OPERATIONS RESEARCH

LECTURE NOTES

Y. İlker Topcu, Prof. Dr.

Y. İlker Topcu, Ph.D. (www.ilkertopcu.net)

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Y. İlker Topcu, Ph.D. (www.ilkertopcu.net)

1. INTRODUCTION TO OR

1.1 TERMINOLOGY

The British/Europeans refer to "operational research", the Americans to "operations research" - but both are often shortened to just "OR" (which is the term we will use).

Another term which is used for this field is "management science" ("MS"). The Americans sometimes combine the terms OR and MS together and say "**OR/MS**" or "ORMS".

Yet other terms sometimes used are "industrial engineering" ("IE"), "decision science" ("DS"), and "problem solving".

In recent years there has been a move towards a standardization upon a single term for the field, namely the term "OR".

"Operations Research (Management Science) is a scientific approach to decision making that seeks to best design and operate a system, usually under conditions requiring the allocation of scarce resources."

A system is an organization of interdependent components that work together to accomplish the goal of the system.

1.2 THE METHODOLOGY OF OR

When OR is used to solve a problem of an organization, the following seven step procedure should be followed:

Step 1. Formulate the Problem

OR analyst first defines the organization's problem. Defining the problem includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

Step 2. Observe the System

Next, the analyst collects data to estimate the values of parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and evaluate (in Step 4) a mathematical model of the organization's problem.

Step 3. Formulate a Mathematical Model of the Problem

The analyst, then, develops a mathematical model (in other words an idealized representation) of the problem. In this class, we describe many mathematical techniques that can be used to model systems.

Step 4. Verify the Model and Use the Model for Prediction

The analyst now tries to determine if the mathematical model developed in Step 3 is an accurate representation of reality. To determine how well the model fits reality, one determines how valid the model is for the current situation.

Step 5. Select a Suitable Alternative

Given a model and a set of alternatives, the analyst chooses the alternative (if there is one) that best meets the organization's objectives.

Sometimes the set of alternatives is subject to certain restrictions and constraints. In many situations, the best alternative may be impossible or too costly to determine.

Step 6. Present the Results and Conclusions of the Study

In this step, the analyst presents the model and the recommendations from Step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the organization choose the decision maker(s) choose the one that best meets her/his/their needs.

After presenting the results of the OR study to the decision maker(s), the analyst may find that s/he does not (or they do not) approve of the recommendations. This may result from incorrect definition of the problem on hand or from failure to involve decision maker(s) from the start of the project. In this case, the analyst should return to Step 1, 2, or 3.

Step 7. Implement and Evaluate Recommendation

If the decision maker(s) has accepted the study, the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations are enabling decision maker(s) to meet her/his/their objectives.

1.3   HISTORY OF OR

(Prof. Beasley's lecture notes)

OR is a relatively new discipline. Whereas 70 years ago it would have been possible to study mathematics, physics or engineering (for example) at university it would not have been possible to study OR, indeed the term OR did not exist then. It was only

really in the late 1930's that operational research began in a systematic fashion, and it started in the UK.

Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained.

It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defense of Britain, was first created. It lacked however any effective fighter aircraft - no Hurricanes or Spitfires had come into service - and no radar data was yet fed into its very elementary warning and control system.

It had become clear that radar would create a whole new series of problems in fighter direction and control so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground based observer data for fighter interception, was the start of OR.

The first of three major pre-war air-defense exercises was carried out in the summer of 1937. The experimental radar station at Bawdsey Research Station was brought into operation and the information derived from it was fed into the general air-defense warning and control system. From the early warning point of view this exercise was encouraging, but the tracking information obtained from radar, after filtering and transmission through the control and display network, was not very satisfactory.

In July 1938 a second major air-defense exercise was carried out. Four additional radar stations had been installed along the coast and it was hoped that Britain now had an aircraft location and control system greatly improved both in coverage and effectiveness. Not so! The exercise revealed, rather, that a new and serious problem had arisen. This was the need to coordinate and correlate the additional, and often conflicting, information received from the additional radar stations. With the out-break of war apparently imminent, it was obvious that something new - drastic if necessary - had to be attempted. Some new approach was needed.

Accordingly, on the termination of the exercise, the Superintendent of Bawdsey Research Station, A.P. Rowe, announced that although the exercise had again demonstrated the technical feasibility of the radar system for detecting aircraft, its operational achievements still fell far short of requirements. He therefore proposed that a crash program of research into the operational - as opposed to the technical - aspects

of the system should begin immediately. The term "operational research" [RESEARCH into (military) OPERATIONS] was coined as a suitable description of this new branch of applied science. The first team was selected from amongst the scientists of the radar research group the same day.

In the summer of 1939 Britain held what was to be its last pre-war air defense exercise. It involved some 33,000 men, 1,300 aircraft, 110 antiaircraft guns, 700 searchlights, and 100 barrage balloons. This exercise showed a great improvement in the operation of the air defense warning and control system. The contribution made by the OR teams was so apparent that the Air Officer Commander-in-Chief RAF Fighter Command (Air Chief Marshal Sir Hugh Dowding) requested that, on the outbreak of war, they should be attached to his headquarters at Stanmore.

On May 15th 1940, with German forces advancing rapidly in France, Stanmore Research Section was asked to analyze a French request for ten additional fighter squadrons (12 aircraft a squadron) when losses were running at some three squadrons every two days. They prepared graphs for Winston Churchill (the British Prime Minister of the time), based upon a study of current daily losses and replacement rates, indicating how rapidly such a move would deplete fighter strength. No aircraft were sent and most of those currently in France were recalled.

This is held by some to be the most strategic contribution to the course of the war made by OR (as the aircraft and pilots saved were consequently available for the successful air defense of Britain, the Battle of Britain).

In 1941 an Operational Research Section (ORS) was established in Coastal Command which was to carry out some of the most well-known OR work in World War II.

Although scientists had (plainly) been involved in the hardware side of warfare (designing better planes, bombs, tanks, etc) scientific analysis of the operational use of military resources had never taken place in a systematic fashion before the Second World War. Military personnel, often by no means stupid, were simply not trained to undertake such analysis.

These early OR workers came from many different disciplines, one group consisted of a physicist, two physiologists, two mathematical physicists and a surveyor. What such people brought to their work were "scientifically trained" minds, used to querying assumptions, logic, exploring hypotheses, devising experiments, collecting data, analyzing numbers, etc. Many too were of high intellectual caliber (at least four wartime

OR personnel were later to win Nobel prizes when they returned to their peacetime disciplines).

By the end of the war OR was well established in the armed services both in the UK and in the USA.

OR started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations. The objective was to find the most effective utilization of limited military resources by the use of quantitative techniques.

Following the end of the war OR spread, although it spread in different ways in the UK and USA.

You should be clear that the growth of OR since it began (and especially in the last 30 years) is, to a large extent, the result of the increasing power and widespread availability of computers. Most (though not all) OR involves carrying out a large number of numeric calculations. Without computers this would simply not be possible.

2. BASIC OR CONCEPTS

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

We can also define a mathematical model as consisting of:

➤ *Decision variables*, which are the unknowns to be determined by the solution to the model.

➤ *Constraints* to represent the physical limitations of the system

➤ An *objective* function

➤ An *optimal solution* to the model is the identification of a set of variable values which are feasible (satisfy all the constraints) and which lead to the optimal value of the objective function.

An optimization model seeks to find values of the decision variables that optimize (maximize or minimize) an objective function among the set of all values for the decision variables that satisfy the given constraints.

Two Mines Example

The Two Mines Company own two different mines that produce an ore which, after being crushed, is graded into three classes: high, medium and low-grade. The company has contracted to provide a smelting plant with 12 tons of high-grade, 8 tons of medium-grade and 24 tons of low-grade ore per week. The two mines have different operating characteristics as detailed below.

Mine	Cost per day (£'000)	Production (tons/day)		
		High	Medium	Low
X	180	6	3	4
Y	160	1	1	6

Consider that mines cannot be operated in the weekend. How many days per week should each mine be operated to fulfill the smelting plant contract?

Guessing

To explore the Two Mines problem further we might simply guess (i.e. use our judgment) how many days per week to work and see how they turn out.

- work one day a week on *X*, one day a week on *Y*

This does not seem like a good guess as it results in only 7 tones a day of high-grade, insufficient to meet the contract requirement for 12 tones of high-grade a day. We say that such a solution is *infeasible*.

- work 4 days a week on *X*, 3 days a week on *Y*

This seems like a better guess as it results in sufficient ore to meet the contract. We say that such a solution is *feasible*. However it is quite expensive (costly).

We would like a solution which supplies what is necessary under the contract at minimum cost. Logically such a minimum cost solution to this decision problem must exist. However even if we keep guessing we can never be sure whether we have found this minimum cost solution or not. Fortunately our structured approach will enable us to find the minimum cost solution.

Solution

What we have is a verbal description of the Two Mines problem. What we need to do is to translate that verbal description into an *equivalent* mathematical description.

In dealing with problems of this kind we often do best to consider them in the order:

- Variables
- Constraints
- Objective

This process is often called *formulating* the problem (or more strictly formulating a mathematical representation of the problem).

Variables

These represent the "decisions that have to be made" or the "unknowns".

We have two decision variables in this problem:

x = number of days per week mine *X* is operated

y = number of days per week mine *Y* is operated

Note here that $x \geq 0$ and $y \geq 0$.

Constraints

It is best to first put each constraint into words and then express it in a mathematical form.

ore production constraints - balance the amount produced with the quantity required under the smelting plant contract

Ore

High $6x + 1y \geq 12$

Medium $3x + 1y \geq 8$

Low $4x + 6y \geq 24$

days per week constraint - we cannot work more than a certain maximum number of days a week e.g. for a 5 day week we have

$x \leq 5$

$y \leq 5$

Inequality constraints

Note we have an inequality here rather than an equality. This implies that we may produce more of some grade of ore than we need. In fact we have the general rule: given a choice between an equality and an inequality choose the inequality

For example - if we choose an equality for the ore production constraints we have the three equations $6x+y=12$, $3x+y=8$ and $4x+6y=24$ and there are no values of x and y which satisfy all three equations (the problem is therefore said to be "over-constrained"). For example the values of x and y which satisfy $6x+y=12$ and $3x+y=8$ are $x=4/3$ and $y=4$, but these values do not satisfy $4x+6y=24$.

The reason for this general rule is that choosing an inequality rather than an equality gives us more flexibility in optimizing (maximizing or minimizing) the objective (deciding values for the decision variables that optimize the objective).

Implicit constraints

Constraints such as days per week constraint are often called implicit constraints because they are implicit in the definition of the variables.

Objective

Again in words our objective is (presumably) to minimize cost which is given by

$180x + 160y$

Hence we have the ***complete mathematical representation*** of the problem:

minimize
$$180x + 160y$$
subject to
$$6x + y \geq 12$$
$$3x + y \geq 8$$
$$4x + 6y \geq 24$$
$$x \leq 5$$
$$y \leq 5$$
$$x, y \geq 0$$

Some notes

The mathematical problem given above has the form

- all variables continuous (i.e. can take fractional values)

- a single objective (maximize or minimize)
- the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown (e.g. 24, $4x$, $6y$ are linear terms but xy or x^2 is a non-linear term)

Any formulation which satisfies these three conditions is called a *linear program* (LP). We have (implicitly) assumed that it is permissible to work in fractions of days - problems where this is not permissible and variables must take integer values will be dealt with under *Integer Programming* (IP).

Discussion

This problem was a *decision problem.*

We have taken a real-world situation and constructed an equivalent mathematical representation - such a representation is often called a mathematical *model* of the real-world situation (and the process by which the model is obtained is called *formulating* the model).

Just to confuse things the mathematical model of the problem is sometimes called the *formulation* of the problem.

Having obtained our mathematical model we (hopefully) have some quantitative method which will enable us to numerically solve the model (i.e. obtain a numerical solution) - such a quantitative method is often called an *algorithm* for solving the model. Essentially an algorithm (for a particular model) is a set of instructions which, when followed in a step-by-step fashion, will produce a numerical solution to that model.

Our model has an *objective*, that is something which we are trying to *optimize*.

Having obtained the numerical solution of our model we have to translate that solution back into the real-world situation.

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

3.   LINEAR PROGRAMMING

It can be recalled from the Two Mines example that the conditions for a mathematical model to be a linear program (LP) were:

- all variables continuous (i.e. can take fractional values)
- a single objective (minimize or maximize)
- the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown.

LP's are important - this is because:

- many practical problems can be formulated as LP's
- there exists an algorithm (called the *simplex* algorithm) which enables us to solve LP's numerically relatively easily

We will return later to the simplex algorithm for solving LP's but for the moment we will concentrate upon formulating LP's.

Some of the major application areas to which LP can be applied are:

- Work scheduling
- Production planning & Production process
- Capital budgeting
- Financial planning
- Blending (e.g. Oil refinery management)
- Farm planning
- Distribution
- Multi-period decision problems
  - Inventory model
  - Financial models
  - Work scheduling

Note that the key to formulating LP's is practice. However a useful hint is that common objectives for LP's are maximize profit/minimize cost.

There are four basic assumptions in LP:

- Proportionality
  - The contribution to the objective function from each decision variable is proportional to the value of the decision variable (The contribution to the objective function from making four soldiers (4×$3=$12) is exactly four times the contribution to the objective function from making one soldier ($3))
  - The contribution of each decision variable to the LHS of each constraint is proportional to the value of the decision variable (It takes exactly three times as many finishing hours (2hrs×3=6hrs) to manufacture three soldiers as it takes to manufacture one soldier (2 hrs))
- Additivity
  - The contribution to the objective function for any decision variable is independent of the values of the other decision variables (No matter what the value of train (x_2), the manufacture of soldier (x_1) will always contribute $3x_1$ dollars to the objective function)
  - The contribution of a decision variable to LHS of each constraint is independent of the values of other decision variables (No matter what the value of x_1, the manufacture of x_2 uses x_2 finishing hours and x2 carpentry hours)
    - 1st *implication*: The value of objective function is the sum of the contributions from each decision variables.
    - 2nd *implication*: LHS of each constraint is the sum of the contributions from each decision variables.
- Divisibility
  - Each decision variable is allowed to assume fractional values. If we actually can not produce a fractional number of decision variables, we use IP (It is acceptable to produce 1.69 trains)
- Certainty
  - Each parameter is known with certainty

3.1   FORMULATING LP

3.1.1 Giapetto Example

(Winston 3.1, p. 49)

Giapetto's wooden soldiers and trains. Each soldier sells for $27, uses $10 of raw materials and takes $14 of labor & overhead costs. Each train sells for $21, uses $9 of raw materials, and takes $10 of overhead costs. Each soldier needs 2 hours finishing and 1 hour carpentry; each train needs 1 hour finishing and 1 hour carpentry. Raw materials are unlimited, but only 100 hours of finishing and 80 hours of carpentry are available each week. Demand for trains is unlimited; but at most 40 soldiers can be sold each week. How many of each toy should be made each week to maximize profits?

Answer

Decision variables completely describe the decisions to be made (in this case, by Giapetto). Giapetto must decide how many soldiers and trains should be manufactured each week. With this in mind, we define:

> x_1 = the number of soldiers produced per week
>
> x_2 = the number of trains produced per week

Objective function is the function of the decision variables that the decision maker wants to maximize (revenue or profit) or minimize (costs). Giapetto can concentrate on maximizing the total weekly profit (z).

Here profit equals to (weekly revenues) – (raw material purchase cost) – (other variable costs). Hence Giapetto's objective function is:

> Maximize $z = 3x_1 + 2x_2$

Constraints show the restrictions on the values of the decision variables. Without constraints Giapetto could make a large profit by choosing decision variables to be very large. Here there are three constraints:

> Finishing time per week
>
> Carpentry time per week
>
> Weekly demand for soldiers

Sign restrictions are added if the decision variables can only assume nonnegative values (Giapetto can not manufacture negative number of soldiers or trains!)

All these characteristics explored above give the following **Linear Programming** (LP) model

> max $z = 3x_1 + 2x_2$ (The Objective function)
>
> s.t. $2x_1 + x_2 \leq 100$ (Finishing constraint)
>
> $x_1 + x_2 \leq 80$ (Carpentry constraint)
>
> $x_1 \leq 40$ (Constraint on demand for soldiers)
>
> $x_1, x_2 \geq 0$ (Sign restrictions)

A value of (x_1, x_2) is in the **feasible region** if it satisfies all the constraints and sign restrictions.

