Basic system Models

- **Objectives:**
- **Devise Models from basic building blocks of mechanical, electrical, fluid and thermal systems**
- **Recognize analogies between mechanical, electrical, fluid and thermal systems**

Basic system Models

- **Mathematical Models**
- **Mechanical system building blocks**
  - Rotational systems
  - Building up a mechanical system

Electrical system building blocks

- Building up a model for electrical systems
- Electrical and mechanical analogyies

Fluid system building blocks

Thermal system building blocks

Mathematical Models

- In order to understand the behavior of systems, mathematical models are needed. Such a model is created using equations and can be used to enable predictions to be made of the behavior of a system under specific conditions.
- The basics for any mathematical model is provided by the fundamental physical laws that govern the behavior of the system.
- This chapter deals with basic building blocks and how to combine such blocks to build a mathematical system model.

Mechanical system building blocks

The models used to represent mechanical systems have the basic building blocks of:

Springs: represent the stiffness of a system

Dashpots: dashpots are the forces opposing motion, i.e. friction or damping

Masses: the inertia or resistance to acceleration

All these building blocks can be considered to have a force as an input and a displacement as an output

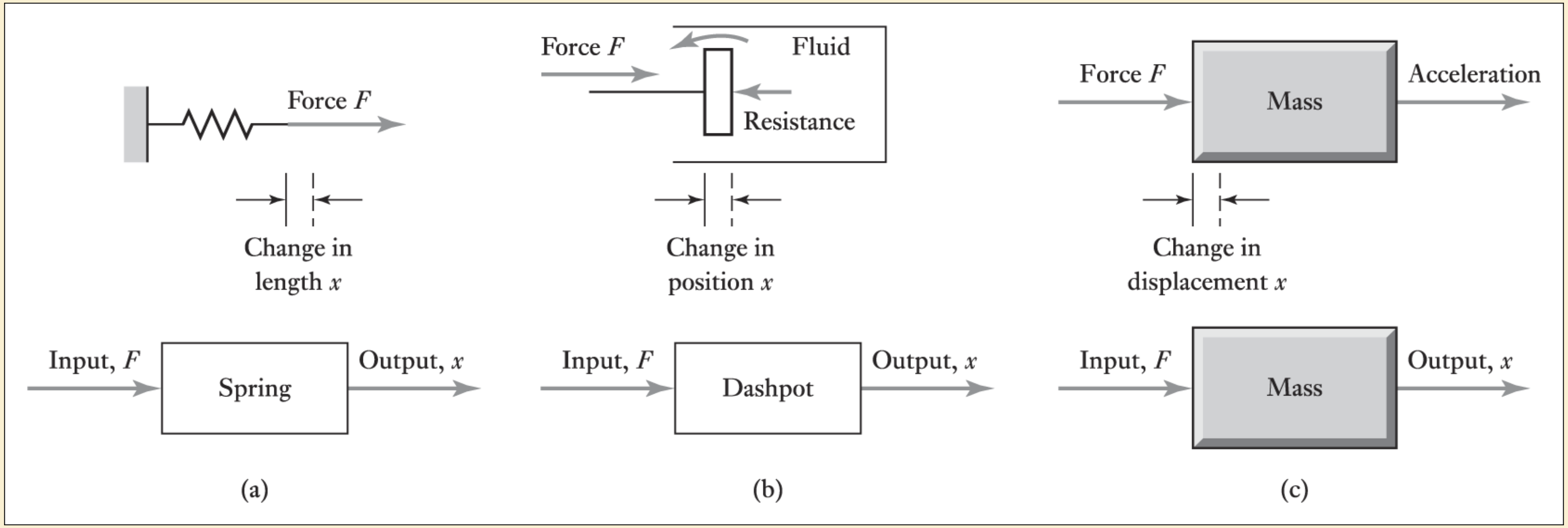


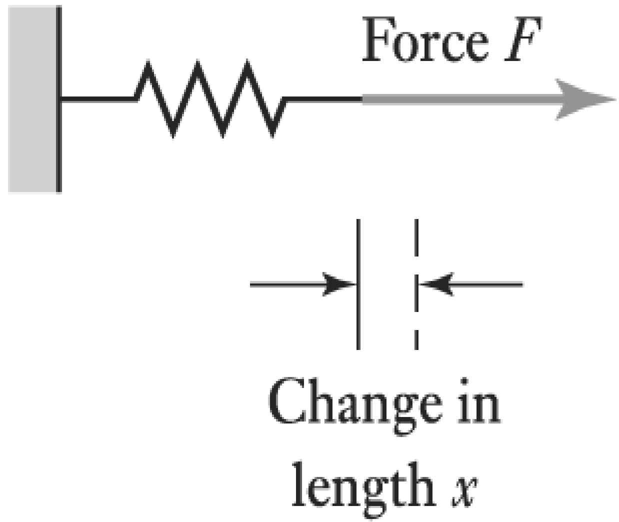
Figure 10.1 Mechanical systems: (a) spring, (b) dashpot, (c) mass

Mech. sys blocks: Spring

- The stiffness of a spring is described by:

$$F = k.x$$

The object applying the force to stretch the spring is also acted on by a force **(Newton's third law),** this force will be in the opposite direction and equal in size to the force used to stretch the spring



