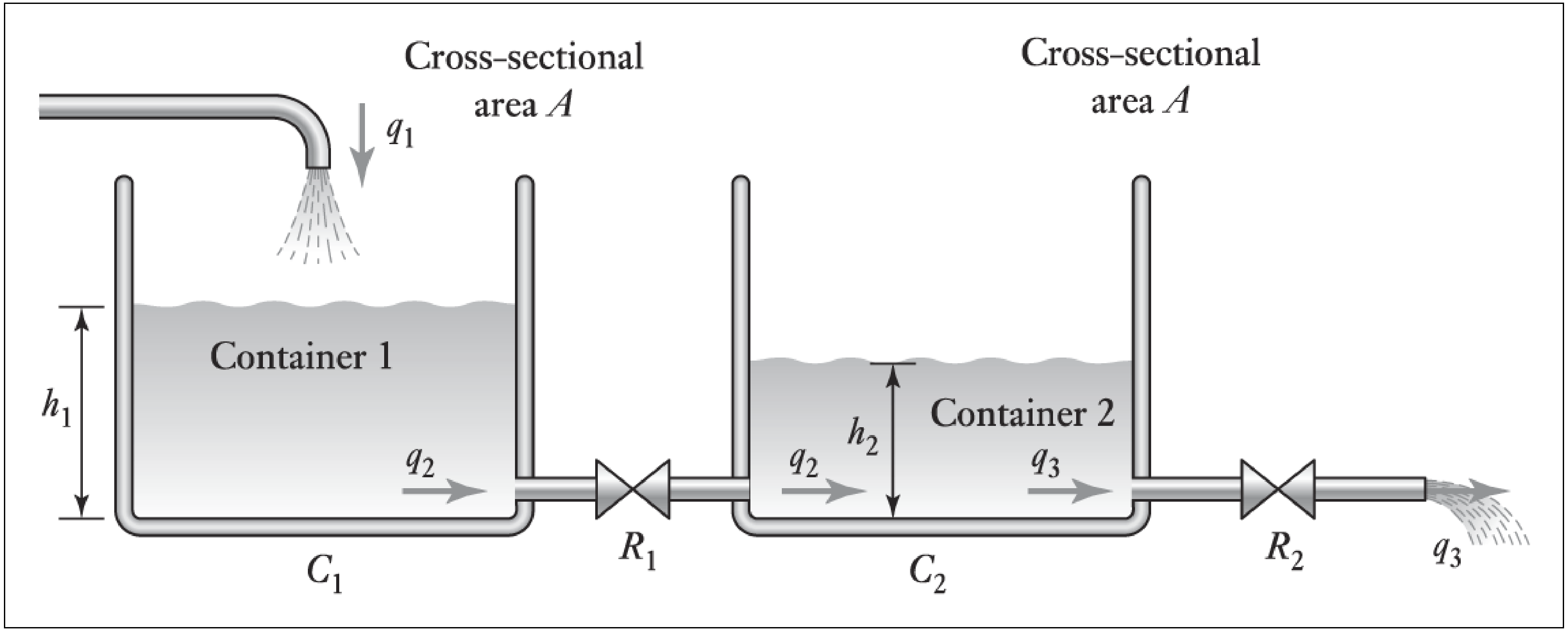


Figure 10.16 A fluid system

Solution2:

Container 1 is a capacitor and thus

$$q_1 - q_2 = C_1 \frac{dp}{dt}$$

where $p = h_1 \rho g$ and $C_1 = A_1/\rho g$ and so

$$q_1 - q_2 = A_1 \frac{dh_1}{dt}$$

The rate at which liquid leaves the container q_2 equals the rate at which it leaves the valve R_1. Thus for the resistor,

$$p_1 - p_2 = R_1 q_2$$

The pressures are $h_1 \rho g$ and $h_2 \rho g$. Thus

$$(h_1 - h_2)\rho g = R_1 q_2$$

Using the value of q_2 given by this equation and substituting it into the earlier equation gives