Graphically and computationally we see the solution is (x_1, x_2) = (20, 60) at which z = 180. (**Optimal solution**)

Report

The maximum profit is $180 by making 20 soldiers and 60 trains each week. Profit is limited by the carpentry and finishing labor available. Profit could be increased by buying more labor.

3.1.2 Advertisement Example

(Winston 3.2, p.61)

Dorian makes luxury cars and jeeps for high-income men and women. It wishes to advertise with 1 minute spots in comedy shows and football games. Each comedy spot costs $50K and is seen by 7M high-income women and 2M high-income men. Each football spot costs $100K and is seen by 2M high-income women and 12M high-income men. How can Dorian reach 28M high-income women and 24M high-income men at the least cost?

Answer

The decision variables are

> x_1 = the number of comedy spots
>
> x_2 = the number of football spots

The model of the problem:

> min $z = 50x_1 + 100x_2$
>
> st $7x_1 + 2x_2 \geq 28$
>
> $2x_1 + 12x_2 \geq 24$
>
> $x_1, x_2 \geq 0$

The graphical solution is z = 320 when (x_1, x_2) = (3.6, 1.4). From the graph, in this problem rounding up to (x_1, x_2) = (4, 2) gives the best *integer* solution.

Report

The minimum cost of reaching the target audience is $400K, with 4 comedy spots and 2 football slots. The model is dubious as it does not allow for saturation after repeated viewings.

3.1.3 Diet Example

(Winston 3.4., p. 70)

Ms. Fidan's diet requires that all the food she eats come from one of the four "basic food groups". At present, the following four foods are available for consumption: brownies, chocolate ice cream, cola, and pineapple cheesecake. Each brownie costs 0.5$, each scoop of chocolate ice cream costs 0.2$, each bottle of cola costs 0.3$, and each pineapple cheesecake costs 0.8$. Each day, she must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. The nutritional content per unit of each food is shown in Table. Formulate an LP model that can be used to satisfy her daily nutritional requirements at minimum cost.

	Calories	Chocolate (ounces)	Sugar (ounces)	Fat (ounces)
Brownie	400	3	2	2
Choc. ice cream (1 scoop)	200	2	2	4
Cola (1 bottle)	150	0	4	1
Pineapple cheesecake (1 piece)	500	0	4	5

Answer

The decision variables:

x_1: number of brownies eaten daily

x_2: number of scoops of chocolate ice cream eaten daily

x_3: bottles of cola drunk daily

x_4: pieces of pineapple cheesecake eaten daily

The objective function (the total cost of the diet in cents):

$$\min w = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

Constraints:

$$400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 \quad \text{(daily calorie intake)}$$

$$3x_1 + 2x_2 \geq 6 \quad \text{(daily chocolate intake)}$$

$$2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 \quad \text{(daily sugar intake)}$$

$$2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 \quad \text{(daily fat intake)}$$

$$x_i \geq 0, \ i = 1, 2, 3, 4 \quad \text{(Sign restrictions!)}$$

Report

The minimum cost diet incurs a daily cost of 90 cents by eating 3 scoops of chocolate and drinking 1 bottle of cola ($w = 90$, $x_2 = 3$, $x_3 = 1$)

3.1.4 Post Office Example

(Winston 3.5, p.74)

A PO requires different numbers of employees on different days of the week. Union rules state each employee must work 5 consecutive days and then receive two days off. Find the minimum number of employees needed.

	Mon	Tue	Wed	Thur	Fri	Sat	Sun
Staff Needed	17	13	15	19	14	16	11

Answer

The decision variables are x_i (# of employees starting on day i)

Mathematically we must

$$
\begin{array}{rl}
\min z = & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\
\text{s.t.} & x_1 \qquad\qquad\quad + x_4 + x_5 + x_6 + x_7 \geq 17 \\
& x_1 + x_2 \qquad\qquad\quad + x_5 + x_6 + x_7 \geq 13 \\
& x_1 + x_2 + x_3 \qquad\qquad\quad + x_6 + x_7 \geq 15 \\
& x_1 + x_2 + x_3 + x_4 \qquad\qquad\quad + x_7 \geq 19 \\
& x_1 + x_2 + x_3 + x_4 + x_5 \qquad\qquad \geq 14 \\
& \quad\;\; x_2 + x_3 + x_4 + x_5 + x_6 \qquad \geq 16 \\
& \qquad\;\; x_3 + x_4 + x_5 + x_6 + x_7 \geq 11 \\
& x_t \geq 0,\ \forall t
\end{array}
$$

The solution is $(x_i) = (4/3, 10/3, 2, 22/3, 0, 10/3, 5)$ giving $z = 67/3$.

We could round this up to $(x_i) = (2, 4, 2, 8, 0, 4, 5)$ giving $z = 25$ (may be wrong!). However restricting the decision var.s to be integers and using Lindo again gives $(x_i) = (4, 4, 2, 6, 0, 4, 3)$ giving $z = 23$.

3.1.5 Sailco Example

(Winston 3.10, p. 99)

Sailco must determine how many sailboats to produce in the next 4 quarters. The demand is known to be 40, 60, 75, and 25 boats. Sailco must meet its demands. At the beginning of the 1st quarter Sailco starts with 10 boats in inventory. Sailco can produce up to 40 boats with regular time labor at $400 per boat, or additional boats at $450 with overtime labor. Boats made in a quarter can be used to meet that quarter's demand or held in inventory for the next quarter at an extra cost of $20.00 per boat.

Answer

The decision variables are for t = 1,2,3,4

    x_t = # of boats in quarter t built in regular time

    y_t = # of boats in quarter t built in overtime

For convenience, introduce variables:

    i_t = # of boats in inventory at the end quarter t

    d_t = demand in quarter t

We are given that	$d_1 = 40, d_2 = 60, d_3 = 75, d_4 = 25, i_0 = 10$
	$x_t \leq 40, \forall t$
By logic	$i_t = i_{t-1} + x_t + y_t - d_t, \forall t.$
Demand is met iff	$i_t \geq 0, \forall t$
(Sign restrictions	$x_t, y_t \geq 0, \forall t$)

We need to minimize total cost z subject to these three sets of conditions where

$$z = 400 (x_1 + x_2 + x_3 + x_4) + 450 (y_1 + y_2 + y_3 + y_4) + 20 (i_1 + i_2 + i_3 + i_4)$$

Report:

Lindo reveals the solution to be $(x_1, x_2, x_3, x_4) = (40, 40, 40, 25)$ and $(y_1, y_2, y_3, y_4) = (0, 10, 35, 0)$ and the minimum cost of $78450.00 is achieved by the schedule

		Q_1	Q_2	Q_3	Q_4
Regular time (x_t)		40	40	40	25
Overtime (y_t)		0	10	35	0
Inventory (i_t)	10	10	0	0	0
Demand (d_t)		40	60	75	25

3.1.6 Customer Service Level Example

(Winston 3.12, p. 108)

CSL services computers. Its demand (hours) for the time of skilled technicians in the next 5 months is

t	Jan	Feb	Mar	Apr	May
d_t	6000	7000	8000	9500	11000

It starts with 50 skilled technicians at the beginning of January. Each technician can work 160 hrs/month. To train a new technician they must be supervised for 50 hrs by an experienced technician for a period of one month time. Each experienced technician is paid $2K/mth and a trainee is paid $1K/mth. Each month 5% of the skilled technicians leave. CSL needs to meet demand and minimize costs.

Answer

The decision variable is

x_t = # to be trained in month t

We must minimize the total cost. For convenience let

y_t = # experienced tech. at start of tth month

d_t = demand during month t

Then we must

min z = 2000 (y_1+...+ y_5) + 1000 (x_1 +...+ x_5)

subject to

$160y_t - 50x_t \geq d_t$ for t = 1,..., 5

y_1 = 50, d_1 = 6000, d_2 = 7000, d_3 = 8000, d_4 = 9500, d_5 = 11000

y_t = .95y_{t-1} + x_{t-1} for t = 2,3,4,5

$x_t, y_t \geq 0$

3.2 SOLVING LP

3.2.1 LP Solutions: Four Cases

When an LP is solved, one of the following four cases will occur:

1. The LP has a **unique optimal solution**.
2. The LP has **alternative (multiple) optimal solutions**. It has more than one (actually an infinite number of) optimal solutions
3. The LP is **infeasible**. It has no feasible solutions (The feasible region contains no points).
4. The LP is **unbounded**. In the feasible region there are points with arbitrarily large (in a max problem) objective function values.

3.2.2 The Graphical Solution

Any LP with only two variables can be solved graphically

Example 1. Giapetto

(*Winston 3.1, p. 49*)

Since the Giapetto LP has two variables, it may be solved graphically.

Answer

The feasible region is the set of all points satisfying the constraints.

$$\max z = 3x_1 + 2x_2$$

$$\begin{aligned}
\text{s.t.} \quad & 2x_1 + x_2 \le 100 && \text{(Finishing constraint)} \\
& x_1 + x_2 \le 80 && \text{(Carpentry constraint)} \\
& x_1 \le 40 && \text{(Demand constraint)} \\
& x_1, x_2 \ge 0 && \text{(Sign restrictions)}
\end{aligned}$$

The set of points satisfying the LP is bounded by the five sided polygon DGFEH. Any point **on** or **in** the interior of this polygon (the shade area) is in the **feasible region**.

Having identified the feasible region for the LP, a search can begin for the *optimal solution* which will be the point in the feasible region with the *largest z*-value (maximization problem).

To find the optimal solution, a line on which the points have the same *z*-value is graphed. In a max problem, such a line is called an *isoprofit* line while in a min problem, this is called the *isocost* line. (*The figure shows the isoprofit lines for z = 60, z = 100, and z = 180*).
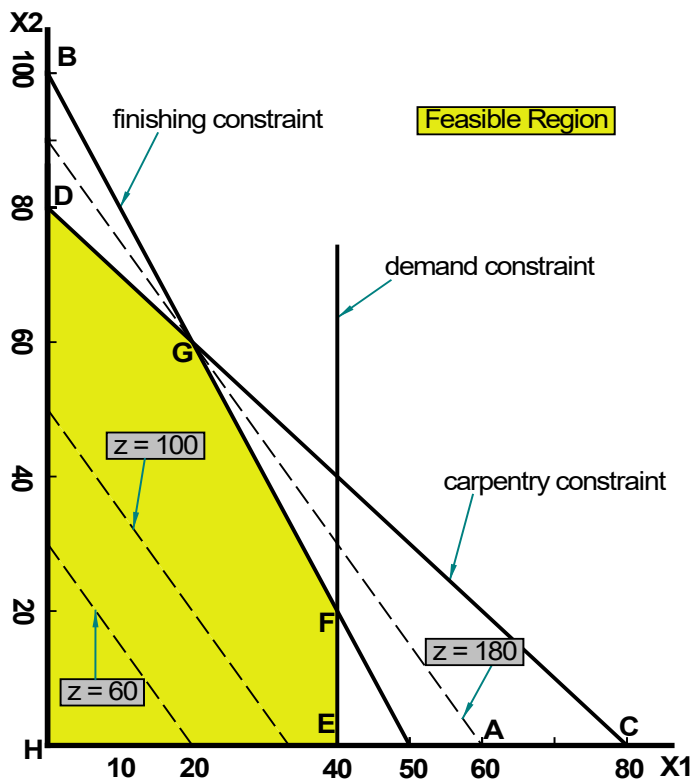
In the unique optimal solution case, isoprofit line last hits a point (vertex - corner) before leaving the feasible region.

The optimal solution of this LP is point G where $(x_1, x_2) = (20, 60)$ giving $z = 180$.

A constraint is ***binding*** (active, tight) if the left-hand and right-hand side of the constraint are equal when the optimal values of the decision variables are substituted into the constraint.

A constraint is ***nonbinding*** (inactive) if the left-hand side and the right-hand side of the constraint are unequal when the optimal values of the decision variables are substituted into the constraint.

In Giapetto LP, the finishing and carpentry constraints are binding. On the other hand the demand constraint for wooden soldiers is nonbinding since at the optimal solution $x_1 < 40$ ($x_1 = 20$).

Example 2. Advertisement

(*Winston 3.2, p. 61*)

Since the Advertisement LP has two variables, it may be solved graphically.

Answer

The feasible region is the set of all points satisfying the constraints.

min $z = 50x_1 + 100x_2$

s.t. $7x_1 + 2x_2 \geq 28$ (high income women)

         $2x_1 + 12x_2 \geq 24$ (high income men)

         $x_1, x_2 \geq 0$
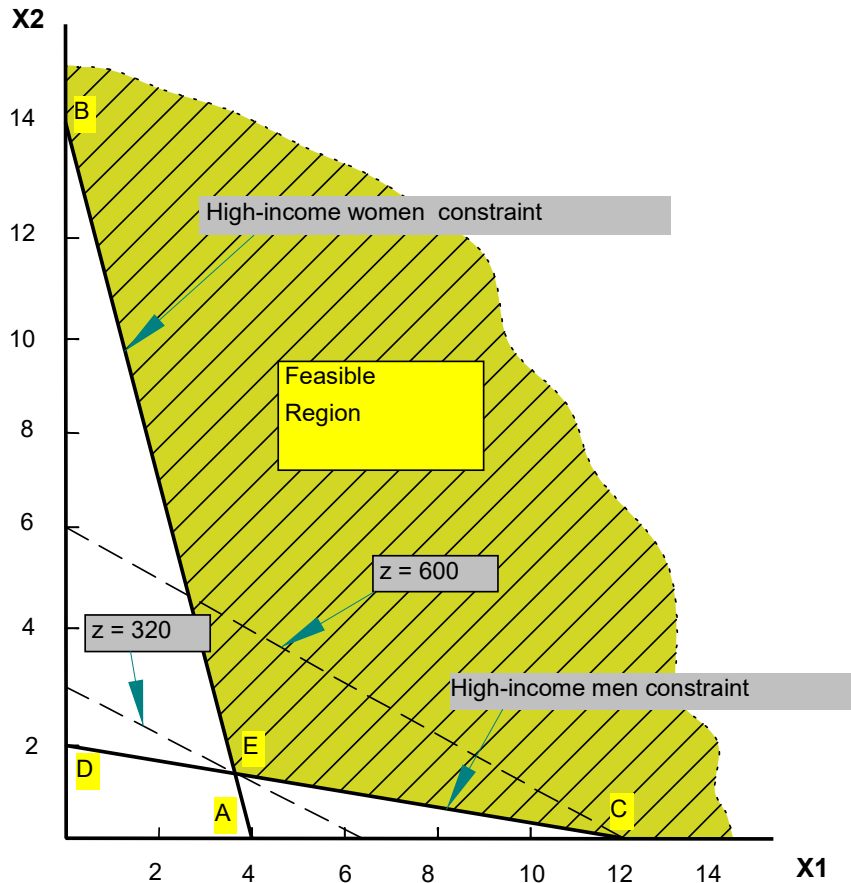
Since Dorian wants to minimize total advertising costs, the optimal solution to the problem is the point in the feasible region with the smallest z value.

An isocost line with the smallest z value passes through point E and is the optimal solution at $x_1 = 3.6$ and $x_2 = 1.4$ giving $z = 320$.

Both the high-income women and high-income men constraints are satisfied, both constraints are binding.