k is the stiffness constant

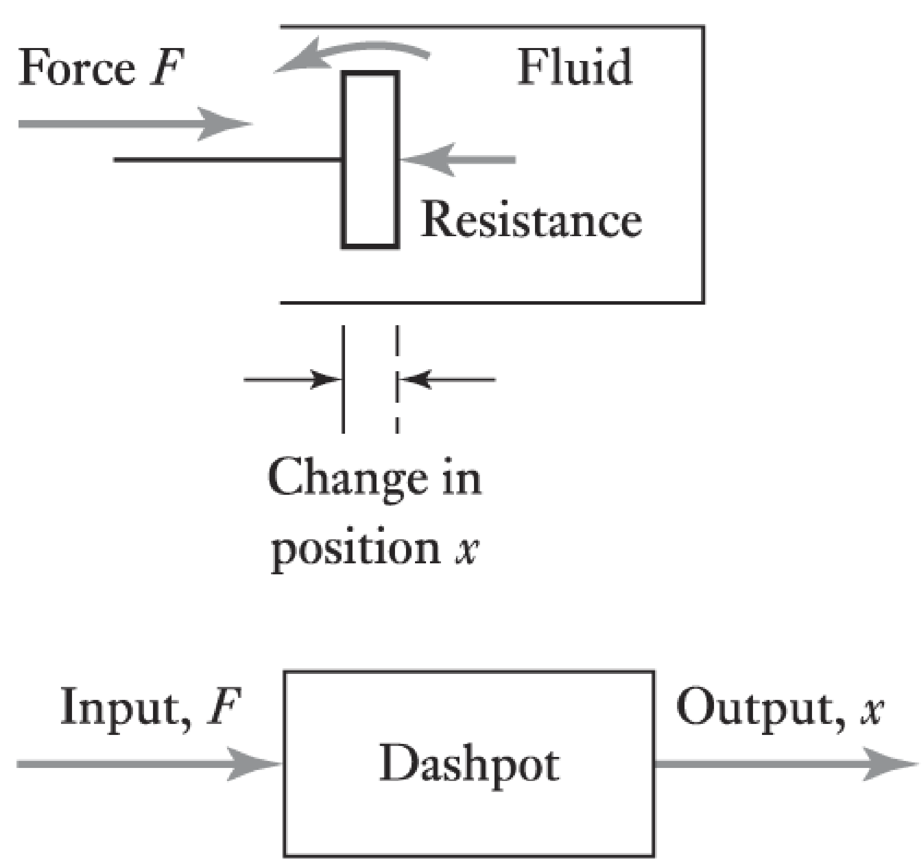
Mech. sys blocks: Dashpots

$$F = c\frac{dx}{dt} = cv$$

c : speed of the body

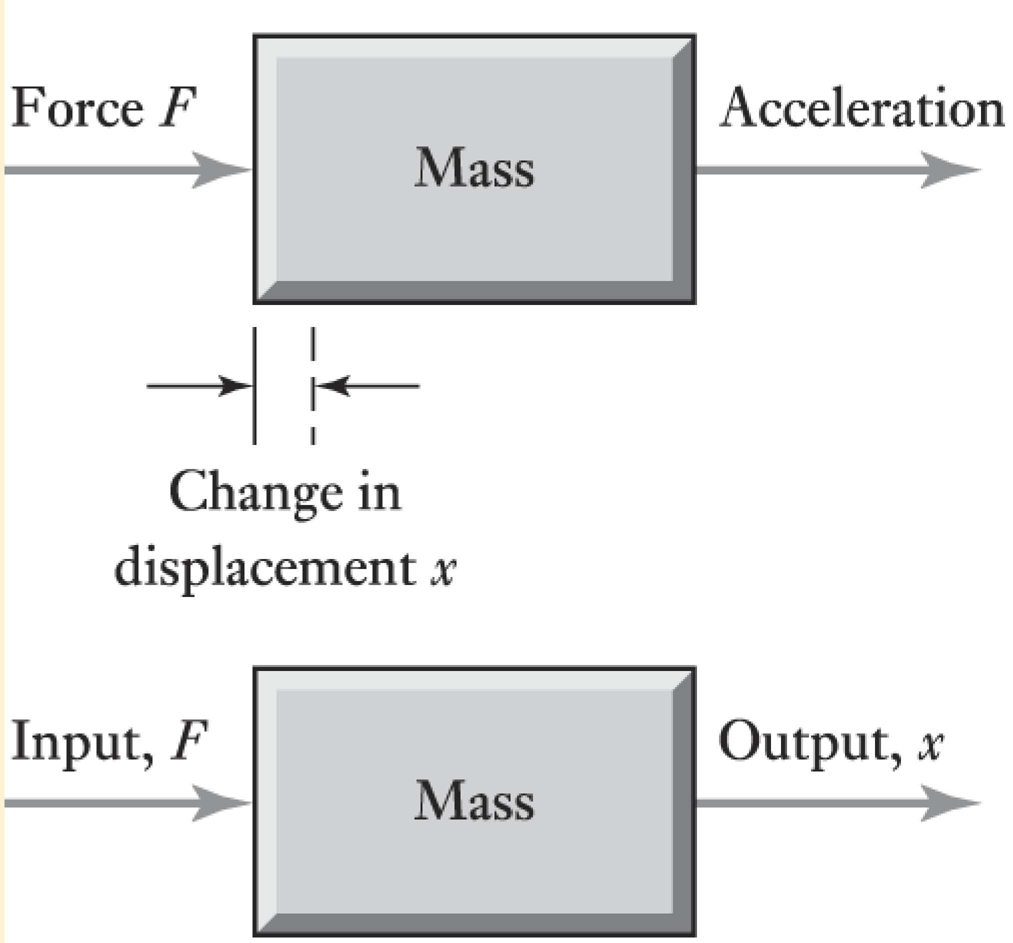
It is a type of forces when we push an object through a fluid or move an object against friction forces.

Thus the relation between the displacement x of the piston, i.e. the output and the force as input is a relationship depending on the rate of change of the output



Mech. sys blocks: Masses

$$F = m\frac{d^2x}{dt^2} = m\frac{dv}{dt}$$



- **F=ma**

m: mass, a: acceleration

Energy in basic mechanical blocks

- The spring when stretched stores energy, the energy being released when the spring springs back to its original length.

The energy stored when there is an extension x is:

$$E = kx^2/2 = E = \frac{1}{2}\frac{F^2}{k}$$

Energy stored in the mass when its moving with a velocity v, its called kinetic energy, and released when it stops moving:

$$E = mv^2/2$$

No stored energy in dashpot, it dissipates energy=cv^2

Basic Blocks or Rotational System

- For rotational system, the equivalent three building blocks are:

a **Torsion spring**, a rotary damper, and **the moment of inertia**

With such building blocks, the inputs are torque and the outputs angle rotated

With a torsional spring

$$T = k\theta$$

With a rotary damper a disc is rotated in a fluid and the resistive torque T is:

$$T = c\frac{\mathrm{d}\theta}{\mathrm{d}t} = c\omega$$

The moment of inertia has the property that the greater the moment of inertia I, the greater the torque needed to produce an angular acceleration

$$T = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = I\frac{\mathrm{d}\omega}{\mathrm{d}t}$$

Energy in rotary system

- The stored energy in rotary system:
- **For torsional spring:** $$E = \frac{1}{2}\frac{T^2}{k}$$
- **Energy stored in mass rotating is :**

$$E = \frac{1}{2} I\omega^2$$

- **The power dissipated by rotary damper when rotating with angular velocity ω is:**

$$P = c\omega^2$$

Summary of Mechanical building blocks

Building block	Describing equation	Energy stored or power dissipated
Translational		
Spring	$F = kx$	$E = \frac{1}{2}\frac{F^2}{k}$
Dashpot	$F = c\frac{dx}{dt} = cv$	$P = cv^2$
Mass	$F = m\frac{d^2x}{dt^2} = m\frac{dv}{dt}$	$E = \frac{1}{2}mv^2$
Rotational		
Spring	$T = k\theta$	$E = \frac{1}{2}\frac{T^2}{k}$
Rotational damper	$T = c\frac{d\theta}{dt} = c\omega$	$P = c\omega^2$
Moment of inertia	$T = I\frac{d^2\theta}{dt^2} = I\frac{d\omega}{dt}$	$E = \frac{1}{2}I\omega^2$