$$q_1 - \frac{(h_1 - h_2)\rho g}{R_1} = A_1 \frac{dh_1}{dt}$$

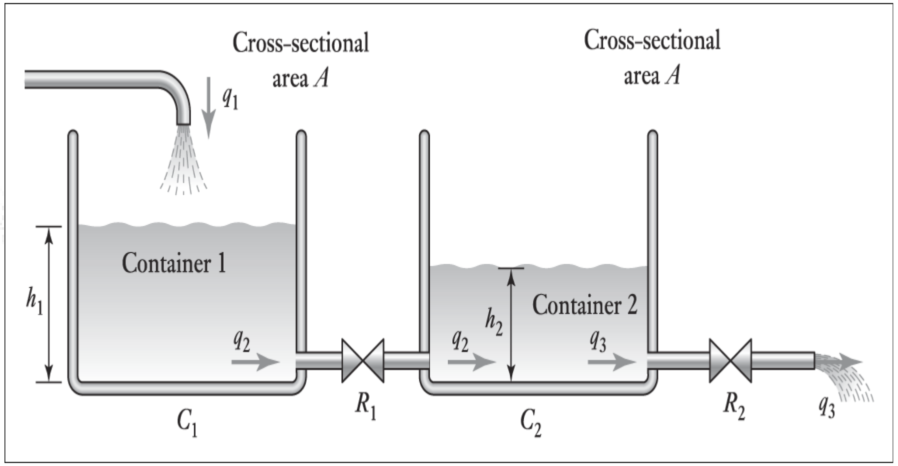
This equation describes how the height of the liquid in container 1 depends on the input rate of flow.

For container 2 a similar set of equations can be derived. Thus for the capacitor C_2,

$$q_2 - q_3 = C_2 \frac{dp}{dt}$$

where $p = h_2 \rho g$ and $C_2 = A_2/\rho g$ and so

$$q_2 - q_3 = A_2 \frac{dh_2}{dt}$$



The rate at which liquid leaves the container q_3 equals the rate at which it leaves the valve R_2. Thus for the resistor,

$$p_2 - 0 = R_2 q_3$$

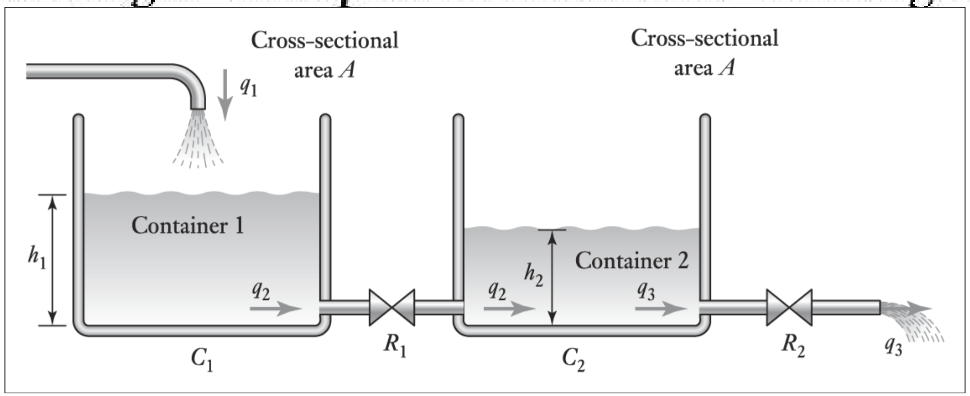
This assumes that the liquid exits into the atmosphere. Thus, using the value of q_3 given by this equation and substituting it into the earlier equation gives

$$q_2 - \frac{h_2 \rho g}{R_2} = A_2 \frac{dh_2}{dt}$$

Substituting for q_2 in this equation using the value given by the equation derived for the first container gives

$$\frac{(h_1 - h_2)\rho g}{R_1} - \frac{h_2 \rho g}{R_2} = A_2 \frac{dh_2}{dt}$$

This equation describes how the height of liquid in container 2 changes.



Model for a fluid system Example3:

A bellows is an example of a simple pneumatic system (Figure 10.15). Resistance is provided by a constriction which restricts the rate of flow of gas into the bellows and capacitance is provided by the bellows itself. Inertance can be neglected since the flow rate changes only slowly.