Example 3. Two Mines

min $180x_1 + 160x_2$

st $6x_1 + x_2 \geq 12$

$$3x_1 + x_2 \geq 8$$

$$4x_1 + 6x_2 \geq 24$$

$$x_1 \leq 5$$

$$x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

Answer
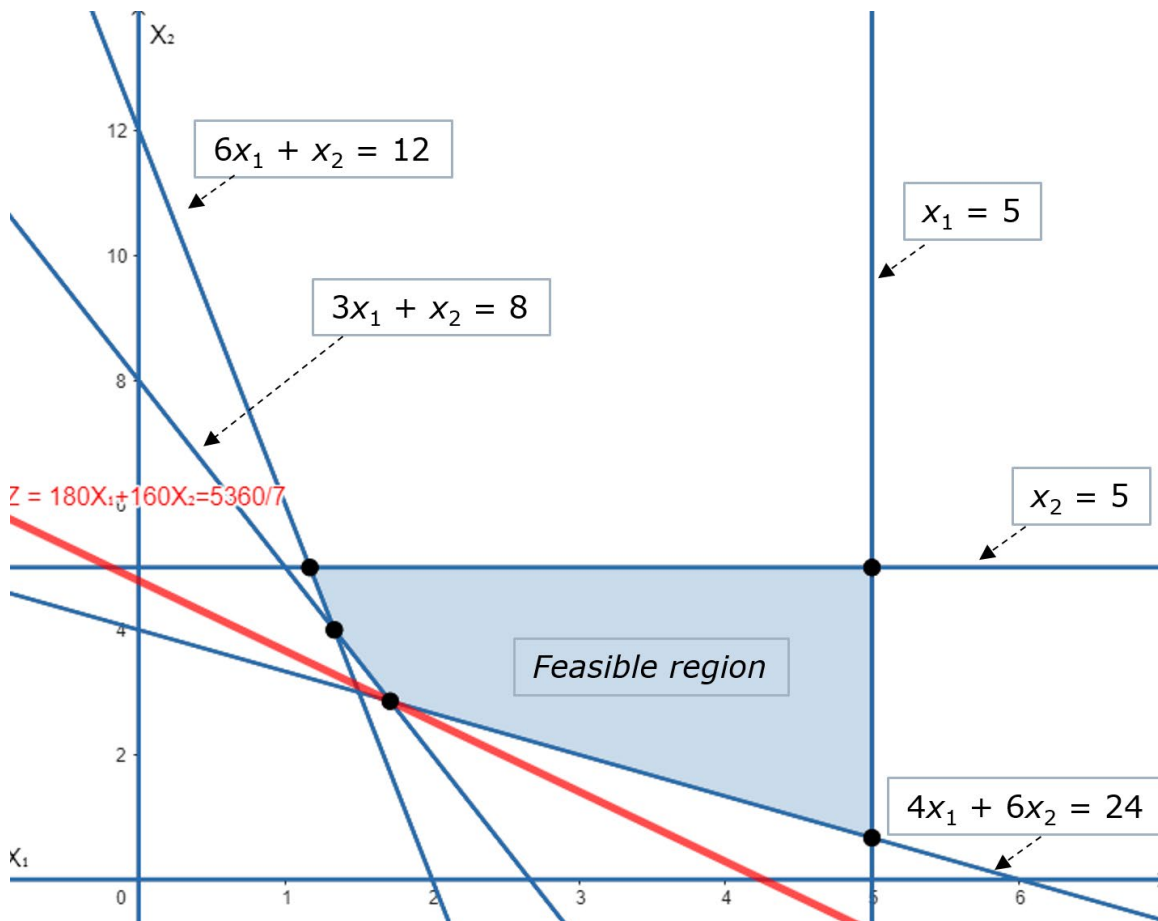
Opt. sol'n: Total cost is 765.71. 1.71 days mine *X* and 2.86 days mine *Y* are operated.

Example 4. Modified Giapetto

max $z = 4x_1 + 2x_2$

s.t. $2x_1 + x_2 \leq 100$ (Finishing constraint)

$x_1 + x_2 \leq 80$ (Carpentry constraint)

$x_1 \qquad \leq 40$ (Demand constraint)

$x_1, x_2 \geq 0$ (Sign restrictions)

Answer

Points on the line between points G (20, 60) and F (40, 20) are the ***alternative optimal solutions*** (see figure below).

Thus, for $0 \le c \le 1$,

$$c \ [20 \ 60] + (1 - c) \ [40 \ 20] \ = \ \mathbf{[40 - 20c, \ 20 + 40c]}$$

will be optimal

For all optimal solutions, the optimal objective function value is 200.
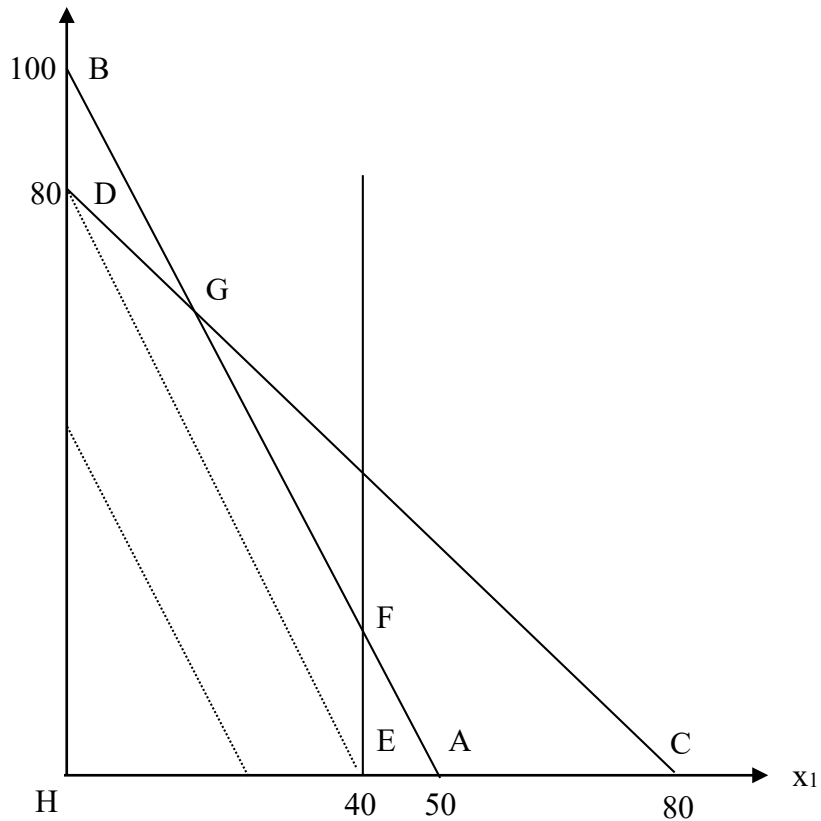
Example 5. Modified Giapetto (v. 2)

Add constraint $x_2 \ge 90$ (Constraint on demand for trains).

Answer

No feasible region: *Infeasible LP*

Example 6. Modified Giapetto (v. 3)

Only use constraint $x_2 \ge 90$

Answer

Isoprofit line never lose contact with the feasible region: *Unbounded LP*

3.2.3 The Simplex Algorithm

Note that in the examples considered at the graphical solution, the unique optimal solution to the LP occurred at a vertex (corner) of the feasible region. In fact it is true that for *any* LP the optimal solution occurs at a vertex of the feasible region. This fact is the key to the simplex algorithm for solving LP's.

Essentially the simplex algorithm starts at one vertex of the feasible region and moves (at each iteration) to another (adjacent) vertex, improving (or leaving unchanged) the objective function as it does so, until it reaches the vertex corresponding to the optimal LP solution.

The simplex algorithm for solving linear programs (LP's) was developed by Dantzig in the late 1940's and since then a number of different versions of the algorithm have been developed. One of these later versions, called the *revised simplex* algorithm (sometimes known as the "product form of the inverse" simplex algorithm) forms the basis of most modern computer packages for solving LP's.

Steps

1. Convert the LP to standard form
2. Obtain a basic feasible solution (bfs) from the standard form
3. Determine whether the current bfs is optimal. If it is optimal, stop.
4. If the current bfs is not optimal, determine which nonbasic variable should become a basic variable and which basic variable should become a nonbasic variable to find a new bfs with a better objective function value
5. Go back to Step 3.

Related concepts:

- Standard form: all constraints are equations and all variables are nonnegative
- bfs: any basic solution where all variables are nonnegative
- Nonbasic variable: a chosen set of variables where variables equal to 0
- Basic variable: the remaining variables that satisfy the system of equations at the standard form

Example 1. Dakota Furniture

(Winston 4.3, p. 134)

Dakota Furniture makes desks, tables, and chairs. Each product needs the limited resources of lumber, carpentry and finishing; as described in the table. At most 5 tables can be sold per week. Maximize weekly revenue.

Resource	Desk	Table	Chair	Max Avail.
Lumber (board ft.)	8	6	1	48
Finishing hours	4	2	1.5	20
Carpentry hours	2	1.5	.5	8
Max Demand	unlimited	5	unlimited	
Price ($)	60	30	20	

LP Model:

Let x_1, x_2, x_3 be the number of desks, tables and chairs produced.

Let the weekly profit be $z. Then, we must

$$\max z = 60x_1 + 30x_2 + 20x_3$$
$$\text{s.t.} \quad 8x_1 + 6x_2 + x_3 \leq 48$$
$$4x_1 + 2x_2 + 1.5x_3 \leq 20$$
$$2x_1 + 1.5x_2 + .5x_3 \leq 8$$
$$x_2 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

Solution with Simplex Algorithm

First introduce slack variables and convert the LP to the standard form and write a canonical form

R_0	z	$-60x_1$	$-30x_2$	$-20x_3$					$= 0$
R_1		$8x_1$	$+ 6x_2$	$+ x_3$	$+ s_1$				$= 48$
R_2		$4x_1$	$+ 2x_2$	$+1.5x_3$		$+ s_2$			$= 20$
R_3		$2x_1$	$+ 1.5x_2$	$+ .5x_3$			$+ s_3$		$= 8$
R_4			x_2					$+ s_4$	$= 5$

$$x_1, x_2, x_3, s_1, s_2, s_3, s_4 \geq 0$$

Obtain a starting bfs.

As $(x_1, x_2, x_3) = 0$ is feasible for the original problem, the below given point where three of the variables equal 0 (the **non-basic variables**) and the four other variables (the **basic variables**) are determined by the four equalities is an obvious bfs:

$$x_1 = x_2 = x_3 = 0, s_1 = 48, s_2 = 20, s_3 = 8, s_4 = 5.$$

.

<u>Determine whether the current bfs is optimal.</u>

Determine whether there is any way that z can be increased by increasing some nonbasic variable.

If each nonbasic variable has a nonnegative coefficient in the objective function row (**row 0**), current bfs is optimal.

However, here all nonbasic variables have negative coefficients: It is not optimal.

<u>Find a new bfs</u>

- z increases most rapidly when x_1 is made non-zero; i.e. x_1 is the **entering variable**.

- Examining R_1, x_1 can be increased only to 6. More than 6 makes $s_1 < 0$. Similarly R_2, R_3, and R_4, give limits of 5, 4, and no limit for x_1 (**ratio test**). The smallest ratio is the largest value of the entering variable that will keep all the current basic variables nonnegative. Thus by R_3, x_1 can only increase to $x_1 = 4$ when s_3 becomes 0. We say s_3 is the **leaving variable** and R_3 is the **pivot equation**.

- Now we must rewrite the system so the values of the basic variables can be read off.

The new *pivot equation* ($R_3/2$) is

R_3': $x_1 + .75x_2 + .25x_3 + \quad .5s_3 \quad = 4$

Then use R_3' to eliminate x_1 in all the other rows.

R0'=R0+60R3', R1'=R1-8R3', R2'=R2-4R3', R4'=R4

R_0'	z	$+ 15x_2$	$- 5x_3$		$+ 30s_3$	$= 240$	$z = 240$
R_1'			$- x_3$	$+ s_1$	$- 4s_3$	$= 16$	$s_1 = 16$
R_2'		$- x_2$	$+ .5x_3$	$+ s_2$	$- 2s_3$	$= 4$	$s_2 = 4$
R_3'	x_1	$+ .75x_2$	$+ .25x_3$		$+ .5s_3$	$= 4$	$x_1 = 4$
R_4'		x_2			$+ s_4$	$= 5$	$s_4 = 5$

The new bfs is $x_2 = x_3 = s_3 = 0$, $x_1 = 4$, $s_1 = 16$, $s_2 = 4$, $s_4 = 5$ making $z = 240$.

<u>Check optimality of current bfs. Repeat steps until an optimal solution is reached</u>

- We increase z fastest by making x_3 non-zero (i.e. x_3 enters).
- x_3 can be increased to at most $x_3 = 8$, when $s_2 = 0$ (i.e. s_2 leaves.)

Rearranging the pivot equation gives

R_2'' $- 2x_2 + x_3 + 2s_2 - 4s_3 = 8$ ($R_2' \times 2$).

Row operations with R_2'' eliminate x_3 to give the new system

$R_0'' = R_0' + 5R_2''$, $R_1'' = R_1' + R_2''$, $R_3'' = R_3' - .5R_2''$, $R_4'' = R_4'$

The bfs is now $x_2 = s_2 = s_3 = 0$, $x_1 = 2$, $x_3 = 8$, $s_1 = 24$, $s_4 = 5$ making $z = 280$.

Each nonbasic variable has a nonnegative coefficient in row 0 ($5x_2$, $10s_2$, $10s_3$).

THE CURRENT SOLUTION IS OPTIMAL

Report: Dakota furniture's optimum weekly profit would be 280$ if they produce 2 desks and 8 chairs.

This was once written as a tableau.

(***Use tableau format for each operation in all HW and exams!!!***)

max $z = 60x_1 + 30x_2 + 20x_3$

s.t. $8x_1 + 6x_2 + x_3 \le 48$

$4x_1 + 2x_2 + 1.5x_3 \le 20$

$2x_1 + 1.5x_2 + .5x_3 \le 8$

$x_2 \le 5$

$x_1, x_2, x_3 \ge 0$

Initial tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	-60	-30	-20	0	0	0	0	0	$z = 0$	
0	8	6	1	1	0	0	0	48	$s_1 = 48$	6
0	4	2	1.5	0	1	0	0	20	$s_2 = 20$	5
0	2	1.5	0.5	0	0	1	0	8	$s_3 = 8$	4
0	0	1	0	0	0	0	1	5	$s_4 = 5$	-

First tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	15	-5	0	0	30	0	240	$z = 240$	
0	0	0	-1	1	0	-4	0	16	$s_1 = 16$	-
0	0	-1	0.5	0	1	-2	0	4	$s_2 = 4$	8
0	1	0.75	0.25	0	0	0.5	0	4	$x_1 = 4$	16
0	0	1	0	0	0	0	1	5	$s_4 = 5$	-

Second and optimal tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	5	0	0	10	10	0	280	$z = 280$	
0	0	-2	0	1	2	-8	0	24	$s_1 = 24$	
0	0	-2	1	0	2	-4	0	8	$x_3 = 8$	
0	1	1.25	0	0	-0.5	1.5	0	2	$x_1 = 2$	
0	0	1	0	0	0	0	1	5	$s_4 = 5$	

Example 2. Modified Dakota Furniture

Dakota example is modified: $35/table

new $z = 60 x_1 + 35 x_2 + 20 x_3$

Second and optimal tableau for the modified problem:

| | | ⇓ | | | | | | | | |
z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	**0**	0	0	10	10	0	280	z=280	
0	0	-2	0	1	2	-8	0	24	s_1=24	-
0	0	-2	1	0	2	-4	0	8	x_3=8	-
0	1	1.25	0	0	-0.5	1.5	0	2	x_1=2	2/1.25 ⇒
0	0	1	0	0	0	0	1	5	s_4=5	5/1

Another optimal tableau for the modified problem:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV
1	0	0	0	0	10	10	0	280	z=280
0	1.6	0	0	1	1.2	-5.6	0	27.2	s_1=27.2
0	1.6	0	1	0	1.2	-1.6	0	11.2	x_3=11.2
0	0.8	1	0	0	-0.4	1.2	0	1.6	x_2=1.6
0	-0.8	0	0	0	0.4	-1.2	1	3.4	s_4=3.4

Therefore the optimal solution is as follows:

$z = 280$ and for $0 \le c \le 1$

$$
\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = c \begin{vmatrix} 2 \\ 0 \\ 8 \end{vmatrix} + (1-c) \begin{vmatrix} 0 \\ 1.6 \\ 11.2 \end{vmatrix} = \begin{vmatrix} 2c \\ 1.6 - 1.6c \\ 11.2 - 3.2c \end{vmatrix}
$$

Example 3. Unbounded LPs

| | | | ⇓ | | | | | | |
z	x_1	x_2	x_3	s_1	s_2	z	RHS	BV	Ratio
1	0	2	**-9**	0	12	4	100	z=100	
0	0	1	-6	1	6	-1	20	x_4=20	None
0	1	1	-1	0	1	0	5	x_1=5	None

Since ratio test fails, the LP under consideration is an unbounded LP.