Table 10.1 Mechanical building blocks

Building up a mechanical system

Many systems can be considered to be a mass, a spring and dashpot combined in the way shown below

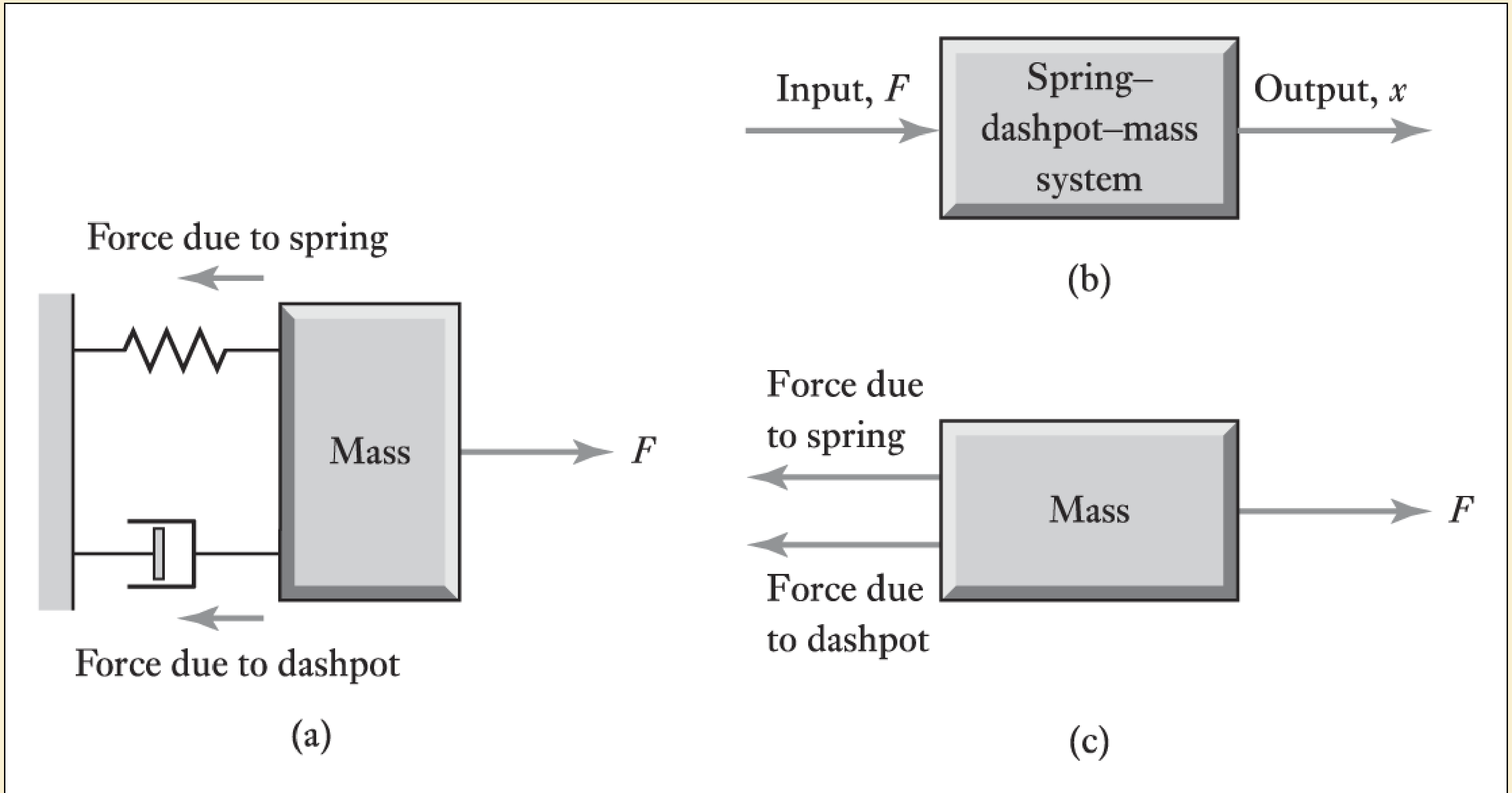


Figure 10.2 (a) Spring–dashpot–mass, (b) system, (c) free-body diagram

Building up a mechanical system

- The net forced applied to the mass m is **F-kx-cv**

V: is the velocity with which the piston (mass) is moving

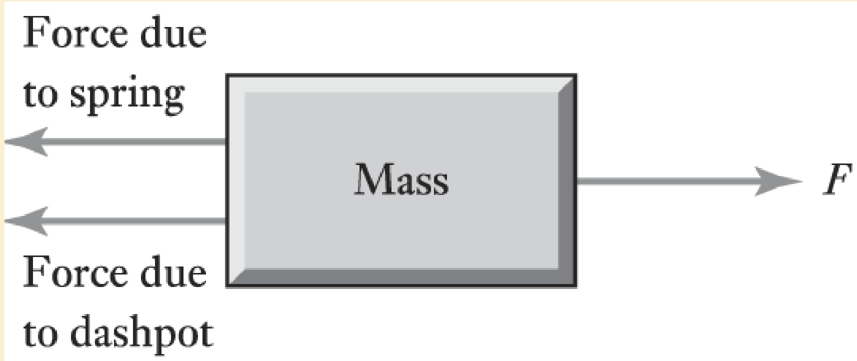
The net fore is the force applied to the mass to cause it to accelerate thus:

net force applied to mass =ma =

$$F - kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$or \quad m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F$$

2nd order differential equation describes the relationship between the input of force F to the system and the output of displacement x



Example of mechanical systems

The model in b can be used for the study of the behavior that could be expected of the vehicle when driven over a rough road and hence as a basis for the design of the vehicle suspension model

The model in C can be used as a part of a larger model to predict how the driver might feel when driving along a road