The mass flow rate into the bellows is given by

$$p_1 - p_2 = R\dot{m}$$

capacitance of the bellows is given by

$$\dot{m}_1 - \dot{m}_2 = (C_1 + C_2)\frac{dp_2}{dt}$$

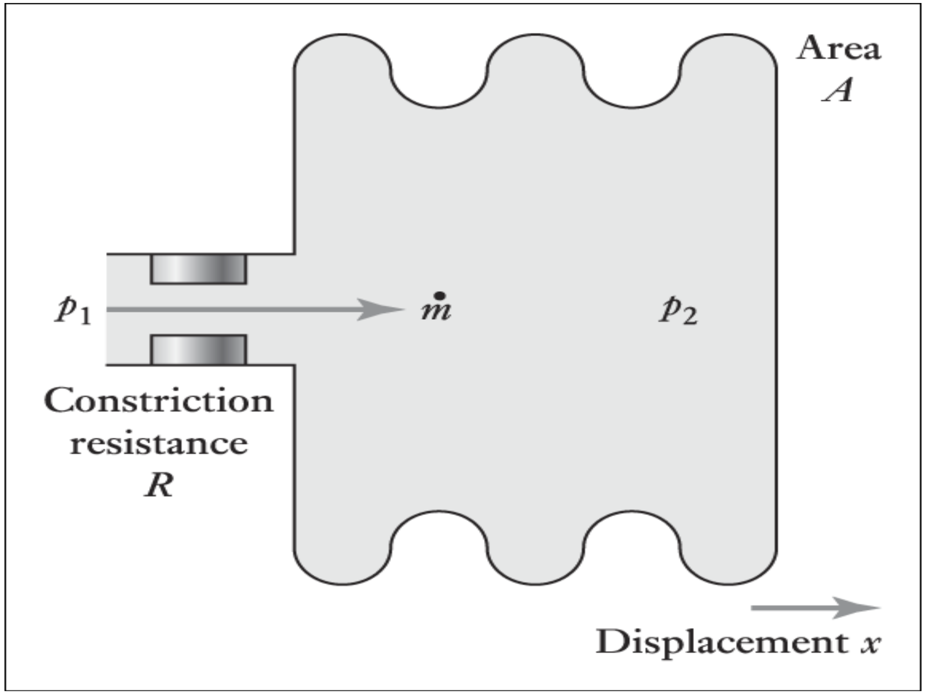


Figure 10.15 A pneumatic system

Model for a fluid system Example3:

The mass flow rate entering the bellows is given by the equation for the resistance and the mass leaving the bellows is zero. Thus

$$\frac{p_1 - p_2}{R} = (C_1 + C_2)\frac{dp_2}{dt}$$

Hence

$$p_1 = R(C_1 + C_2)\frac{dp_2}{dt} + p_2$$

This equation describes how the pressure in the bellows p_2 varies with time when there is an input of a pressure p_1.

Variables

- q : volumetric flow rate **[m³/sec]** (current)
- V: volume **[m³]** (charge)
- p : pressure **[N/m²]** (voltage)

The analogy between the hydraulic system and the electrical system will be used often. Just as in electrical systems, the flow rate (current) is defined to be the time rate of change (derivative) of volume (charge):

$$q = \frac{d}{dt}V = \dot{V}$$

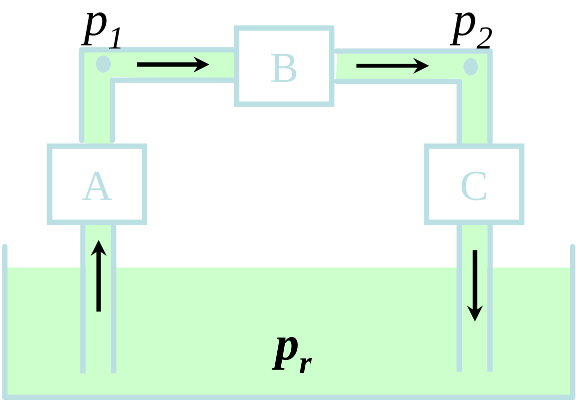
The pressure, p, used in this chapter is the *absolute pressure*. You need to be careful in determining whether the pressure is the absolute pressure or *gauge pressure*, p^*. Gauge pressure is the difference between the absolute pressure and the atmospheric pressure, i.e.