3.2.4 The Big M Method

If an LP has any ≥ or = constraints, a starting bfs may not be readily apparent.

When a bfs is not readily apparent, the Big M method or the two-phase simplex method may be used to solve the problem.

The Big M method is a version of the Simplex Algorithm that first finds a bfs by adding "artificial" variables to the problem. The objective function of the original LP must, of course, be modified to ensure that the artificial variables are all equal to 0 at the conclusion of the simplex algorithm.

Steps

1. Modify the constraints so that the RHS of each constraint is nonnegative (This requires that each constraint with a negative RHS be multiplied by -1. Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed!). After modification, identify each constraint as a ≤, ≥ or = constraint.

2. Convert each inequality constraint to standard form (If constraint i is a ≤ constraint, we add a slack variable s_i; and if constraint i is a ≥ constraint, we subtract an excess variable e_i).

3. Add an artificial variable a_i to the constraints identified as ≥ or = constraints at the end of Step 1. Also add the sign restriction $a_i \geq 0$.

4. Let M denote a very large positive number. If the LP is a min problem, add (for each artificial variable) Ma_i to the objective function. If the LP is a max problem, add (for each artificial variable) $-Ma_i$ to the objective function.

5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Now solve the transformed problem by the simplex (In choosing the entering variable, remember that M is a very large positive number!).

If all artificial variables are equal to zero in the optimal solution, we have found the **optimal solution** to the original problem.

If any artificial variables are positive in the optimal solution, the original problem is **infeasible**!!!

Example 1. Oranj Juice

(Winston 4.10, p. 164)

Bevco manufactures an orange flavored soft drink called Oranj by combining orange soda and orange juice. Each ounce of orange soda contains 0.5 oz of sugar and 1 mg of vitamin C. Each ounce of orange juice contains 0.25 oz of sugar and 3 mg of vitamin C. It costs Bevco 2¢ to produce an ounce of orange soda and 3¢ to produce an ounce of orange juice. Marketing department has decided that each 10 oz bottle of Oranj must contain at least 20 mg of vitamin C and at most 4 oz of sugar. Use LP to determine how Bevco can meet marketing dept.'s requirements at minimum cost.

LP Model:

Let x_1 and x_2 be the quantity of ounces of orange soda and orange juice (respectively) in a bottle of Oranj.

$$\min z = 2x_1 + 3x_2$$

$$
\begin{array}{lll}
\text{s.t.} & 0.5\,x_1 + 0.25\,x_2 \leq 4 & \text{(sugar const.)} \\
& x_1 + 3\,x_2 \geq 20 & \text{(vit. C const.)} \\
& x_1 + x_2 = 10 & \text{(10 oz in bottle)} \\
& x_1,\ x_2 \geq 0 &
\end{array}
$$

Solving Oranj Example with Big M Method

1. Modify the constraints so that the RHS of each constraint is nonnegative

The RHS of each constraint is nonnegative

2. Convert each inequality constraint to standard form

$$
\begin{array}{rcl}
z - 2x_1 - 3x_2 & = & 0 \\
0.5x_1 + 0.25x_2 + s_1 & = & 4 \\
x_1 + 3x_2 - e_2 & = & 20 \\
x_1 + x_2 & = & 10
\end{array}
$$

all variables nonnegative

3. Add a_i to the constraints identified as $>$ or $=$ const.s

$$
\begin{array}{rcll}
z - 2x_1 - 3x_2 & = & 0 & \text{Row 0} \\
0.5x_1 + 0.25x_2 + s_1 & = & 4 & \text{Row 1} \\
x_1 + 3x_2 - e_2 + a_2 & = & 20 & \text{Row 2} \\
x_1 + x_2 + a_3 & = & 10 & \text{Row 3}
\end{array}
$$

all variables nonnegative

4. Add Ma_i to the objective function (min problem)

$$\min z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$

Row 0 will change to

$$z - 2x_1 - 3x_2 - Ma_2 - Ma_3 = 0$$

<u>5. Since each artificial variable are in our starting bfs, they must be eliminated from row 0</u>

New Row 0 = Row 0 + M * Row 2 + M * Row 3 $\Rightarrow$

$z + (2M–2) x_1 + (4M–3) x_2 – M e_2$ = 30M   New Row 0

Initial tableau:

z	x_1	x_2 $\Downarrow$	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	2M-2	**4M-3**	0	-M	0	0	30M	z=30M	
0	0.5	0.25	1	0	0	0	4	s_1=4	16
0	1	3	0	-1	1	0	20	a_2=20	**20/3***
0	1	1	0	0	0	1	10	a_3=10	10

In a min problem, entering variable is the variable that has the "most positive" coefficient in row 0!

First tableau:

z	x_1 $\Downarrow$	x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	**(2M-3)/3**	0	0	(M-3)/3	(3-4M)/3	0	20+3.3M	z	
0	5/12	0	1	1/12	-1/12	0	7/3	s_1	28/5
0	1/3	1	0	-1/3	1/3	0	20/3	x_2	20
0	2/3	0	0	1/3	-1/3	1	10/3	a_3	**5***

Optimal tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV
1	0	0	0	-1/2	(1-2M)/2	(3-2M)/2	25	z=25
0	0	0	1	-1/8	1/8	-5/8	1/4	s_1=1/4
0	0	1	0	-1/2	1/2	-1/2	5	x_2=5
0	1	0	0	1/2	-1/2	3/2	5	x_1=5

Report:

In a bottle of Oranj, there should be 5 oz orange soda and 5 oz orange juice.

In this case the cost would be 25¢.

Example 2. Modified Oranj Juice

Consider Bevco's problem. It is modified so that 36 mg of vitamin C are required. Related LP model is given as follows:

Let x_1 and x_2 be the quantity of ounces of orange soda and orange juice (respectively) in a bottle of Oranj.

$$\min z = 2x_1 + 3x_2$$
s.t. $0.5\, x_1 + 0.25\, x_2 \leq 4$ (sugar const.)
           $x_1 + 3\, x_2 \geq 36$ (vit. C const.)
           $x_1 + x_2 = 10$ (10 oz in bottle)
      $x_1, x_2 \geq 0$

Solving with Big M method:

Initial tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	2M-2	**4M-3**	0	-M	0	0	46M	z=46M	
0	0.5	0.25	1	0	0	0	4	s_1=4	16
0	1	3	0	-1	1	0	36	a_2=36	36/3
0	1	**1**	0	0	0	1	10	a_3=10	**10** $\Rightarrow$

Optimal tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV
1	1-2M	0	0	-M	0	3-4M	30+6M	z=30+6M
0	1/4	0	1	0	0	-1/4	3/2	s_1=3/2
0	-2	0	0	-1	1	-3	6	a_2=6
0	1	1	0	0	0	1	10	x_2=10

An artificial variable (a_2) is BV so the original LP has no feasible solution

Report:

It is impossible to produce Oranj under these conditions.

3.2.5 Unrestricted in Sign Variables

Some variables are allowed to be unrestricted in sign (urs).

To be solved, an LP with urs variables should be transformed into an LP in which all variables are required to be nonnegative.

For each urs variable x_i,

- define two new variables x_i' and x_i''
- in each constraint and in the objective function, substitute $x_i' - x_i''$ for x_i

  $x_i = x_i' - x_i''$

- add the sign restrictions $x_i' \geq 0$ and $x_i'' \geq 0$

Example. Baker

A baker has 30 kg. of flour and 8 packages of yeast. Baking a loaf of bread requires 0.5 kg. of flour and 0.2 package of yeast. Each loaf of bread can be sold for $3. The

baker may purchase additional flour at \$4/kg. or sell leftover flour at the same price. Formulate and solve an LP model to help the baker maximize profits.

Answer

Decision variables:

- x_1: number of loaves of bread baked
- x_2: ounces by which flour supply is increased by transactions

Therefore, if x_2 oz of flour were purchased then $x_2 > 0$, if $-x_2$ ounces of flour were sold then $x_2 < 0$, if no flour was bought or sold then $x_2 = 0$.

LP model would be:

$$max\ z = 3x_1 - 4x_2$$

s.t.

$$0.5x_1 \le 30 + x_2$$
$$0.2x_1 \le 8$$
$$x_1 \ge 0,\ x_2\ urs$$

Use the substitution $x_2 = x_2' - x_2''$

$$max\ z = 3x_1 - 4x_2' + 4x_2''$$

s.t.

$$0.5x_1 \le 30 + x_2' - x_2''$$
$$0.2x_1 \le 8$$
$$x_1, x_2', x_2'' \ge 0$$

Apply Simplex Algorithm.

Convert the problem to standard form:

$$Max\ Z = 3x_1 - 4x_2' + 4x_2''$$
$$0.5x_1 - x_2' + x_2'' + s_1 = 30$$
$$0.2x_1 + s_2 = 8$$
$$x_1, x_2', x_2'', s_1, s_2 \ge 0$$

Move the variables to left-hand side in the objective function:

$$Z - 3x_1 + 4x_2' - 4x_2'' = 0$$

Construct the initial tableau (No need to add artificial variable):

	Z	x_1	x_2'	x_2''	s_1	s_2	RHS	BV	Ratio
R0	1	-3	4	-4	0	0	0	Z=0	
R1	0	0.5	-1	1	1	0	30	s_1=30	30
R2	0	0.2	0	0	0	1	8	s_2=8	-

As we have negative values in R0, this bfs is not optimal.

x_2" enters the basis as it has the lowest R0 value.

R1 has the minimum ratio. Thus, s_1 leaves the basis.

Apply EROs to make x_2" a basic variable in R1:

R1' = R1,   R0' = R0 + 4R1',   R2' = R2

The first tableau:

	z	x_1	x_2'	x_2"	s_1	s_2	RHS	BV	Ratio
R0	1	-1	0	0	4	0	120	Z=120	
R1	0	0.5	-1	1	1	0	30	x_2"=30	60
R2	0	0.2	0	0	0	1	8	s_2=8	40*

As we have negative values in R0, this bfs is not optimal.

x_1 enters the basis as it has the lowest R0 value.

R2 has the minimum ratio. Thus, s_2 leaves the basis.

Apply EROs to make x_1 a basic variable in R2:

R2' = R2/0,2,   R0' = R0 + R2',   R1' = R1 - 0,5R2'

	z	x_1	x_2'	x_2"	s_1	s_2	RHS	BV
R0	1	0	0	0	4	5	160	Z=160
R1	0	0	-1	1	1	-2.5	10	x_2"=10
R2	0	1	0	0	0	5	40	x_1=40

As we don't have any negative values in R0, this bfs is optimal.

Optimal solution: x_1 = 40; x_2'= 0, and x_2"= 10 → x_2 = -10; z = 160

$x_2 = x_2' - x_2$" , x_2 = 0 - 10 = -10.

Report

The baker can earn a profit of $160 by baking and selling 40 loaves of bread. He should also sell the remaining 10 kg. of flour.

3.3   DUALITY

3.3.1 Primal – Dual

Associated with any LP is another LP called the **dual**. Knowledge of the dual provides interesting economic and sensitivity analysis insights. When taking the dual of any LP, the given LP is referred to as the **primal**. If the primal is a max problem, the dual will be a min problem and vice versa.

3.3.2 Finding the Dual of an LP

The dual of a ***normal max*** problem is a ***normal min*** problem.

> *Normal max problem is a problem in which all the variables are required to be nonnegative and all the constraints are ≤ constraints.*
>
> *Normal min problem is a problem in which all the variables are required to be nonnegative and all the constraints are ≥ constraints.*

Similarly, the dual of a normal min problem is a normal max problem.

Finding the Dual of a Normal Max Problem

PRIMAL

$$\max z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

s.t.

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m$$

$$x_j \geq 0 \ (j = 1, 2, \ldots, n)$$

DUAL

$$\min w = b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$$

s.t.

$$a_{11} y_1 + a_{21} y_2 + \ldots + a_{m1} y_m \geq c_1$$
$$a_{12} y_1 + a_{22} y_2 + \ldots + a_{m2} y_m \geq c_2$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$a_{1n} y_1 + a_{2n} y_2 + \ldots + a_{mn} y_m \geq c_n$$

$$y_i \geq 0 \ (i = 1, 2, \ldots, m)$$

Finding the Dual of a Normal Min Problem

PRIMAL

$$\min w = b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$$

s.t.

$$a_{11} y_1 + a_{21} y_2 + \ldots + a_{m1} y_m \geq c_1$$
$$a_{12} y_1 + a_{22} y_2 + \ldots + a_{m2} y_m \geq c_2$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$a_{1n} y_1 + a_{2n} y_2 + \ldots + a_{mn} y_m \geq c_n$$

$$y_i \geq 0 \ (i = 1, 2, \ldots, m)$$

DUAL

$$\max z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

s.t.

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2$$

$$\ldots \quad \ldots \quad \quad \ldots \quad \ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m$$

$$x_j \ge 0 \; (j = 1, 2, \ldots, n)$$

Finding the Dual of a Nonnormal Max Problem

- If the ith primal constraint is a $\ge$ constraint, the corresponding dual variable y_i must satisfy $y_i \le 0$

- If the ith primal constraint is an equality constraint, the dual variable y_i is now unrestricted in sign (urs).

- If the ith primal variable is urs, the ith dual constraint will be an equality constraint

Finding the Dual of a Nonnormal Min Problem

- If the ith primal constraint is a $\le$ constraint, the corresponding dual variable x_i must satisfy $x_i \le 0$

- If the ith primal constraint is an equality constraint, the dual variable x_i is now urs.

- If the ith primal variable is urs, the ith dual constraint will be an equality constraint

Example

Primal model:

max $z = 20x_1 - 4x_2$

s.t. $\quad 5x_1 - \quad x_2 \le 30$

$\quad\quad\quad x_1 + 2x_2 \ge \; 5$

$\quad\quad\quad 2x_1 + 3x_2 = 10$

$\quad\quad\quad x_1 \ge 0, \; x_2$ urs

Dual model:

min w $= 30y_1 + 5y_2 + 10y_3$

s.t. $\quad 5y_1 + \quad y_2 + \quad 2y_3 \ge 20$

$\quad\quad\quad -y_1 + 2y_2 + \quad 3y_3 = -4$

$\quad\quad\quad y_1 \ge 0, \; y_2 \le 0, \; y_3$ urs

3.3.3 The Dual Theorem

The primal and dual have equal optimal objective function values (if the problems have optimal solutions).

Weak duality implies that if for any feasible solution to the primal and an feasible solution to the dual, the *w*-value for the feasible dual solution will be at least as large as the *z*-value for the feasible primal solution → $z \leq w$.

Consequences

- Any feasible solution to the dual can be used to develop a bound on the optimal value of the primal objective function.

- If the primal is unbounded, then the dual problem is infeasible.

- If the dual is unbounded, then the primal is infeasible.

- How to read the optimal dual solution from Row 0 of the optimal tableau if the primal is a max problem:

  'optimal value of dual variable y_i'

  = 'coefficient of s_i in optimal row 0'   (if const. *i* is a ≤ const.)

  = –'coefficient of e_i in optimal row 0'   (if const. *i* is a ≥ const.)

  = 'coefficient of a_i in optimal row 0' – M (if const. *i* is a = const.)

- How to read the optimal dual solution from Row 0 of the optimal tableau if the primal is a min problem:

  'optimal value of dual variable x_i'

  = 'coefficient of s_i in optimal row 0'   (if const. *i* is a ≤ const.)

  = –' coefficient of e_i in optimal row 0'   (if const. *i* is a ≥ const.)

  = 'coefficient of a_i in optimal row 0' + M (if const. *i* is a = const.)