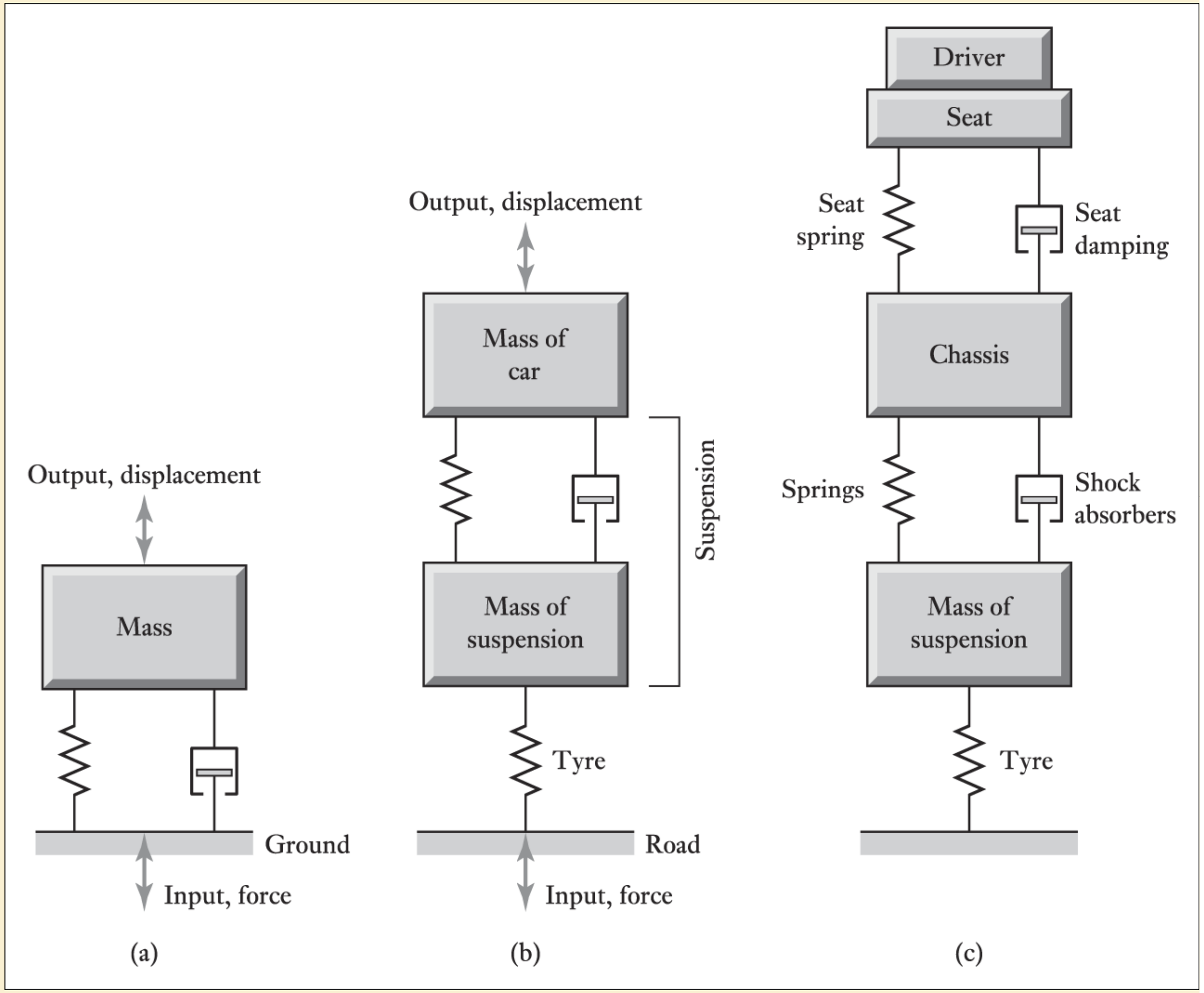


Figure 10.3 Model for (a) a machine mounted on the ground, (b) the chassis of a car as a result of a wheel moving along a road, (c) the driver of a car as it is driven along a road

Analysis of mechanical systems

The analysis of such systems is carried out by drawing a free-body diagram for each mass in the system, thereafter the system equations can be derived

The net force applied to the mass is F minus the resisting forces exerted by each of the springs. Since these are k_1x and k_2x, then

$$\text{net force} = F - k_1x - k_2x$$

Since the net force causes the mass to accelerate, then

$$\text{net force} = m\frac{d^2x}{dt^2}$$

Hence

$$m\frac{d^2x}{dt^2} + (k_1 + k_2)x = F$$

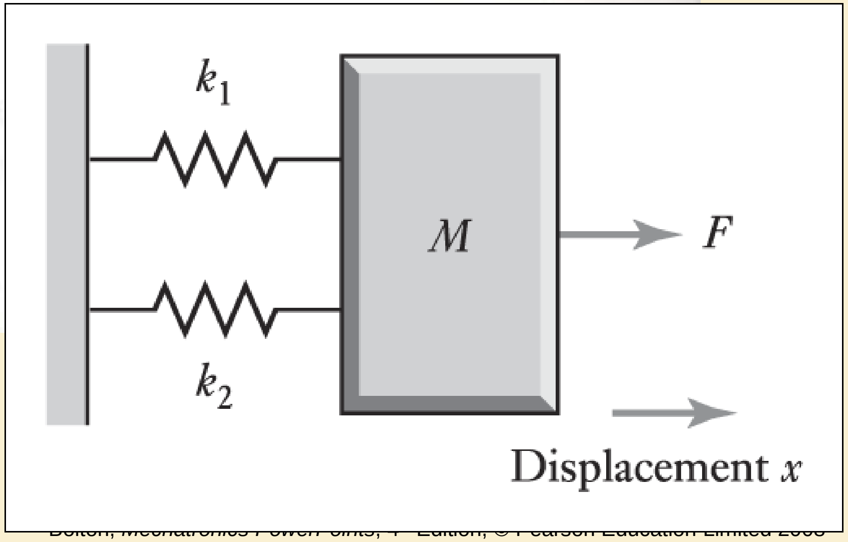
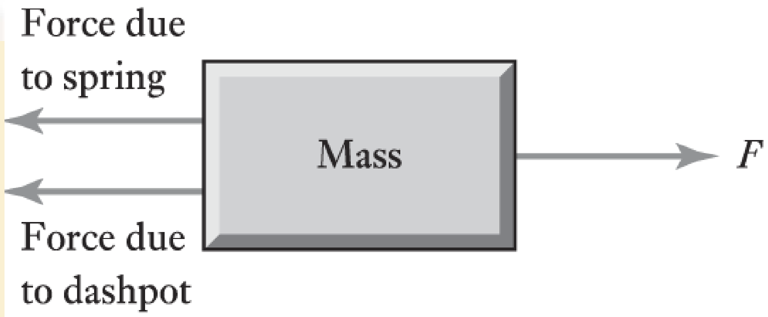


Figure 10.4 Example

- **Procedure to obtain the differential equation relating the inputs to the outputs for a mechanical system consisting of a number of components can be written as follows**

1 Isolate the various components in the system and draw free-body diagrams for each.
2 Hence, with the forces identified for a component, write the modelling equation for it.
3 Combine the equations for the various system components to obtain the system differential equation.