$$p^* = p - p_{atmospheric}$$

Interconnection Laws

- **Compatibility Law**
  - The sum of the pressure drops around a loop must be zero.
  - Similar to the Kirchhoff's voltage law.

$$\sum_{\substack{\text{Closed}\\ \text{Loop}}} \Delta p_j = \sum_{\substack{\text{Closed}\\ \text{Loop}}} p_{ij} = 0$$

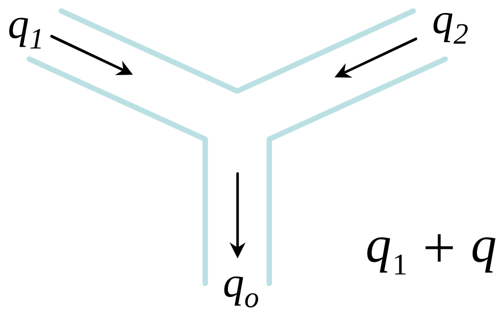


$$p_{r1} + p_{12} + p_{2r} = 0$$

- **Continuity Law**
  - The algebraic sum of the flow rates at any junction in the loop is zero.
  - This is the consequence of the conservation of mass.
  - Similar to the Kirchhoff's current law.

$$\sum_{\substack{\text{Any}\\ \text{Node}}} q_j = 0$$

$$\text{or} \quad \sum q_{IN} = \sum q_{OUT}$$



$$q_1 + q_2 = q_o$$

Thermal System building Blocks

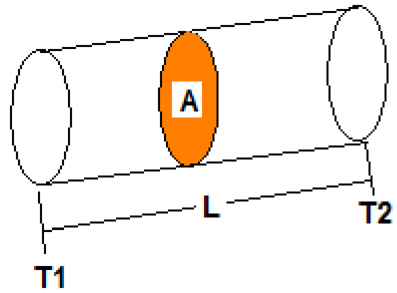
- Two basic building blocks: Resistance & capacitance
- The Thermal Resistance: is defined by the relation

$$q = \frac{T_2 - T_1}{R}$$

q: rate of heat flow

T2-T1: Temperature difference

R: Thermal resistance



The value of R depends on the mode of heat transfer

Conduction Mode:

$$R = \frac{L}{Ak}$$

K: thermal conductivity of the material through which conduction is taken place

L: length of the material

Convection Mode: in liquid and gasses

$$R = \frac{1}{Ah}$$

A: is the surface area across which there is temperature difference;
h: coefficient of heat transfer

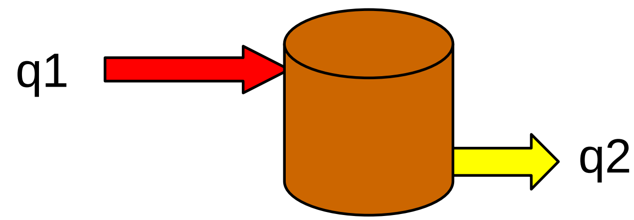
Thermal System building Blocks

- **Thermal capacitance**: is a measure of the store of internal energy in a system. It is defined by the following equation

$$q_1 - q_2 = C\frac{dT}{dt}$$

$q_1 - q_2$: rate of change of internal energy

C=cm is the thermal capacitance, **m** is the mass and **c** is the specific heat capacity



Building block	Describing equation	Energy stored
Capacitance	$q_1 - q_2 = C\dfrac{dT}{dt}$	$E = CT$
Resistance	$q = \dfrac{T_1 - T_2}{R}$	

Table 10.4 Thermal building blocks

Building up a Model for a Thermal system

Consider a thermometer at temperature T which has just been inserted in a liquid at temperature T_L (Figure 10.17).