Example

LP and its optimal tableau is given as:

$$\max z = 3x_1 + 2x_2 + 5x_3$$
$$\text{s.t.} \quad x_1 + 3x_2 + 2x_3 \leq 15$$
$$2x_2 - x_3 \geq 5$$
$$2x_1 + x_2 - 5x_3 = 10$$
$$x_1, x_2, x_3 \geq 0$$

z	x_1	x_2	x_3	s_1	e_2	a_2	a_3	rhs	Basic Variable
1	0	0	0	$\frac{51}{23}$	$\frac{58}{23}$	$M - \frac{58}{23}$	$M + \frac{9}{23}$	$\frac{565}{23}$	$z = \frac{565}{23}$
0	0	0	1	$\frac{4}{23}$	$\frac{5}{23}$	$-\frac{5}{23}$	$-\frac{2}{23}$	$\frac{15}{23}$	$x_3 = \frac{15}{23}$
0	0	1	0	$\frac{2}{23}$	$-\frac{9}{23}$	$\frac{9}{23}$	$-\frac{1}{23}$	$\frac{65}{23}$	$x_2 = \frac{65}{23}$
0	1	0	0	$\frac{9}{23}$	$\frac{17}{23}$	$-\frac{17}{23}$	$\frac{7}{23}$	$\frac{120}{23}$	$x_1 = \frac{120}{23}$

Use this information to find the optimal solution to the dual LP

Answer

Dual LP:

$$\min w = 15y_1 + 5y_2 + 10y_3$$
$$\text{s.t.} \quad y_1 \quad\quad + 2y_3 \geq 3$$
$$3y_1 + 2y_2 + \ y_3 \geq 2$$
$$2y_1 - \ y_2 - 5y_3 \geq 5$$
$$y_1 \geq 0, y_2 \leq 0, y_3 \text{ urs}$$

According to dual theorem consequences;

'opt. value of dual var. y_i'

= 'coef. of s_i in opt. row 0'   (if const. i is a $\leq$ const.)

= –'coef. of e_i in opt. row 0'   (if const. i is a $\geq$ const.)

= 'coef. of a_i in opt. row 0' – M   (if const. i is a = const.)

We look at Row 0 of the optimal tableau of the primal model, and find the optimal dual solution as:

$w = 565/23$, $y_1 = 51/23$, $y_2 = -58/23$, $y_3 = 9/23$

Example

LP and its optimal tableau is given as:

$$\min w = 3y_1 + 2y_2 + y_3$$
$$\text{s.t.} \quad y_1 + y_2 + \ y_3 \geq 4$$
$$y_2 - \ y_3 \leq 2$$
$$y_1 + y_2 + 2y_3 = 6$$
$$y_1, y_2, y_3 \geq 0$$

w	y_1	y_2	y_3	e_1	s_2	a_1	a_3	rhs
1	−1	0	0	−3	0	$3 - M$	$-1 - M$	6
0	1	1	0	−2	0	2	−1	2
0	−1	0	0	3	1	−3	2	2
0	0	0	1	1	0	−1	1	2

Use this information to find the optimal solution to the dual LP

Answer

Dual LP:

$$\max z = 4x_1 + 2x_2 + 6x_3$$
$$\text{s.t.} \quad x_1 \qquad + \ x_3 \le 3$$
$$x_1 + x_2 + \ x_3 \le 2$$
$$x_1 - x_2 + 2x_3 \le 1$$
$$x_1 \ge 0, \ x_2 \le 0, \ x_3 \text{ urs}$$

According to dual theorem consequences;

'opt. value of dual var. x_i'

= 'coef. of s_i in opt. row 0'   (if const. i is a $\le$ const.)

= –'coef. of e_i in opt. row 0'   (if const. i is a $\ge$ const.)

= 'coef. of a_i in opt. row 0' – M   (if const. i is a = const.)

We look at Row 0 of the optimal tableau of the primal model, and find the optimal dual solution as:

$z = 6$, $x_1 = 3$, $x_2 = 0$, $x_3 = -1$

3.3.4 Economic Interpretation

When the primal is a normal max problem, the dual variables are related to the value of resources available to the decision maker. For this reason, dual variables are often referred to as *resource shadow prices*.

Example

PRIMAL

Let x_1, x_2, x_3 be the number of desks, tables and chairs produced. Let the weekly profit be \$$z$. Then, we must

$$\max z = 60x_1 + 30x_2 + 20x_3$$
$$\text{s.t.} \quad 8x_1 + \ 6x_2 + \quad x_3 \le \ 48 \text{ (Lumber constraint)}$$
$$4x_1 + \ 2x_2 + 1.5x_3 \le \ 20 \text{ (Finishing hour constraint)}$$
$$2x_1 + 1.5x_2 + 0.5x_3 \le \quad 8 \text{ (Carpentry hour constraint)}$$
$$x_1, \ x_2, \ x_3 \ge 0$$

DUAL

Suppose an entrepreneur wants to purchase all of Dakota's resources.

In the dual problem y_1, y_2, y_3 are the resource prices (price paid for one board ft of lumber, one finishing hour, and one carpentry hour).

\$$w$ is the cost of purchasing the resources.

Resource prices must be set high enough to induce Dakota to sell. i.e. total purchasing cost equals total profit.

min $w = 48y_1 + 20y_2 + 8y_3$

s.t. $\quad 8y_1 + 4y_2 + 2y_3 \geq 60$ (Desk constraint)

$\qquad 6y_1 + 2y_2 + 1.5y_3 \geq 30$ (Table constraint)

$\qquad y_1 + 1.5y_2 + 0.5y_3 \geq 20$ (Chair constraint)

$\quad y_1, y_2, y_3 \geq 0$

3.4 SENSITIVITY ANALYSIS

3.4.1 Reduced Cost

For any nonbasic variable, the reduced cost for the variable is the amount by which the nonbasic variable's objective function coefficient must be improved before that variable will become a basic variable in some optimal solution to the LP.

If the objective function coefficient of a nonbasic variable x_k is improved by its reduced cost, then the LP will have alternative optimal solutions at least one in which x_k is a basic variable, and at least one in which x_k is not a basic variable.

If the objective function coefficient of a nonbasic variable x_k is improved by more than its reduced cost, then any optimal solution to the LP will have x_k as a basic variable and $x_k > 0$.

Reduced cost of a basic variable is zero (see definition)!

3.4.2 Shadow Price

We define the shadow price for the ith constraint of an LP to be the amount by which the optimal z value is "improved" (increased in a max problem and decreased in a min problem) if the RHS of the ith constraint is increased by 1.

This definition applies only if the change in the RHS of the constraint leaves the current basis optimal!

A ≥ constraint will always have a nonpositive shadow price; a ≤ constraint will always have a nonnegative shadow price.

3.4.3 Conceptualization

$$\max z = 6\,x_1 + x_2 + 10x_3$$
$$x_1 + x_3 \leq 100$$
$$x_2 \leq 1$$

All variables ≥ 0

This is a very easy LP model and can be solved manually without utilizing Simplex.

$x_2 = 1$ (This variable does not exist in the first constraint. In this case, as the problem is a maximization problem, the optimum value of the variable equals the RHS value of the second constraint).

$x_1 = 0$, $x_3 = 100$ (These two variables do exist only in the first constraint and as the objective function coefficient of x_3 is greater than that of x_1, the optimum value of x_3 equals the RHS value of the first constraint).

Hence, the optimal solution is as follows:

$z = 1001$, $[x_1, x_2, x_3] = [0, 1, 100]$

Similarly, sensitivity analysis can be executed manually.

Reduced Cost

As x_2 and x_3 are in the basis, their reduced costs are 0.

In order to have x_1 enter in the basis, we should make its objective function coefficient as great as that of x_3. In other words, improve the coefficient as 4 (10-6). New objective function would be (max $z = 10x_1 + x_2 + 10x_3$) and there would be at least two optimal solutions for $[x_1, x_2, x_3]$: $[0, 1, 100]$ and $[100, 1, 0]$.

Therefore reduced cost of x_1 equals 4.

If we improve the objective function coefficient of x_1 more than its reduced cost, there would be a unique optimal solution: $[100, 1, 0]$.

Shadow Price

If the RHS of the first constraint is increased by 1, new optimal solution of x_3 would be 101 instead of 100. In this case, new z value would be 1011.

If we use the definition: 1011 - 1001 = 10 is the shadow price of the first constraint.

Similarly the shadow price of the second constraint can be calculated as 1 (please find it).

3.4.4 Utilizing Lindo Output for Sensitivity

NOTICE: The objective function which is regarded as Row 0 in Simplex is accepted as Row 1 in Lindo.

Therefore the first constraint of the model is always second row in Lindo!!!

```
MAX    6 X1 + X2 + 10 X3
SUBJECT TO
   2)  X1 + X3 <=  100
   3)  X2 <=  1
END

LP OPTIMUM FOUND AT STEP    1

   OBJECTIVE FUNCTION VALUE
   1)    1001.000
```

VARIABLE	VALUE	REDUCED COST
X1	0.000000	4.000000
X2	1.000000	0.000000
X3	100.000000	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	10.000000
3)	0.000000	1.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1	6.000000	4.000000	INFINITY
X2	1.000000	INFINITY	1.000000
X3	10.000000	INFINITY	4.000000

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	100.000000	INFINITY	100.000000
3	1.000000	INFINITY	1.000000

Lindo output reveals the reduced costs of x_1, x_2, and x_3 as 4, 0, and 0 respectively.

In the maximization problems, the reduced cost of a non-basic variable can also be read from the allowable increase value of that variable at obj. coefficient ranges. Here, the corresponding value of x_1 is 4.

In the minimization problems, the reduced cost of a non-basic variable can also be read from the allowable decrease value of that variable at obj. coefficient ranges.

The same Lindo output reveals the shadow prices of the constraints in the "dual price" section:

Here, the shadow price of the first constraint (Row 2) equals 10.

The shadow price of the second constraint (Row 3) equals 1.

3.4.5 Some important equations

If the change in the RHS of the constraint leaves the current basis optimal (within the allowable RHS range), the following equations can be used to calculate new objective function value:

for maximization problems

- new obj. fn. value = old obj. fn. value + (new RHS – old RHS) × shadow price

for minimization problems

- new obj. fn. value = old obj. fn. value – (new RHS – old RHS) × shadow price

For Lindo example, as the allowable increases in RHS ranges are infinity for each constraint, we can increase RHS of them as much as we want. But according to allowable decreases, RHS of the first constraint can be decreased by 100 and that of second constraint by 1.

Lets assume that new RHS value of the first constraint is 60.

As the change is within allowable range, we can use the first equation (max. problem):

$$z_{new} = 1001 + (60 - 100) 10 = 601.$$

3.4.6 Utilizing Simplex for Sensitivity

In Dakota furniture example; x_1, x_2, and x_3 were representing the number of desks, tables, and chairs produced.

The LP formulated for profit maximization:

$$
\begin{array}{llllll}
\max z = & 60\,x_1 & + 30\,x_2 & + 20x_3 \\
& 8\,x_1 & + 6\,x_2 & + x_3 & + s_1 & & = 48 & \text{Lumber} \\
& 4\,x_1 & + 2\,x_2 & +1.5\,x_3 & & + s_2 & = 20 & \text{Finishing} \\
& 2\,x_1 & +1.5\,x_2 & + .5\,x_3 & & & + s_3 & = 8 & \text{Carpentry} \\
& & x_2 & & & & & + s_4 = 5 & \text{Demand}
\end{array}
$$

The optimal solution was:

$$
\begin{array}{llllll}
z & +5\,x_2 & & +10\,s_2 & +10\,s_3 & = 280 \\
& -2\,x_2 & +s_1 & +2\,s_2 & -8\,s_3 & = 24 \\
& -2\,x_2 & + x_3 & +2\,s_2 & -4\,s_3 & = 8 \\
+ x_1 & + 1.25\,x_2 & & -.5\,s_2 & +1.5\,s_3 & = 2 \\
& x_2 & & & + s_4 & = 5
\end{array}
$$

Analysis 1

Suppose available finishing time changes from $20 \rightarrow 20+\delta$, then we have the system:

$$
\begin{array}{llllll}
z' = & 60\,x_1' & + 30\,x_2' & + 20\,x_3' \\
& 8\,x_1' & + 6\,x_2' & + x_3' & + s_1' & & = 48 \\
& 4\,x_1' & + 2\,x_2' & +1.5\,x_3' & & + s_2' & = 20+\delta \\
& 2\,x_1' & +1.5\,x_2' & + .5\,x_3' & & & + s_3' & = 8 \\
& & + x_2' & & & & & + s_4' = 5
\end{array}
$$

or equivalently:

$$
\begin{array}{llllll}
z' = & 60\,x_1' & + 30\,x_2' & + 20\,x_3' \\
& 8\,x_1' & + 6\,x_2' & + x_3' & + s_1' & & = 48
\end{array}
$$

$$4x_1' + 2x_2' + 1.5x_3' + (s_2'-\delta) = 20$$
$$2x_1' + 1.5x_2' + .5x_3' + s_3' = 8$$
$$+ x_2' + s_4' = 5$$

That is z', x_1', x_2', x_3', x_4', s_1', $s_2'-\delta$, s_3', s_4' satisfy the original problem, and hence (1)

Substituting in:

$$z' + 5x_2' + 10(s_2'-\delta) + 10s_3' = 280$$
$$-2x_2' + s_1' + 2(s_2'-\delta) - 8s_3' = 24$$
$$-2x_2' + x_3' + 2(s_2'-\delta) - 4s_3' = 8$$
$$+ x_1' + 1.25x_2' - .5(s_2'-\delta) + 1.5s_3' = 2$$
$$x_2' + s_4' = 5$$

and thus

$$z' + 5x_2' + 10s_2' + 10s_3' = 280+10\delta$$
$$-2x_2' + s_1' + 2s_2' - 8s_3' = 24+2\delta$$
$$-2x_2' + x_3' + 2s_2' - 4s_3' = 8+2\delta$$
$$+ x_1' + 1.25x_2' - .5s_2' + 1.5s_3' = 2-.5\delta$$
$$x_2' + s_4' = 5$$

For $-4 \leq \delta \leq 4$, the new system maximizes z'. In this range RHS values are non-negative.

As δ increases, revenue increases by 10δ. Therefore, the **shadow price** of finishing labor is $10 per hr. (This is valid for up to 4 extra hours or 4 fewer hours).

Analysis 2

What happens if revenue from desks changes to $60+\gamma$? For small γ, revenue increases by 2γ (as we are making 2 desks currently). But how large an increase is possible?

The new revenue is:

$$z' = (60+\gamma)x_1 + 30x_2 + 20x_3 = z+\gamma x_1$$
$$= (280 - 5x_2 - 10s_2 - 10s_3) + \gamma(2 - 1.25x_2 + .5s_2 - 1.5s_3)$$
$$= 280 + 2\gamma - (5 + 1.25\gamma)x_2 - (10-.5\gamma)s_2 - (10 + 1.5\gamma)s_3$$

So the top line in the final system would be:

$$z' + (5 + 1.25\gamma)x_2 + (10 - .5\gamma)s_2 + (10 + 1.5\gamma)s_3 = 280 + 2\gamma$$

Provided all terms in this row are ≥ 0, we are still optimal.

For $-4 \leq \gamma \leq 20$, the current production schedule is still optimal.

Analysis 3

If revenue from a non-basic variable changes, the revenue is

$z' = 60x_1 + (30 + \gamma)x_2 + 20x_3 = z + \gamma x_2$

$= 280 - 5x_2 - 10s_2 - 10s_3 + \gamma x_2$

$= 280 - (5 - \gamma)x_2 - 10s_2 - 10s_3$

The current solution is optimal for $\gamma \leq 5$. But when $\gamma > 5$ or the revenue per table is increased past \$35, it becomes better to produce tables. We say the **reduced cost** of tables is \$5.00.

3.4.7 Interpretation of Dual Prices

For maximization problems:

- The shadow price of the ith constraint = the optimal value of the ith dual variable

For minimization problems:

- The shadow price of the ith constraint = -(the optimal value of the ith dual variable)

Example

PRIMAL: x_1, x_2, x_3 be the number of desks, tables and chairs produced

$\max z = 60x_1 + 30 x_2 + 20 x_3$

s.t.
$8x_1 + 6 x_2 + x_3 \leq 48$ (lumber)

$4x_1 + 2 x_2 + 1.5 x_3 \leq 20$ (finishing)

$2x_1 + 1.5 x_2 + .5 x_3 \leq 8$ (carpentry)

$x_1, x_2, x_3 \geq 0$

DUAL: y_1, y_2, y_3 are the resource prices (price paid for one board ft of lumber, one finishing hour, and one carpentry hour)

$\min w = 48y_1 + 20y_2 + 8y_3$

s.t.
$8y_1 + 4y_2 + 2y_3 \geq 60$ (Desk constraint)

$6y_1 + 2y_2 + 1.5y_3 \geq 30$ (Table constraint)

$y_1 + 1.5y_2 + 0.5y_3 \geq 20$ (Chair constraint)

$y_1, y_2, y_3 \geq 0$

In this case, the optimal value of the dual variables may be read at the Lindo output of primal model:

OBJECTIVE FUNCTION VALUE

1)   280.0000 ➔ **w**

VARIABLE	VALUE	REDUCED COST
X1	2.000000	0.000000
X2	0.000000	5.000000
X3	8.000000	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES	
2)	24.000000	0.000000	→ y_1
3)	0.000000	10.000000	→ y_2
4)	0.000000	10.000000	→ y_3

3.4.8 Duality and Sensitivity Analysis

We may utilize three types of sensitivity analysis by using "duaity and sensitivity":

- Changing the Objective Function Coefficient of a NBV
- Changing the column of a NBV
- Adding a new activity

Changing the Objective Function Coefficient of a NBV

Example

c_2 is the price at which a table is sold. For what values of c_2 will the current basis remain optimal?