Example: derive the differential equations for the system in Figure

Consider the free body diagram

For the mass m2 we can write

$$\text{net force} = F - k_2(x_3 - x_2)$$

This force will cause the mass to accelerate and so

$$F - k_2(x_3 - x_2) = m_2\frac{d^2x_3}{dt}$$

For the free body diagram of mass m1 we can write

$$\text{net force} = k_1(x_2 - x_1) - k_2(x_3 - x_2)$$

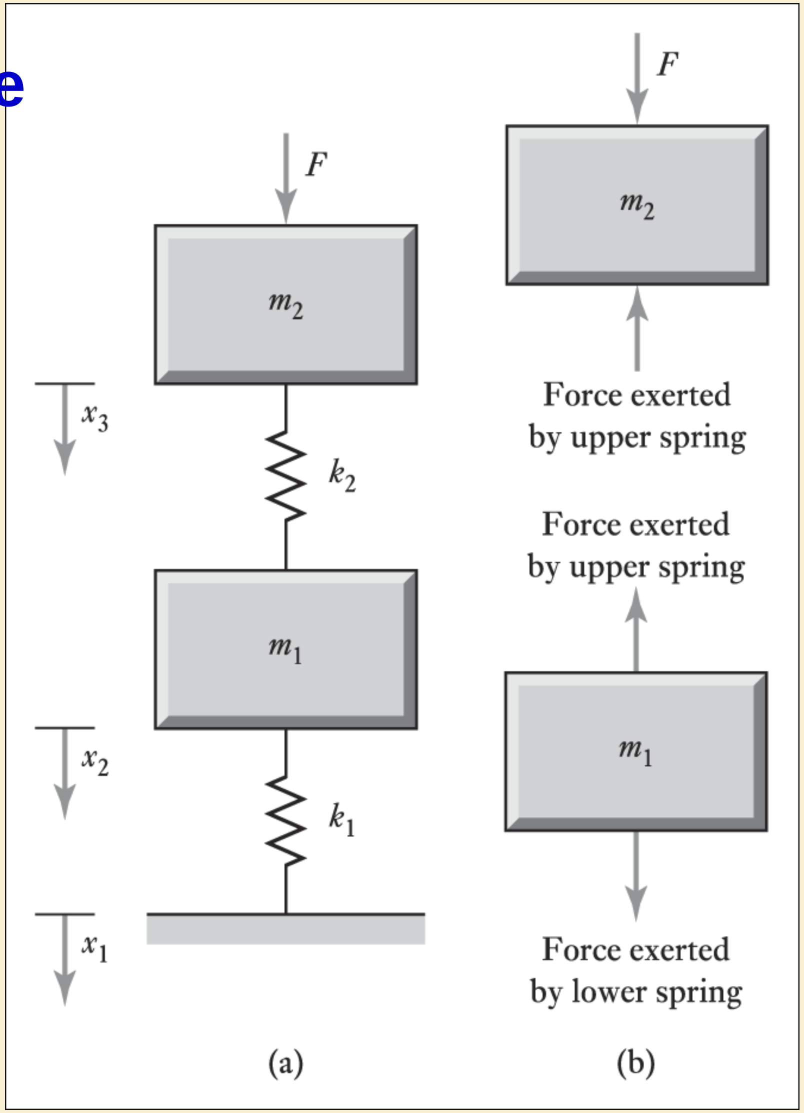
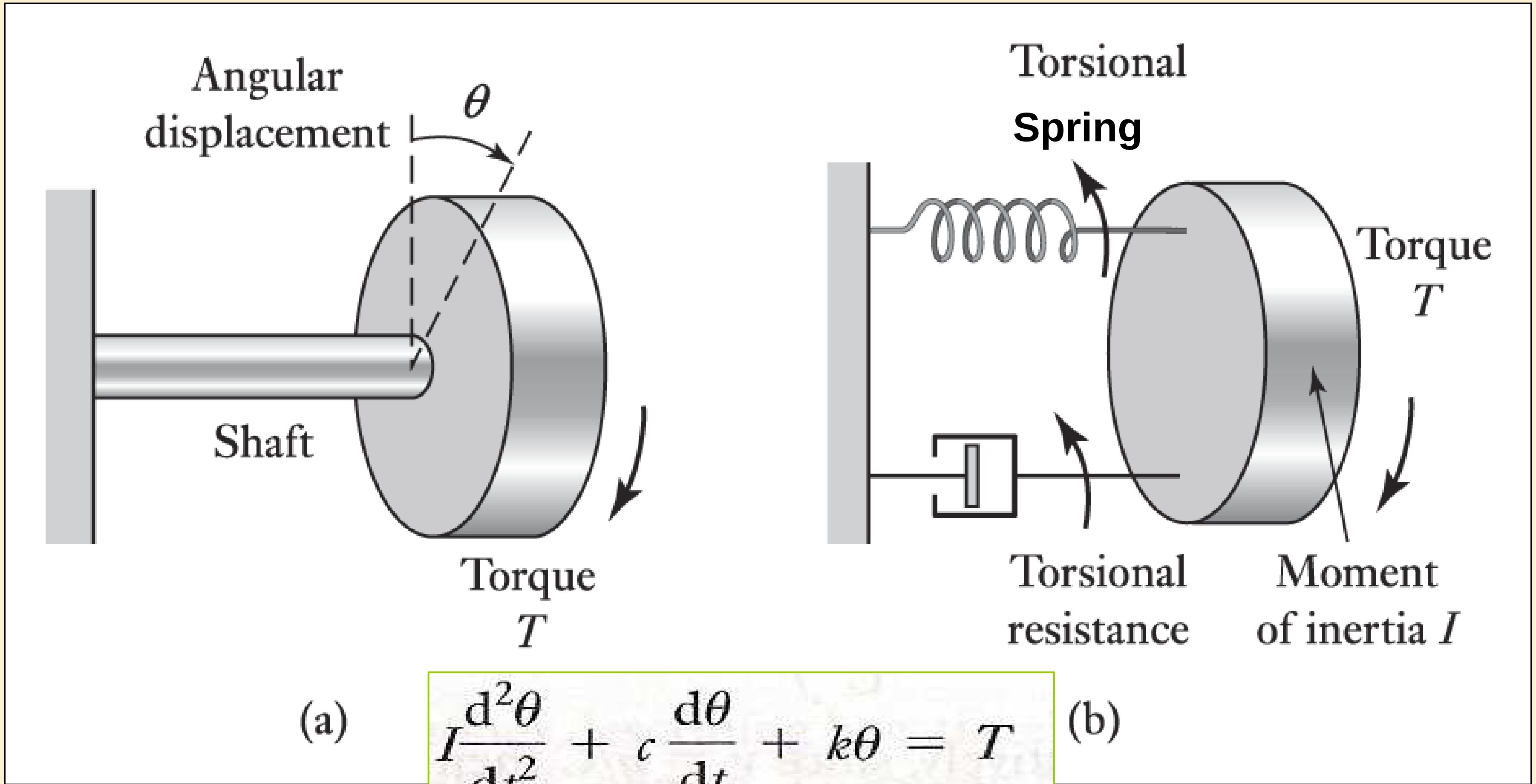