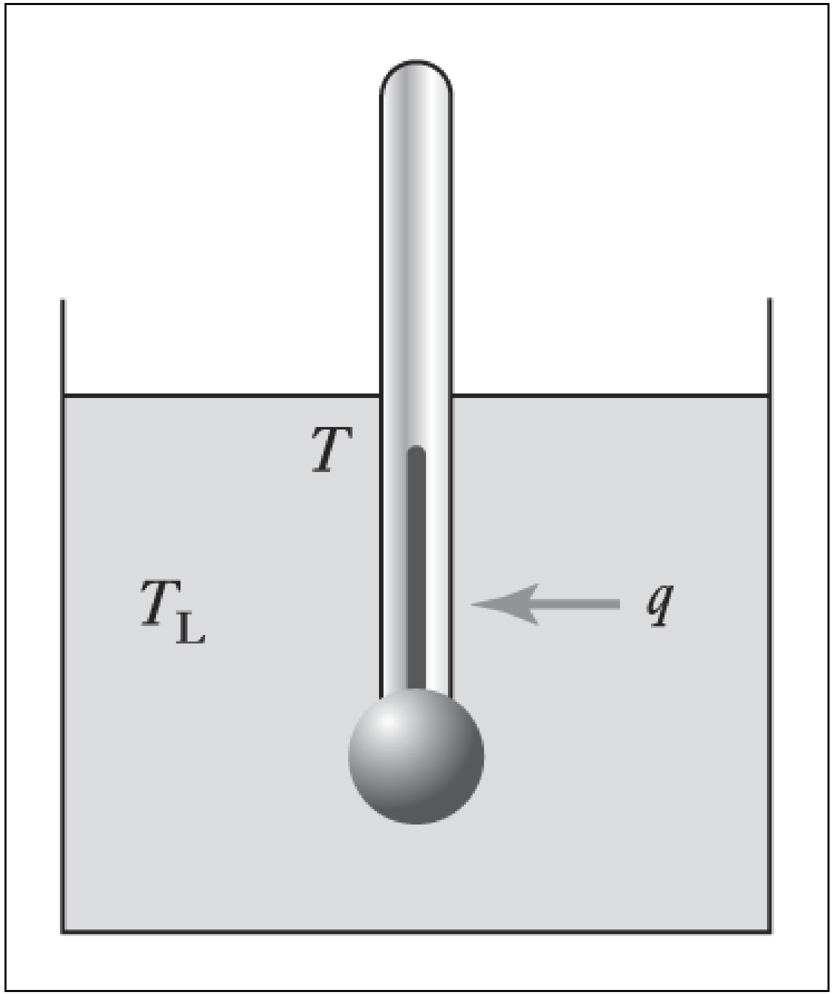


Figure 10.17 A thermal system

If the thermal resistance to heat flow from the liquid to the thermomet is R, then

$$q = \frac{T_L - T}{R}$$

where q is the net rate of heat flow from liquid to thermometer. The therm capacitance C of the thermometer is given by the equation

$$q_1 - q_2 = C\frac{dT}{dt}$$

Since there is only a net flow of heat from the liquid to the thermomet $q_1 = q$ and $q_2 = 0$. Thus

$$q = C\frac{dT}{dt}$$

Substituting this value of q in the earlier equation gives

$$C\frac{dT}{dt} = \frac{T_L - T}{R}$$

Rearranging this equation gives

$$RC\frac{dT}{dt} + T = T_L$$

This equation, a first-order differential equation, describes how the temperat

Thermal System: Example

consider Figure 10.18 which shows a thermal system consisting of an electric fire in a room. The fire emits heat at the rate q_1 and the room loses heat at the rate q_2. Assuming that the air in the room is at a uniform temperature T and that there is no heat storage in the walls of the room, derive an equation describing how the room temperature will change with time.