Answer

The 2nd (table) dual constraint:

$$6y_1 + 2y_2 + 1.5y_3 \geq c_2$$

$$y_1 = 0, y_2 = 10, y_3 = 10$$

The current basis remains optimal if c_2 satisfies

$$6 (0) + 2 (10) + 1.5 (10) \geq c_2 \rightarrow c_2 \leq 35$$

The current basis is no longer optimal if $c_2 > 35$

Changing the column of a NBV

Example

Suppose a table sells for $43 and uses 5 board feet of lumber, 2 finishing hours, and 2 carpentry hours. Does the current basis remain optimal?

Answer

$$\max z = 60x_1 + 30 x_2 + 20 x_3$$

$$\text{s.t.} \quad 8x_1 + 6 x_2 + x_3 \leq 48 \text{ (lumber)}$$

$$4x_1 + 2x_2 + 1.5x_3 \leq 20 \text{ (finishing)}$$
$$2x_1 + 1.5x_2 + .5x_3 \leq 8 \text{ (carpentry)}$$
$$x_1, x_2, x_3 \geq 0$$

$$\max z = 60x_1 + \textcolor{red}{43}x_2 + 20x_3$$

s.t.
$$8x_1 + \textcolor{red}{5}x_2 + x_3 \leq 48 \text{ (lumber)}$$
$$4x_1 + 2x_2 + 1.5x_3 \leq 20 \text{ (finishing)}$$
$$2x_1 + \textcolor{red}{2}x_2 + .5x_3 \leq 8 \text{ (carpentry)}$$
$$x_1, x_2, x_3 \geq 0$$

$$5y_1 + 2y_2 + 2y_3 \textcolor{red}{\geq?} 43 \text{ (New table constraint in dual)}$$

$40 < 43 \rightarrow$ The current basis is no longer optimal

Each table uses \$40 worth of resources and sells for \$43. So the revenue for the table would be \$3. In this case, table will be basic in the new optimal solution.

Adding a new activity

Example

Suppose Dakota considers manufacturing footstools (x_4). A footstool sells for \$15 and uses 1 board foot of lumber, 1 finishing hour, and 1 carpentry hour. Does the current basis remain optimal?

Answer

$$\max z = 60x_1 + 30x_2 + 20x_3 + \textcolor{red}{15}x_4$$

s.t.
$$8x_1 + 6x_2 + x_3 + \textcolor{red}{x_4} \leq 48 \text{ (lumber)}$$
$$4x_1 + 2x_2 + 1.5x_3 + \textcolor{red}{x_4} \leq 20 \text{ (finishing)}$$
$$2x_1 + 1.5x_2 + .5x_3 + \textcolor{red}{x_4} \leq 8 \text{ (carpentry)}$$
$$x_1, x_2, x_3, x_4 \geq 0$$

$$y_1 + y_2 + y_3 \textcolor{red}{\geq?} 15 \text{ (Footstool constraint in dual)}$$

$20 > 15 \rightarrow$ The current basis remains optimal

Dakota should not make footstools.

3.4.9 The 100% Rule

If the values of more than one parameter of the model changes, we utilize 100% rule. These changes:

- Change in an Objective Function Coefficient
- Change in a Right-Hand Side

Change in an Objective Function Coefficient

CASE 1:

The case where all variables whose objective function coefficients are changed have non-zero reduced costs.

- The current basis remains optimal if and only if the objective function coefficient for each variable remains within the allowable range

- If the objective function coefficient for any variable is outside its allowable range, then the current basis is no longer optimal.

CASE 2:

The case where at least one variable whose objective function coefficient is changed has a reduced cost of zero. 100% rule is utilized:

$$\text{If } \Delta c_j \geq 0, \; r_j \; = \; \frac{\Delta c_j}{I_j} \quad I_j \text{: allowable increase}$$

$$\text{If } \Delta c_j \leq 0, \; r_j \; = \; \frac{-\Delta c_j}{D_j} \quad D_j \text{: allowable decrease}$$

- $\sum r_j \leq 1 \rightarrow$ current basis remains optimal

- $\sum r_j > 1 \rightarrow$ current basis may be or may not be optimal, we can't be sure

Example

If the price of brownie (x_1) changes from 50¢ to 60¢ and the price of cheesecake (x_4) changes from 80¢ to 50¢, does the current basis remain optimal?

What would be the new optimal solution?

Answer

VARIABLE	VALUE	REDUCED COST
X1	0.000000	27.500000
X4	0.000000	50.000000

As both decision variables are NBVs, we are in CASE 1

	OBJ COEFFICIENT RANGES		
VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1	50.000000	INFINITY	27.500000
X4	80.000000	INFINITY	50.000000

New prices are within allowable ranges: <u>current basis remains optimal</u>

Also the optimal *z*-value and optimal value of the decision variables remain unchanged.

Example

If the price of brownie (x_1) changes from 50¢ to 40¢ and the price of cheesecake (x_4) changes from 80¢ to 25¢, does the current basis remain optimal?

What would be the new optimal solution?

Answer

VARIABLE	VALUE	REDUCED COST
X1	0.000000	27.500000
X4	0.000000	50.000000

As both decision variables are NBVs, we are in CASE 1.

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1	50.000000	INFINITY	27.500000
X4	80.000000	INFINITY	50.000000

The price of cheesecake does not remain in its allowable range: <u>current basis is no longer optimal</u>

The problem must be solved again.

Example

The price of desk (x_1) changes from $60 to $70 and the price of chair (x_3) changes from $20 to $18. Does the current basis remain optimal? What would be the new optimal solution?

Answer

VARIABLE	VALUE	REDUCED COST
X1	2.000000	0.000000
X3	8.000000	0.000000

As at least one decision is a BV, we are in CASE 2.

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1	60.000000	20.000000	4.000000
X3	20.000000	2.500000	5.000000

$\Delta c_1 = 10, I_1 = 20, r_1 = \frac{10}{20} = 0.5; \quad \Delta c_3 = -2, D_3 = 5, r_3 = \frac{2}{5} = 0.4$

$r_1 + r_2 + r_3 = 0.9 \leq 1 \rightarrow$ <u>Current basis remains optimal</u>

So the values of the decision variables do not change.

New profit = 280 + 2 (10) + 8 (-2) = 284

Example

The price of desk (x_1) changes from $60 to $58 and the price of table (x_2) changes from $30 to $33. Does the current basis remain optimal?

What would be the new optimal solution?

Answer

VARIABLE	VALUE	REDUCED COST
X1	2.000000	0.000000
X2	0.000000	5.000000

As at least one decision is a BV, we are in CASE 2.

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1	60.000000	20.000000	4.000000
X2	30.000000	5.000000	INFINITY

$\Delta c_1 = -2$, $D_1 = 4$, $r_1 = \frac{2}{4} = 0.5$; $\Delta c_2 = 3$, $I_2 = 5$, $r_3 = \frac{3}{5} = 0.6$

$r_1 + r_2 + r_3 = 1.1 > 1 \rightarrow$

<u>No information about whether the current basis is optimal</u>

Changes in Right-Hand Sides

CASE 1:

The case where all constraints whose Right-Hand Sides are being modified are nonbinding constraints

- The current basis remains optimal if and only if each right hand side remains within its allowable range

- If the right-hand side for any constraint is outside its allowable range, then the current basis is no longer optimal

CASE 2:

The case where at least one of the constraints whose Right-Hand Side is being modified is a binding constraint. 100% rule is utilized:

$\quad$ If $\Delta b_i \geq 0$, $r_i = \frac{\Delta b_i}{I_i}$ I_i: allowable increase

$\quad$ If $\Delta b_i \leq 0$, $r_i = \frac{-\Delta b_i}{D_i}$ D_i: allowable decrease

- $\sum r_i \leq 1 \rightarrow$ current basis remains optimal

- $\sum r_i > 1 \rightarrow$ current basis may be or may not be optimal, we can't be sure

Example

The calorie requirement changes from 500 cal. to 400 cal. and the fat requirement changes from 8 oz. to 10 oz. Does the current basis remain optimal? What would be the new optimal solution?

Answer

ROW	SLACK OR SURPLUS	DUAL PRICES
CALORIE)	250.000000	0.000000
FAT)	5.000000	0.000000

As both constraints are nonbinding, we are in CASE 1

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
CALORIE	500.000000	250.000000	INFINITY
FAT	8.000000	5.000000	INFINITY

New requirements are within allowable ranges: <u>current basis remains optimal</u>

Also the optimal z-value and optimal value of the decision variables remain unchanged.

Example

The calorie requirement changes from 500 cal. to 400 cal. and the fat requirement changes from 8 oz. to 15 oz. Does the current basis remain optimal?

What would be the new optimal solution?

Answer

ROW	SLACK OR SURPLUS	DUAL PRICES
CALORIE)	250.000000	0.000000
FAT)	5.000000	0.000000

As both constraints are nonbinding, we are in CASE 1.

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
CALORIE	500.000000	250.000000	INFINITY
FAT	8.000000	5.000000	INFINITY

The fat requirement is no longer in its allowable range: <u>the current basis is no longer optimal</u>

The problem must be solved again.

Example

Available finishing changes from 20 hrs to 22 hrs and available carpentry changes from 8 hrs to 9 hrs. Does the current basis remain optimal?

What would be the new optimal solution?

Answer

ROW	SLACK OR SURPLUS	DUAL PRICES
finishinq)	0.000000	10.000000
carpentry)	0.000000	10.000000

As at least one of these constraint is binding, we are in CASE 2.

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLLOWABLE DECREASE
finishinq	20.000000	4.000000	4.000000
carpentry	8.000000	2.000000	1.333333

$$\Delta b_2 = 2, \; I_2 = 4, \; r_2 = \frac{2}{4} = 0.5; \quad \Delta b_3 = 1, \; I_3 = 2, r_3 = \frac{1}{2} = 0.5$$

$r_2 + r_3 = 1 \;\rightarrow\;$ <u>Current basis remains optimal</u>

As RHS changed, optimal values of decision variables and objective function may change.

Example

The chocolate requirement changes from 6 oz. to 8 oz. And the sugar requirement changes from 10 oz. to 7 oz. Does the current basis remain optimal?

What would be the new optimal solution?

Answer

ROW	SLACK OR SURPLUS	DUAL PRICES
CHOCO)	0.000000	-2.500000
SUGAR)	0.000000	-7.500000

As at least one of these constraint is binding, we are in CASE 2.

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
CHOCO	6.000000	4.000000	2.857143
SUGAR	10.000000	INFINITY	4.000000

$$\Delta b_2 = 2, \; I_2 = 4, \; r_2 = \frac{2}{4} = 0.5; \quad \Delta b_3 = -3, \; D_3 = 4, r_3 = \frac{3}{4} = 0.75$$

$r_2 + r_3 = 1.25 > 1 \;\rightarrow\;$

<u>100% Rule yields no information about whether the current basis is optimal</u>

4.   TRANSPORTATION PROBLEMS

4.1   FORMULATING TRANSPORTATION PROBLEMS

In general, a transportation problem is specified by the following information:

- A set of m **supply points** from which a good/service is shipped. Supply point i can supply at most s_i units.

- A set of n **demand points** to which the good/service is shipped. Demand point j must receive at least d_j units.

- Each unit produced at supply point i and shipped to demand point j incurs a variable cost of c_{ij}.

The relevant data can be formulated in a **transportation tableau**:

	Demand point 1	Demand point 2	.....	Demand point n	SUPPLY
Supply point 1	c_{11}	c_{12}		c_{1n}	s_1
Supply point 2	c_{21}	c_{22}		c_{2n}	s_2
.....					
Supply point m	c_{m1}	c_{m2}		c_{mn}	s_m
DEMAND	d_1	d_2		d_n	

If total supply equals total demand then the problem is said to be a **balanced transportation problem**.

Let x_{ij} = number of units shipped from supply point i to demand point j

- □ Decision variable x_{ij}: number of units shipped from supply point i to demand point j

then the general LP representation of a transportation problem is

$$\min \sum_i \sum_j c_{ij} x_{ij}$$

s.t. $\sum_j x_{ij} \leq s_i$ (i=1,2, ..., m)   Supply constraints

$\sum_i x_{ij} \geq d_j$ (j=1,2, ..., n)   Demand constraints

$x_{ij} \geq 0$

If a problem has the constraints given above and is a *maximization* problem, it is still a transportation problem.

4.1.1 Formulating Balanced Transportation Problem

Example 1. Powerco

Powerco has three electric power plants that supply the needs of four cities. Each power plant can supply the following numbers of kwh of electricity: plant 1, 35 million; plant 2, 50 million; and plant 3, 40 million. The peak power demands in these cities as follows (in kwh): city 1, 45 million; city 2, 20 million; city 3, 30 million; city 4, 30 million. The costs of sending 1 million kwh of electricity from plant to city is given in the table below. To minimize the cost of meeting each city's peak power demand, formulate a balanced transportation problem in a transportation tableau and represent the problem as a LP model.

From	To			
	City 1	City 2	City 3	City 4
Plant 1	$8	$6	$10	$9
Plant 2	$9	$12	$13	$7
Plant 3	$14	$9	$16	$5

Answer

<u>Representation of the problem as a LP model</u>

x_{ij}: number of (million) kwh produced at plant i and sent to city j.

min $z = 8 x_{11} + 6 x_{12} + 10 x_{13} + 9 x_{14} + 9 x_{21} + 12 x_{22} + 13 x_{23} + 7 x_{24} + 14 x_{31} + 9 x_{32} + 16 x_{33} + 5 x_{34}$

s.t. $x_{11} + x_{12} + x_{13} + x_{14} \leq 35$ (supply constraints)

$x_{21} + x_{22} + x_{23} + x_{24} \leq 50$

$x_{31} + x_{32} + x_{33} + x_{34} \leq 40$

$x_{11} + x_{21} + x_{31} \geq 45$ (demand constraints)

$x_{12} + x_{22} + x_{32} \geq 20$

$x_{13} + x_{23} + x_{33} \geq 30$

$x_{14} + x_{24} + x_{34} \geq 30$

$x_{ij} \geq 0$ (i = 1, 2, 3; j = 1, 2, 3, 4)

Formulation of the transportation problem

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8	6	10	9	35
Plant 2	9	12	13	7	50
Plant 3	14	9	16	5	40
DEMAND	45	20	30	30	125

Total supply & total demand both equal 125: "balanced transport'n problem".

4.1.2 Balancing an Unbalanced Transportation Problem

Excess Supply

If total supply exceeds total demand, we can balance a transportation problem by creating a **dummy demand point** that has a demand equal to the amount of excess supply. Since shipments to the dummy demand point are not real shipments, they are assigned a cost of zero. These shipments indicate unused supply capacity.

Unmet Demand

If total supply is less than total demand, actually the problem has no feasible solution. To solve the problem it is sometimes desirable to allow the possibility of leaving some demand unmet. In such a situation, *a penalty is often associated with unmet demand.* This means that a **dummy supply point** should be introduced.

Example 2. Modified Powerco for Excess Supply

Suppose that demand for city 1 is 40 million kwh. Formulate a balanced transportation problem.

Answer

Total demand is 120, total supply is 125.

To balance the problem, we would add a dummy demand point with a demand of 125 – 120 = 5 million kwh.

From each plant, the cost of shipping 1 million kwh to the dummy is 0.

For details see Table 4.