Figure 10.5 Mass–spring system

Rotary system analysis

The same analysis procedures can also be applied to rotary system, so just one rotational mass block and just the torque acting on the body are considered



$$I\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k\theta = T$$

Figure 10.6 Rotating a mass on the end of a shaft: (a) physical situation, (b) building block model

Electrical system building blocks

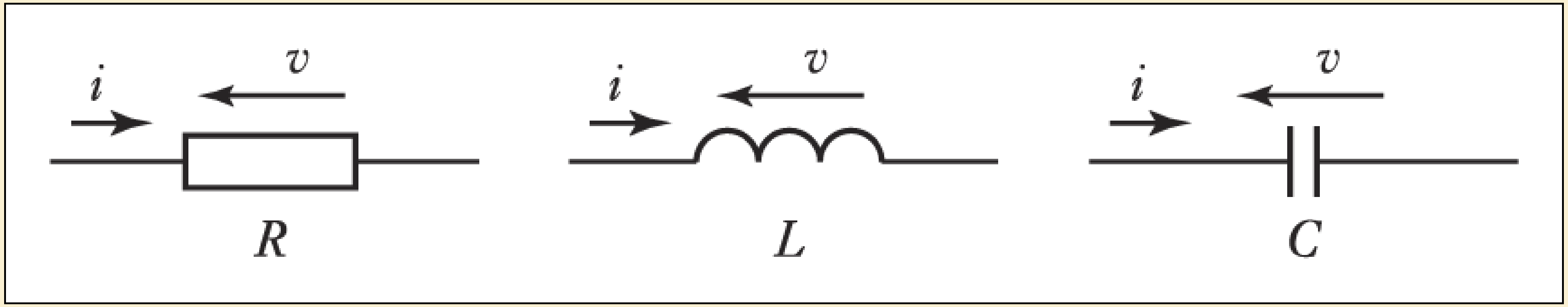


Figure 10.7 Electrical building blocks

Building block	Describing equation	Energy stored or power dissipated
Inductor	$i = \frac{1}{L}\int v\,dt$	$E = \frac{1}{2}Li^2$
	$v = L\frac{di}{dt}$	
Capacitor	$i = C\frac{dv}{dt}$	$E = \frac{1}{2}Cv^2$
Resistor	$i = \frac{v}{R}$	$P = \frac{v^2}{R}$