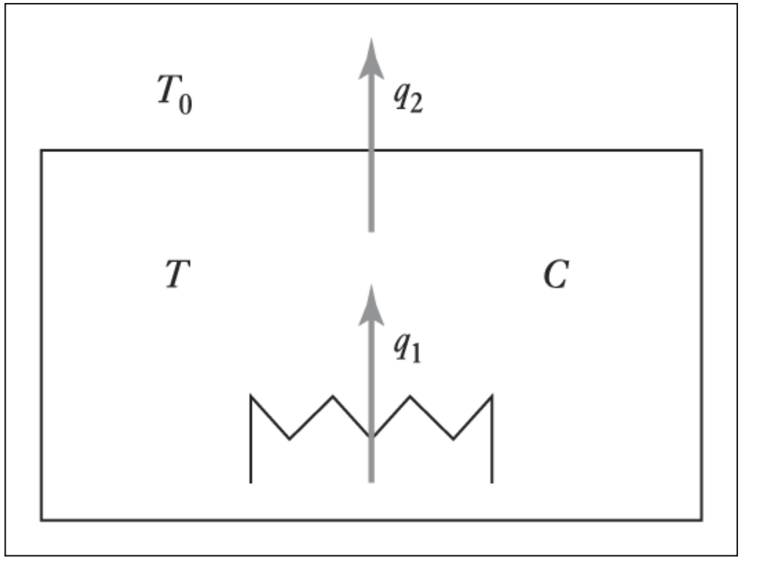


Figure 10.18 Thermal system

Thermal System: Example cont…

If the air in the room has a thermal capacity C then

$$q_1 - q_2 = C\frac{dT}{dt}$$

If the temperature inside the room is T and that outside the room T_0 then

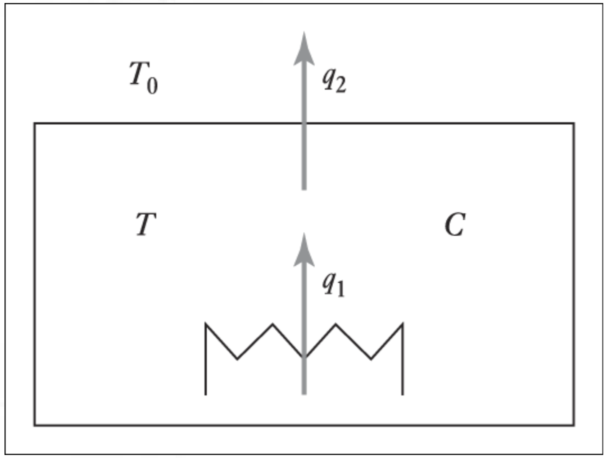
$$q_2 = \frac{T - T_0}{R}$$

where R is the resistivity of the walls. Substituting for q_2 gives

$$q_1 - \frac{T - T_0}{R} = C\frac{dT}{dt}$$

Hence

$$RC\frac{dT}{dt} + T = Rq_1 + T_0$$



	Mechanical (translational)	**Mechanical (rotational)**	**Electrical**	**Fluid (hydraulic)**	**Thermal**
Element	Mass	Moment of inertia	Capacitor	Capacitor	Capacitor
Equation	$F = m\frac{d^2x}{dt^2}$	$T = I\frac{d^2\theta}{dt^2}$			
	$F = m\frac{dv}{dt}$	$T = I\frac{d\omega}{dt}$	$i = C\frac{dv}{dt}$	$q = C\frac{d(p_1 - p_2)}{dt}$	$q_1 - q_2 = C\frac{dT}{dt}$
Energy	$E = \frac{1}{2}mv^2$	$E = \frac{1}{2}I\omega^2$	$E = \frac{1}{2}Cv^2$	$E = \frac{1}{2}C(p_1 - p_2)^2$	$E = CT$
Element	Spring	Spring	Inductor	Inertance	None
Equation	$F = kx$	$T = k\theta$	$v = L\frac{di}{dt}$	$p = L\frac{dq}{dt}$	
Energy	$E = \frac{1}{2}\frac{F^2}{k}$	$E = \frac{1}{2}\frac{T^2}{k}$	$E = \frac{1}{2}Li^2$	$E = \frac{1}{2}Iq^2$	
Element	Dashpot	Rotational damper	Resistor	Resistance	Resistance
Equation	$F = c\frac{dx}{dt} = cv$	$T = c\frac{d\theta}{dt} = c\omega$	$i = \frac{v}{R}$	$q = \frac{p_1 - p_2}{R}$	$q = \frac{T_1 - T_2}{R}$
Power	$P = cv^2$	$P = c\omega^2$	$P = \frac{v^2}{R}$	$P = \frac{1}{R}(p_1 - p_2)^2$	

Table 10.5 System elements