Table 4. Transportation Tableau for Excess Supply

	City 1	City 2	City 3	City 4	Dummy	SUPPLY
Plant 1	8	6	10	9	0	35
Plant 2	9	12	13	7	0	50
Plant 3	14	9	16	5	0	40
DEMAND	40	20	30	30	5	125

Example 3. Modified Powerco for Unmet Demand

Suppose that demand for city 1 is 50 million kwh. For each million kwh of unmet demand, there is a penalty of 80$. Formulate a balanced transportation problem.

Answer

We would add a dummy supply point having a supply of 5 million kwh representing shortage.

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8	6	10	9	35
Plant 2	9	12	13	7	50
Plant 3	14	9	16	5	40
Dummy (Shortage)	80	80	80	80	5
DEMAND	50	20	30	30	130

4.2 FINDING BFS FOR TRANSPORT'N PROBLEMS

For a balanced transportation problem, general LP representation may be written as:

$$\min \sum_i \sum_j c_{ij} x_{ij}$$

s.t. $\sum_j x_{ij} = s_i$ ($i=1,2, ..., m$) Supply constraints

$\sum_i x_{ij} = d_j$ ($j=1,2, ..., n$) Demand constraints

$x_{ij} \geq 0$

To find a bfs to a balanced transportation problem, we need to make the following important observation:

If a set of values for the xij's satisfies all but one of the constraints of a balanced transportation problem, the values for the xij's will automatically satisfy the other constraint.

This observation shows that when we solve a balanced transportation, we may omit from consideration any one of the problem's constraints and solve an LP having $m+n$-1 constraints. We arbitrarily assume that the first supply constraint is omitted from consideration. In trying to find a bfs to the remaining $m+n$-1 constraints, you might think that any collection of $m+n$-1 variables would yield a basic solution. But this is not the case: If the $m+n$-1 variables yield a basic solution, the cells corresponding to this set contain **no loop**.

An ordered sequence of at least four different cells is called a loop if

- Any two consecutives cells lie in either the same row or same column
- No three consecutive cells lie in the same row or column
- The last cell in the sequence has a row or column in common with the first cell in the sequence

There are three methods that can be used to find a bfs for a balanced transportation problem:

1. Northwest Corner method
2. Minimum cost method
3. Vogel's method

4.2.1 Northwest Corner Method

We begin in the upper left corner of the transportation tableau and set x_{11} as large as possible (clearly, x_{11} can be no larger than the smaller of s_1 and d_1).

- If $x_{11}=s_1$, cross out the first row of the tableau. Also change d_1 to d_1-s_1.
- If $x_{11}=d_1$, cross out the first column of the tableau. Change s_1 to s_1-d_1.
- If $x_{11}=s_1=d_1$, cross out either row 1 or column 1 (but not both!).
  - If you cross out row, change d_1 to 0.
  - If you cross out column, change s_1 to 0.

Continue applying this procedure to the most northwest cell in the tableau that does not lie in a crossed out row or column.

Eventually, you will come to a point where there is only one cell that can be assigned a value. Assign this cell a value equal to its row or column demand, and cross out both the cell's row or column.

A bfs has now been obtained.

Example 1.

For example consider a balanced transportation problem given below (We omit the costs because they are not needed to find a bfs!).

				5
				1
				3
2	4	2	1	

Total demand equals total supply (9): this is a balanced transport'n problem.

2				3
▓				1
▓				3
X	4	2	1	

2	3	▓	▓	X
▓				1
▓				3
X	1	2	1	

2	3	▓	▓	X
▓	1	▓	▓	X
▓				3
X	0	2	1	

2	3			X
	1			X
	0	2	1	3
X	0	2	1	

NWC method assigned values to $m+n$-1 (3+4-1 = 6) variables. The variables chosen by NWC method can not form a loop, so a bfs is obtained.

4.2.2 Minimum Cost Method

Northwest Corner method does not utilize shipping costs, so it can yield an initial bfs that has a very high shipping cost. Then determining an optimal solution may require several pivots.

To begin the minimum cost method, find the variable with the smallest shipping cost (call it x_{ij}). Then assign x_{ij} its largest possible value, min $\{s_i, d_j\}$.

As in the NWC method, cross out row i or column j and reduce the supply or demand of the noncrossed-out of row or column by the value of x_{ij}.

Continue like NWC method (instead of assigning upper left corner, the cell with the minimum cost is assigned). See Northwest Corner Method for the details!

Example 2.

2	3	5	6	5
2	1	3	5	10
3	8	4	6	15
12	8	4	6	

2	3	5	6	5
2	**1** 8	3	5	2
3	8	4	6	15
12	X	4	6	

Tableau 1

	D1 (10)	D2 (X)	D3 (4)	D4 (6)	Supply
	2	3	5	6	5
	2 [2]	1 [8]	3	5	X
	3	8	4	6	15

Tableau 2

	D1 (5)	D2 (X)	D3 (4)	D4 (6)	Supply
	2 [5]	3	5	6	X
	2 [2]	1 [8]	3	5	X
	3	8	4	6	15

Tableau 3

	D1 (5)	D2 (X)	D3 (4)	D4 (6)	Supply
	2 [5]	3	5	6	X
	2 [2]	1 [8]	3	5	X
	3 [5]	8	4 [4]	6 [6]	10

4.2.3 Vogel's Method

Begin by computing for each row and column a penalty equal to the difference between the two smallest costs in the row and column. Next find the row or column with the largest penalty. Choose as the first basic variable the variable in this row or column that has the smallest cost. As described in the NWC method, make this variable as large as possible, cross out row or column, and change the supply or demand associated with the basic variable (See Northwest Corner Method for the details!). Now recomputed new penalties (using only cells that do not lie in a crossed out row or column), and repeat the procedure until only one uncrossed cell remains. Set this variable equal to the supply or demand associated with the variable, and cross out the variable's row and column.

Example 3.

			Supply	Row penalty
6	7	8	10	7-6=1
15	80	78	15	78-15=63
Demand	15	5	5	
Column penalty	15-6=9	80-7=**73**	78-8=70	

			Supply	Row penalty
6	**7** 5	8	5	8-6=2
15	80	78	15	78-15=63
Demand	15	X	5	
Column penalty	15-6=9	-	78-8=**70**	

			Supply	Row penalty
6	7 5	**8** 5	X	-
15	80	78	15	-
Demand	15	X	0	
Column penalty	15-6=9	-	-	

			Supply
6	7 5	8 5	X
15 15	80	78 0	15
Demand	15	X	0

4.3   THE TRANSPORTATION SIMPLEX METHOD

Steps of the Method

1. If the problem is unbalanced, balance it

2. Use one of the methods to find a bfs for the problem

3. Use the fact that $u_1 = 0$ and $u_i + v_j = c_{ij}$ for all basic variables to find the u's and v's for the current bfs.

4. If $u_i + v_j - c_{ij} \leq 0$ for all nonbasic variables, then the current bfs is optimal. If this is not the case, we enter the variable with the most positive $u_i + v_j - c_{ij}$ into the basis using the *pivoting procedure*. This yields a new bfs. Return to Step 3.

For a maximization problem, proceed as stated, but replace Step 4 by the following step:

If $u_i + v_j - c_{ij} \geq 0$ for all nonbasic variables, then the current bfs is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis using the *pivoting procedure*. This yields a new bfs. Return to Step 3.

Pivoting procedure

1. Find the loop (there is only one possible loop!) involving the entering variable (determined at step 4 of the transport'n simplex method) and some or all of the basic variables.

2. Counting *only cells in the loop*, label those that are an even number (0, 2, 4, and so on) of cells away from the entering variable as *even cells*. Also label those that are an odd number of cells away from the entering variable as *odd cells*.

3. Find the odd cell whose variable assumes the smallest value. Call this value Φ. The variable corresponding to this odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by Φ and increase the value of each even cell by Φ. The values of variables not in the loop remain unchanged. The pivot is now complete. If $\Phi = 0$, the entering variable will equal 0, and odd variable that has a current value of 0 will leave the basis.

Example 1. Powerco

The problem is balanced (total supply equals total demand).

When the NWC method is applied to the Powerco example, the bfs in the following table is obtained (check: there exist $m+n-1=6$ basic variables).

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 **35**	6	10	9	35
Plant 2	9 **10**	12 **20**	13 **20**	7	50
Plant 3	14	9	16 **10**	5 **30**	40
DEMAND	45	20	30	30	125

$u1 = 0$

$u1 + v1 = 8$ yields $v1 = 8$

$u2 + v1 = 9$ yields $u2 = 1$

$u2 + v2 = 12$ yields $v2 = 11$

$u2 + v3 = 13$ yields $v3 = 12$

$u3 + v3 = 16$ yields $u3 = 4$

$u3 + v4 = 5$ yields $v4 = 1$

For each nonbasic variable, we now compute $\hat{c}_{ij} = u_i + v_j - c_{ij}$

$\hat{c}_{12} = 0 + 11 - 6 = 5$

$\hat{c}_{13} = 0 + 12 - 10 = 2$

$\hat{c}_{14} = 0 + 1 - 9 = -8$

$\hat{c}_{24} = 1 + 1 - 7 = -5$

$\hat{c}_{31} = 4 + 8 - 14 = -2$

$\hat{c}_{32} = 4 + 11 - 9 = 6$

Since $\hat{c}_{32}$ is the most positive one, we would next enter x_{32} into the basis: Each unit of x_{32} that is entered into the basis will decrease Powerco's cost by \$6.

The loop involving x_{32} is (3,2)-(3,3)-(2,3)-(2,2). $\Phi = 10$ (see table)

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 **35**	6	10	9	35
Plant 2	9 **10**	12 **20–Φ**	13 **20+Φ**	7	50
Plant 3	14	9 **Φ**	16 **10–Φ**	5 **30**	40
DEMAND	45	20	30	30	125

x_{33} would leave the basis. New bfs is shown at the following table:

u_i/v_j	8	11	12	7	SUPPLY
0	8 35	6	10	9	35
1	9 10	12 10	13 30	7	50
-2	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

$$\hat{c}_{12} = 5,\ \hat{c}_{13} = 2,\ \hat{c}_{14} = -2,\ \hat{c}_{24} = 1,\ \hat{c}_{31} = -8,\ \hat{c}_{33} = -6$$

Since $\hat{c}_{12}$ is the most positive one, we would next enter x_{12} into the basis.

The loop involving x_{12} is (1,2)-(2,2)-(2,1)-(1,1). $\Phi = 10$ (see table)

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 **35–Φ**	6 **Φ**	10	9	35
Plant 2	9 **10+Φ**	12 **10–Φ**	13 30	7	50
Plant 3	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

x_{22} would leave the basis. New bfs is shown at the following table:

u_i/v_j	8	6	12	2	SUPPLY
0	8 25	6 10	10	9	35
1	9 20	12	13 30	7	50
3	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

$$\hat{c}_{13} = 2,\ \hat{c}_{14} = -7,\ \hat{c}_{22} = -5,\ \hat{c}_{24} = -4,\ \hat{c}_{31} = -3,\ \hat{c}_{33} = -1$$

Since $\hat{c}_{13}$ is the most positive one, we would next enter x_{13} into the basis.

The loop involving x_{13} is (1,3)-(2,3)-(2,1)-(1,1). $\Phi = 25$ (see table)

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 **25–Φ**	6 10	10 **Φ**	9	35
Plant 2	9 **20+Φ**	12	13 **30–Φ**	7	50
Plant 3	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

x_{11} would leave the basis. New bfs is shown at the following table:

u_i/v_j	6	6	10	2	SUPPLY
0	8	6 10	10 25	9	35
3	9 45	12	13 5	7	50
3	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

$\hat{c}_{11} = -2$, $\hat{c}_{14} = -7$, $\hat{c}_{22} = -3$, $\hat{c}_{24} = -2$, $\hat{c}_{31} = -5$, $\hat{c}_{33} = -3$

Since all $\hat{c}_{ij}$'s are negative, an optimal solution has been obtained.

Report

45 million kwh of electricity would be sent from plant 2 to city 1.

10 million kwh of electricity would be sent from plant 1 to city 2. Similarly, 10 million kwh of electricity would be sent from plant 3 to city 2.

25 million kwh of electricity would be sent from plant 1 to city 3. 5 million kwh of electricity would be sent from plant 2 to city 3.

30 million kwh of electricity would be sent from plant 3 to city 4 and

Total shipping cost is:

$$z = .9\,(45) + 6\,(10) + 9\,(10) + 10\,(25) + 13\,(5) + 5\,(30) = \$\,1020$$

4.4   TRANSSHIPMENT PROBLEMS

Sometimes a point in the shipment process can both receive goods from other points and send goods to other points. This point is called as **transshipment point** through which goods can be transshipped on their journey from a supply point to demand point. Shipping problem with this characteristic is a transshipment problem.

The optimal solution to a transshipment problem can be found by converting this transshipment problem to a transportation problem and then solving this transportation problem.

Remark

As stated in "Formulating Transportation Problems", we define a **supply point** to be a point that can send goods to another point but cannot receive goods from any other point.

Similarly, a **demand point** is a point that can receive goods from other points but cannot send goods to any other point.

Steps

1. If the problem is unbalanced, balance it

    Let s = total available supply (or demand) for balanced problem

2. Construct a transportation tableau as follows

    A row in the tableau will be needed for each supply point and transshipment point

    A column will be needed for each demand point and transshipment point

    Each supply point will have a supply equal to its original supply

    Each demand point will have a demand equal to its original demand

    Each transshipment point will have a supply equal to "that point's original supply + s"

    Each transshipment point will have a demand equal to "that point's original demand + s"

3. Solve the transportation problem

Example 1. Bosphorus

(Based on Winston 7.6.)

Bosphorus manufactures LCD TVs at two factories, one in Istanbul and one in Bruges. The Istanbul factory can produce up to 150 TVs per day, and the Bruges factory can produce up to 200 TVs per day. TVs are shipped by air to customers in London and Paris. The customers in each city require 130 TVs per day. Because of the deregulation of air fares, Bosphorus believes that it may be cheaper to first fly some TVs to Amsterdam or Munchen and then fly them to their final destinations. The costs of flying a TV are shown at the table below. Bosphorus wants to minimize the total cost of shipping the required TVs to its customers.

€			To			
From	Istanbul	Bruges	Amsterdam	Munchen	London	Paris
Istanbul	0	-	8	13	25	28
Bruges	-	0	15	12	26	25
Amsterdam	-	-	0	6	16	17
Munchen	-	-	6	0	14	16
London	-	-	-	-	0	-
Paris	-	-	-	-	-	0

Answer:

In this problem Amsterdam and Munchen are *transshipment points*.

Step 1. Balancing the problem

Total supply = 150 + 200 = 350

Total demand = 130 + 130 = 260

Dummy's demand = 350 – 260 = 90

s = 350 (total available supply or demand for balanced problem)

Step 2. Constructing a transportation tableau

Transshipment point's demand = Its original demand + s = 0 + 350 = 350

Transshipment point's supply = Its original supply + s = 0 + 350 = 350

	Amsterdam	Munchen	London	Paris	Dummy	Supply
Istanbul	8	13	25	28	0	150
Bruges	15	12	26	25	0	200
Amsterdam	0	6	16	17	0	350
Munchen	6	0	14	16	0	350
Demand	350	350	130	130	90	

Step 3. Solving the transportation problem

	Amsterdam	Munchen	London	Paris	Dummy	Supply
Istanbul	8 / 130	13	25	28	0 / 20	150
Bruges	15	12	26	25 / 130	0 / 70	200
Amsterdam	0 / 220	6	16 / 130	17	0	350
Munchen	6	0 / 350	14	16	0	350
Demand	350	350	130	130	90	1050

Report:

Bosphorus should produce 130 TVs at Istanbul, ship them to Amsterdam, and transship them from Amsterdam to London.

The 130 TVs produced at Bruges should be shipped directly to Paris.

The total shipment is 6370 Euros.

4.5 ASSIGNMENT PROBLEMS

There is a special case of transportation problems where each supply point should be assigned to a demand point and each demand should be met. This certain class of problems is called as "assignment problems". For example determining which employee or machine should be assigned to which job is an assignment problem.