Table 10.2 Electrical building blocks

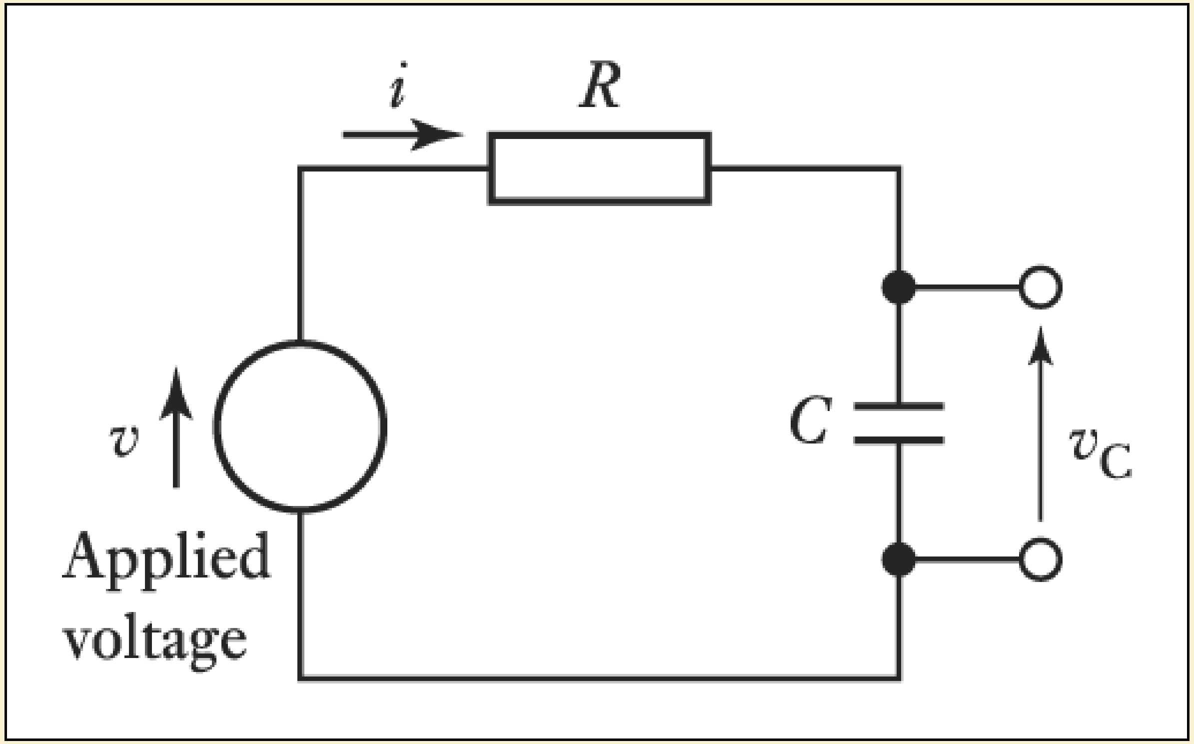


Figure 10.8 Resistor–capacitor system

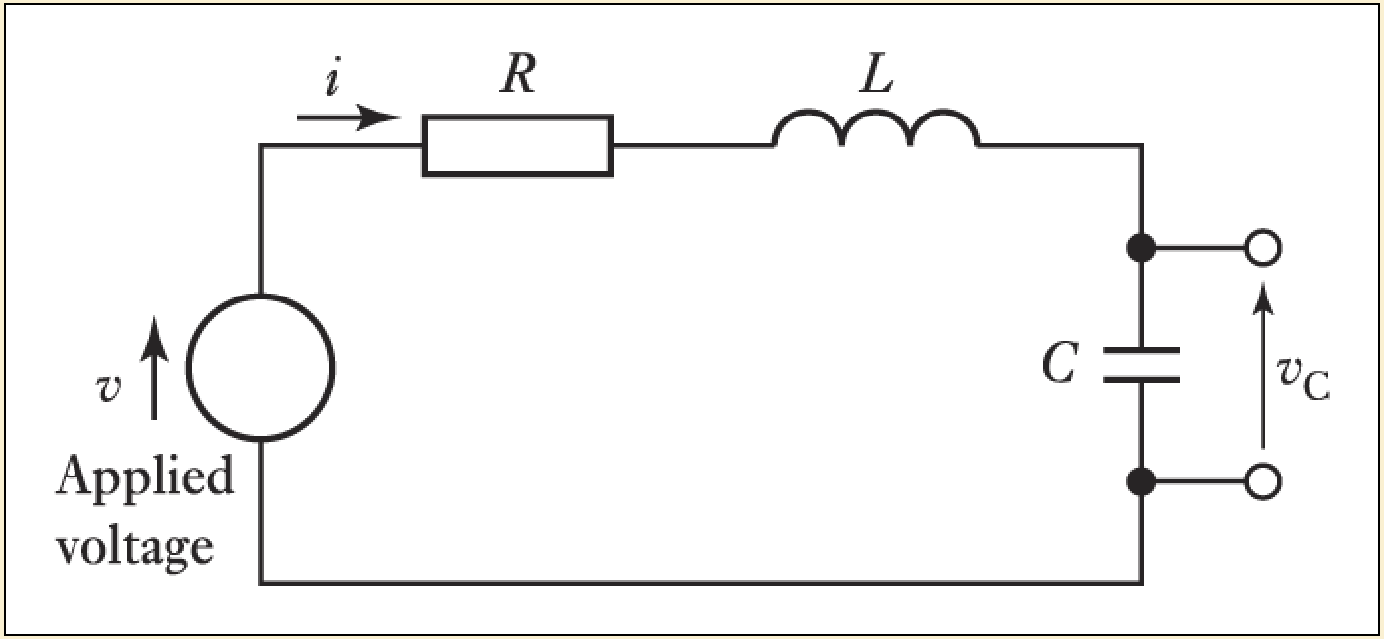


Figure 10.9 Resistor–inductor–capacitor system

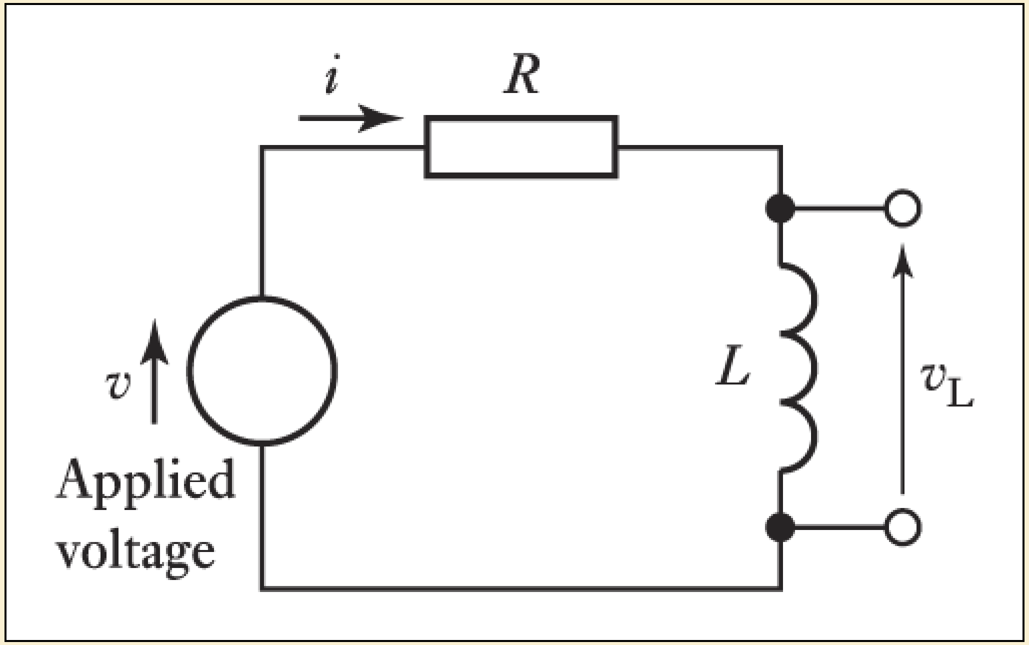
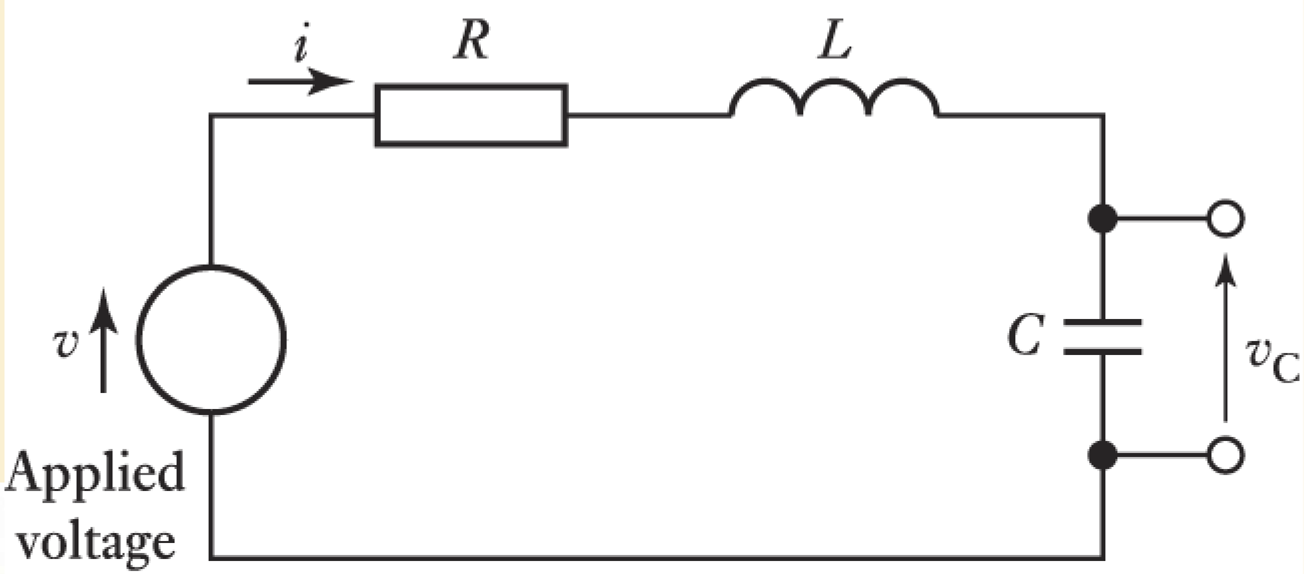


Figure 10.10 Resistor–inductor system

Electrical System Model
Resistor–capacitor–inductor system



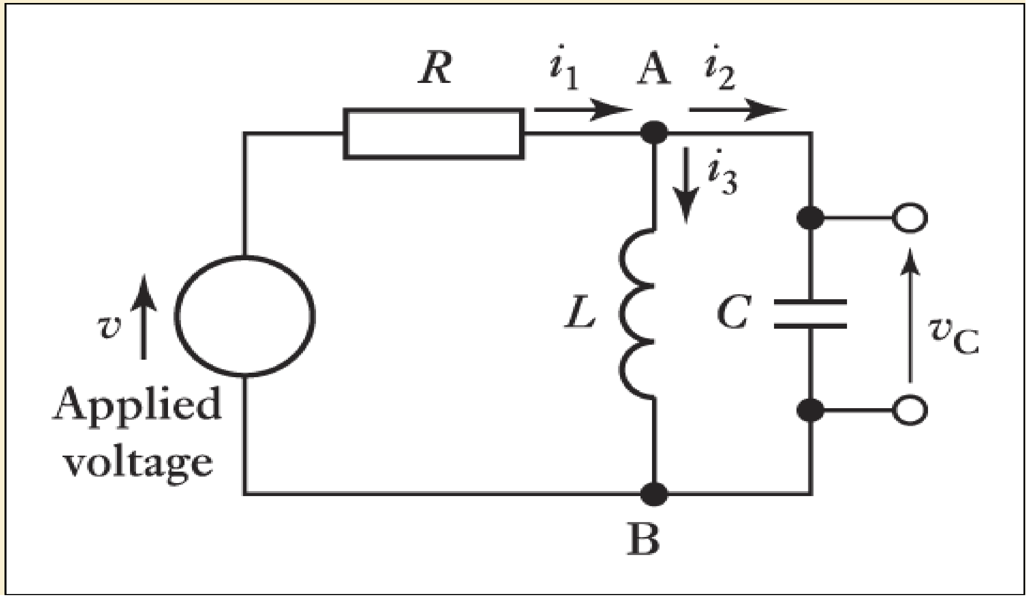
$$v = iR + L\frac{di}{dt} + v_C$$

But $i = C(dv_C/dt)$ and so

$$\frac{di}{dt} = C\frac{d(dv_C/dt)}{dt} = C\frac{d^2v_C}{dt^2}$$

Hence

$$v = RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2} + v_C$$



Electrical and Mechanical Analogy

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F \quad \text{and} \quad RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2} + v_C = v$$

The analogy between current and force is the one most often used. However, another set of analogies can be drawn between potential difference and force.

F	**I**
Velocity	**Volt**
C dashpot	**1/R**
Spring	**inductor**
Mass	**capacitor**

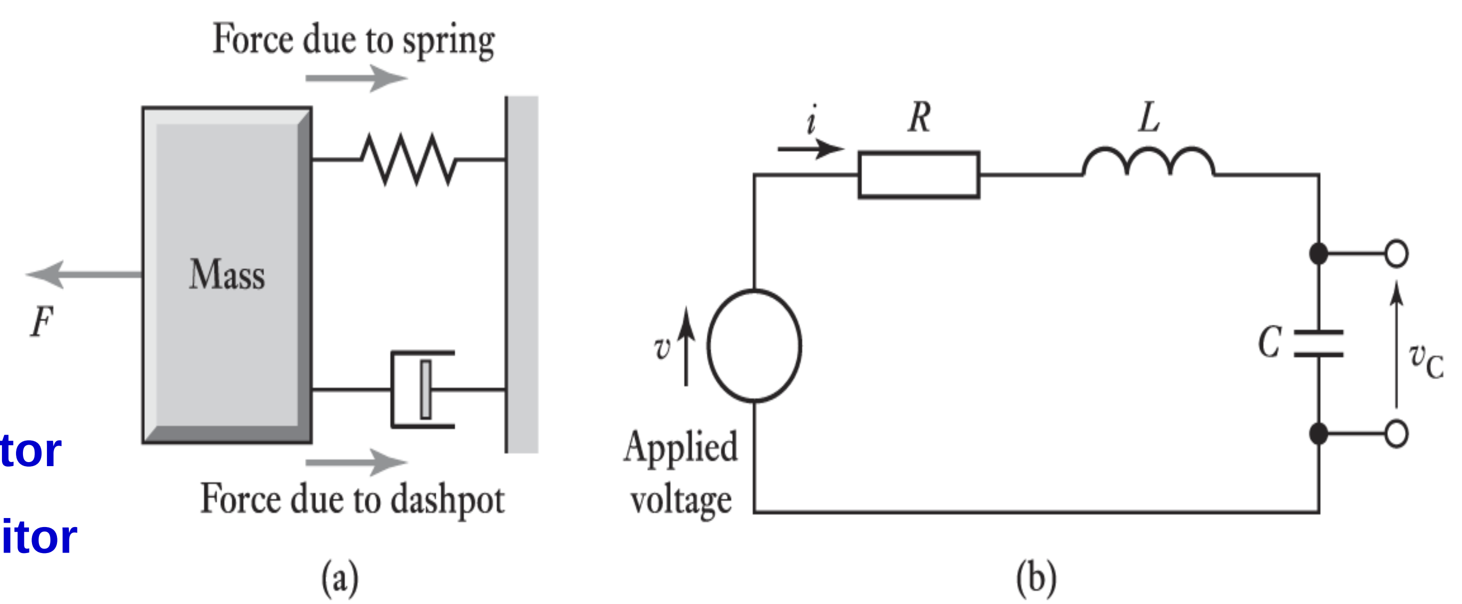


Figure 10.12 Analogous systems

	Mechanical (translational)	Mechanical (rotational)	Electrical
Element	Mass	Moment of inertia	Capacitor
Equation	$F = m\frac{d^2x}{dt^2}$	$T = I\frac{d^2\theta}{dt^2}$	
	$F = m\frac{dv}{dt}$	$T = I\frac{d\omega}{dt}$	$i = C\frac{dv}{dt}$
Energy	$E = \frac{1}{2}mv^2$	$E = \frac{1}{2}I\omega^2$	$E = \frac{1}{2}Cv^2$
Element	Spring	Spring	Inductor
Equation	$F = kx$	$T = k\theta$	$v = L\frac{di}{dt}$
Energy	$E = \frac{1}{2}\frac{F^2}{k}$	$E = \frac{1}{2}\frac{T^2}{k}$	$E = \frac{1}{2}Li^2$