4.5.1 LP Representation

An assignment problem is characterized by knowledge of the cost of assigning each supply point to each demand point: c_{ij}

On the other hand, a 0-1 integer variable x_{ij} is defined as follows

x_{ij} = 1 if supply point i is assigned to meet the demands of demand point j

x_{ij} = 0 if supply point i is not assigned to meet the demands of point j

In this case, the general LP representation of an assignment problem is

$$\min \sum_i \sum_j c_{ij} x_{ij}$$

s.t. $\sum_j x_{ij} = 1$ (i=1,2, ..., m)   Supply constraints

$\sum_i x_{ij} = 1$ (j=1,2, ..., n)   Demand constraints

x_{ij} = 0 or x_{ij} = 1

4.5.2 Hungarian Method

Since all the supplies and demands for any assignment problem are integers, all variables in optimal solution of the problem must be integers. Since the RHS of each constraint is equal to 1, each x_{ij} must be a nonnegative integer that is no larger than 1, so each x_{ij} must equal 0 or 1.

Ignoring the x_{ij} = 0 or x_{ij} = 1 restrictions at the LP representation of the assignment problem, we see that we confront with a balanced transportation problem in which each supply point has a supply of 1 and each demand point has a demand of 1.

However, the high degree of degeneracy in an assignment problem may cause the Transportation Simplex to be an inefficient way of solving assignment problems.

For this reason and the fact that the algorithm is even simpler than the Transportation Simplex, the Hungarian method is usually used to solve assignment problems.

Remarks

1. To solve an assignment problem in which the goal is to maximize the objective function, multiply the profits matrix through by –1 and solve the problem as a **minimization** problem.

2. If the number of rows and columns in the cost matrix are unequal, the assignment problem is **unbalanced**. Any assignment problem should be balanced by the addition of one or more dummy points before it is solved by the Hungarian method.

Steps

1. Find the minimum cost each row of the $m*m$ cost matrix.

2. Construct a new matrix by subtracting from each cost the minimum cost in its row

3. For this new matrix, find the minimum cost in each column

4. Construct a new matrix (reduced cost matrix) by subtracting from each cost the minimum cost in its column

5. Draw the minimum number of lines (horizontal and/or vertical) that are needed to cover all the zeros in the reduced cost matrix. If m lines are required, an optimal solution is available among the covered zeros in the matrix. If fewer than m lines are needed, proceed to next step

6. Find the smallest cost (k) in the reduced cost matrix that is uncovered by the lines drawn in Step 5

7. Subtract k from each uncovered element of the reduced cost matrix and add k to each element that is covered by two lines. Return to Step 5

Example 1. Flight Crew

(Based on Winston 7.5.)

Four captain pilots (CP1, CP2, CP3, CP4) has evaluated four flight officers (FO1, FO2, FO3, FO4) according to perfection, adaptation, morale motivation in a 1-20 scale (1: very good, 20: very bad). Evaluation grades are given in the table. Flight Company wants to assign each flight officer to a captain pilot according to these evaluations. Determine possible flight crews.

	FO1	FO2	FO3	FO4
CP1	2	4	6	10
CP2	2	12	6	5
CP3	7	8	3	9
CP4	14	5	8	7

Answer:

Step 1. For each row in the table we find the minimum cost: 2, 2, 3, and 5 respectively

Step 2 & 3. We subtract the row minimum from each cost in the row. For this new matrix, we find the minimum cost in each column

0	2	4	8
0	10	4	3
4	5	0	6
9	0	3	2

| Column minimum | 0 | 0 | 0 | 2 |

Step 4. We now subtract the column minimum from each cost in the column obtaining reduced cost matrix.

0	2	4	6
0	10	4	1
4	5	0	4
9	0	3	0

Step 5. As shown, lines through row 3, row 4, and column 1 cover all the zeros in the reduced cost matrix. The minimum number of lines for this operation is 3. Since fewer than four lines are required to cover all the zeros, solution is not optimal: we proceed to next step.

0	2	4	6
0	10	4	1
4	5	0	4
9	0	3	0

Step 6 & 7. The smallest uncovered cost equals 1. We now subtract 1 from each uncovered cost, add 1 to each twice-covered cost, and obtain

0	1	3	5
0	9	3	0
5	5	0	4
10	0	3	0

Four lines are now required to cover all the zeros: An optimal s9olution is available.

Observe that the only covered 0 in column 3 is x_{33}, and in column 2 is x_{42}. As row 5 can not be used again, for column 4 the remaining zero is x_{24}. Finally we choose x_{11}.

Report:

CP1 should fly with FO1; CP2 should fly with FO4; CP3 should fly with FO3; and CP4 should fly with FO2.

Example 2. Maximization problem

	F	G	H	I	J
A	6	3	5	8	10
B	2	7	6	3	2
C	5	8	3	4	6
D	6	9	3	1	7
E	2	2	2	2	8

Report:

Optimal profit = 36

Assigments: A-I, B-H, C-G, D-F, E-J

Alternative optimal sol'n: A-I, B-H, C-F, D-G, E-J

5. INTEGER PROGRAMMING

When formulating Linear Programs (LPs) we often found that, strictly, certain variables should have been regarded as taking integer values but, for the sake of convenience, we let them take fractional values reasoning that the variables were likely to be so large that any fractional part could be neglected.

While this is acceptable in some situations, in many cases it is not, and in such cases, we must find a numeric solution in which the variables take integer values.

Problems in which this is the case are called *integer programs* (*IP*'s) and the subject of solving such programs is called *integer programming* (also referred to by the initials *IP*).

IP's occur frequently because many decisions are essentially discrete (such as yes/no, do/do not) in that one or more options must be chosen from a finite set of alternatives.

An IP in which all variables are required to be integers is called a *pure IP* problem.

If some variables are restricted to be integer and some are not then the problem is a *mixed IP* problem.

The case where the integer variables are restricted to be 0 or 1 comes up surprising often. Such problems are called *pure (mixed) 0-1 programming* problems or *pure (mixed) binary IP* problems.

For any IP we can generate an LP by taking the same objective function and same constraints but with the requirement that variables are integer replaced by appropriate continuous constraints:

"$x_i \geq 0$ and integer" can be replaced by $x_i \geq 0$

"$x_i = 0$ or 1" can be replaced by $x_i \geq 0$ and $x_i \leq 1$

The LP obtained by omitting all integer or 0-1 constraints on variables is called *LP Relaxation of the IP* (*LR*).

5.1 FORMULATING IP

Practical problems can be formulated as IPs. For instance budgeting problems, knapsack problems, fixed charge production and location problems, set covering problems, etc.

5.1.1 Budgeting Problems

Example 1. Capital Budgeting *(Winston 9.2, p. 478 – modified)*

Stock is considering four investments. Each investment yields a determined NPV ($8,000, $11,000, $6,000, $4,000). Each investment requires at certain cash flow at the present time ($5,000, $7,000, $4,000, $3,000). Currently Stock has $14,000 available for investment.

Formulate an IP whose solution will tell Stock how to maximize the NPV obtained from the four investments.

Answer

Begin by defining a variable for each decision that Stockco must make.

In this case, we will use a 0-1 variable x_j for each investment:

If x_j is 1 then Stock will make investment j.

If it is 0, Stock will not make the investment.

This leads to the 0-1 programming problem:

$$\max z = 8x_1 + 11x_2 + 6x_3 + 4x_4$$
$$\text{s.t.} \quad 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14$$
$$x_j = 0 \text{ or } 1 \quad (j = 1,2,3,4)$$

Comment

Now, a straightforward "bang for buck" (taking ratios of objective coefficient over constraint coefficient) suggests that investment 1 is the best choice.

Ignoring integrality constraints, the optimal linear programming solution is:

$x_1 = x_2 = 1$, $x_3 = 0.5$, and $x_4 = 0$ for a value of $22K

Unfortunately, this solution is not integer. Rounding x_3 down to 0:

$x_1 = x_2 = 1$, $x_3 = x_4 = 0$ for a value of $19K

There is a better integer solution (optimal solution):

$x_1 = 0$, $x_2 = x_3 = x_4 = 1$ for a value of $21K

This example shows that rounding does not necessarily give an optimal value.

Example 2. Multiperiod Capital Budgeting

There are four possible projects, which each run for three years and have the following characteristics:

Which projects would you choose in order to maximize the total return?

| | | Capital requirements | | |
Project	Return	Year1	Year2	Year3
1	0.2	0.5	0.3	0.2
2	0.3	1	0.5	0.2
3	0.5	1.5	1.5	0.3
4	0.1	0.1	0.4	0.1
Available capital		3.1	2.5	0.4

Answer

We will use a 0-1 variable x_j for each project:

x_j is 1 if we decide to do project j;

x_j is 0 otherwise (i.e. not do project j).

This leads to the 0-1 programming problem:

$$\max \quad 0.2\, x_1 + 0.3\, x_2 + 0.5\, x_3 + 0.1\, x_4$$

$$\text{s.t.} \quad 0.5\, x_1 + 1\ \ x_2 + 1.5\, x_3 + 0.1\, x_4 \leq 3.1$$

$$0.3\, x_1 + 0.8\, x_2 + 1.5\, x_3 + 0.4\, x_4 \leq 2.5$$

$$0.2\, x_1 + 0.2\, x_2 + 0.3\, x_3 + 0.1\, x_4 \leq 0.4$$

$$x_j = 0 \text{ or } 1 \quad j = 1, \ldots, 4$$

Example 3. Capital Budgeting Extension

There are a number of additional constraints Stock might want to add.

Logical restrictions can be enforced using 0-1 variables:

Stock can only make two investments

$x_1 + x_2 + x_3 + x_4 \leq 2$

Any choice of three or four investments will have $x_1 + x_2 + x_3 + x_4 \geq 3$

If investment 2 is made, investment 4 must also be made

$x_2 \leq x_4$ or $x_2 - x_4 \leq 0$

If x_2 is 1, then x_4 is also 1; if x_2 is 0, then there is no restriction for x_4 (x_4 is 0 or 1)

If investment 1 is made, investment 3 cannot be made

$x_1 + x_3 \leq 1$

If x_1 is 1, then x_3 is 0; if x_1 is 0, then there is no restriction for x_3 (x_3 is 0 or 1)

Either investment 1 or investment 2 must be made (only one of them, not both)

$x_1 + x_2 = 1$

If x_1 is 1, then x_2 is 0 (only investment 1 is made); if x_1 is 0, then x_2 is 1 (only investment 2 is made)

5.1.2 Knapsack Problems

Any IP that has only one constraint is referred to as a knapsack problem.

Furthermore, the coefficients of this constraint and the objective are all non-negative.

The traditional story is that: There is a knapsack. There are a number of items, each with a size and a value. The objective is to maximize the total value of the items in the knapsack.

Knapsack problems are nice because they are (usually) easy to solve.

General representation of the knapsack problem:

$$\max z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
$$\text{s.t.} \quad a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$$
$$x_i = 0 \text{ or } 1 \ (i = 1, 2, \ldots, n)$$

where c_i is the value of item i, a_i is the weight of item i, and b is the total weight capacity. For instance, the following is a knapsack problem:

$$\text{Maximize} \quad 8\ x_1 + 11\ x_2 + 6\ x_3 + 4\ x_4$$
$$\text{Subject to} \quad 5\ x_1 + 7\ x_2 + 4\ x_3 + 3\ x_4 \leq 14$$
$$x_j = 0 \text{ or } 1 \quad j = 1, \ldots 4$$

5.1.3 Fixed Charge Problems

There is a cost associated with performing an activity at a nonzero level that does not depend on the level of the activity.

An important trick can be used to formulate many production and location problems involving the idea of a fixed charge as IP.

Example 4. Gandhi *(Winston 9.2, p. 480)*

Gandhi Co makes shirts, shorts, and pants using the limited labor and cloth described below.

In addition, the machinery to make each product must be rented.

	Shirts	Shorts	Pants	Total Avail.
Labor (hrs/wk)	3	2	6	150
Cloth (m^2/wk)	4	3	4	160
Rent for machine ($/wk)	200	150	100	
Variable unit cost	6	4	8	
Sale Price	12	8	15	

Answer

Let x_j be number of clothing produced.

Let y_j be 1 if any clothing j is manufactured and 0 otherwise.

Profit = Sales revenue – Variable Cost – Costs of renting machinery

For example the profit from shirts is

$$z_1 = (12 - 6) \, x_1 - 200 \, y_1$$

Since supply of labor and cloth is limited, Gandhi faces two constraints.

To ensure $x_j > 0$ forces $y_j = 1$, we include the additional constraints

$$x_j \le M_j \, y_j$$

From the cloth constraint at most 40 shirts can be produced ($M_1 = 40$), so the additional constraint for shirts is not an additional limit on x_1 (If M_1 were not chosen large (say $M_1 = 10$), then the additional constraint for shirts would unnecessarily restrict the value of x_1).

From the cloth constraint at most 53 shorts can be produced ($M_2 = 53$)

From the labor constraint at most 25 pants can be produced ($M_3 = 25$)

We thus get the mixed (binary) integer problem:

$$\max \quad 6\,x_1 + 4\,x_2 + 7\,x_3 - 200\,y_1 - 150\,y_2 - 100\,y_3$$

$$\text{s.t.} \quad 3\,x_1 + 2\,x_2 + 6\,x_3 \le 150 \qquad \text{(Labor constraint)}$$

$$4\,x_1 + 3\,x_2 + 4\,x_3 \le 160 \qquad \text{(Cloth constraint)}$$

$$x_1 \le 40\,y_1 \qquad \text{(Shirt production constraint)}$$

$$x_2 \le 53\,y_2 \qquad \text{(Short production constraint)}$$

$$x_3 \le 25\,y_3 \qquad \text{(Pant production constraint)}$$

$$x_1, x_2, x_3 \ge 0 \text{ and integer}$$

$$y_1, y_2, y_3 = 0 \text{ or } 1$$

Example 5. Warehouse problem

ATK-White are planning where to open warehouse (WH) and how to transport products from warehouses to costumers. 5 possible WH locations are considered to serve 4 customers. There is a fixed cost of opening and operating WHs (f_i). There is also transportation cost of transporting products from WHs to customers (c_{ij}). If a WH is opened, its capacity (K_i) cannot be exceeded. All demands of the customers (D_j) should be met.

Formulate an IP to minimize the total cost of ATK-White (WH opening costs and transportation costs) to meet the demands of the customers.

	Cust. 1	Cust. 2	Cust. 3	Cust. 4	Fixed cost (TL)	Capacity
WH -1	8TL	6TL	10TL	9TL	220,000	50,000
WH -2	9TL	12TL	13TL	7TL	280,000	60,000
WH -3	14TL	9TL	16TL	5TL	150,000	45,000
WH -4	18TL	16TL	10TL	9TL	290,000	80,000
WH -5	10TL	6TL	13TL	8TL	490,000	120,000
Demand	40,000	30,000	25,000	75,000		

Answer

<u>Define decision variables:</u>

x_{ij}: Amount of product transported from WH-i to Customer j. (i=1,2,3,4,5; j=1,2,3,4)

y_i: Binary variable for opening WH-i. (i=1,2,3,4,5)

y_i = 1 if WH-i is opened, y_i = 0 if WH-i is not opened.

Objective is to minimize the sum of WH opening cost and total transportation cost.

$$Min\ Z = \sum_{i=1}^{5} f_i y_i + \sum_{i=1}^{5}\sum_{j=}^{4} c_{ij}x_{ij}$$

Subject to the following constraints:

$$\sum_{j=1}^{4} x_{ij} \le K_i y_i \qquad \forall i$$

$$\sum_{i=1}^{5} x_{ij} \ge D_j \qquad \forall j$$

$$x_{ij} \ge 0 \quad \forall i,j, \qquad y_i \in \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad \forall i$$

5.1.4 Either-Or Constraints

Given two constraints

$f(x_1, x_2,…, x_n) \le 0$ (1)

$g(x_1, x_2,…, x_n) \le 0$ (2)

ensure that at least one is satisfied (1 or 2) by adding either-or-constraints:

$f(x_1, x_2,…, x_n) \le M\ y$

$g(x_1, x_2,…, x_n) \le M\ (1 - y)$

Here *y* is a 0-1 variable, and M is a number chosen large enough to ensure that both constraints are satisfied for all values of decision variables that satisfy the other constraints in the problem:

- o If $y = 0$, then (1) and possibly (2) must be satisfied.
- o If $y = 1$, then (2) and possibly (1) must be satisfied.

Example 1. Warehouse problem – Either-Or extension

Consider the Warehouse problem. Suppose that ATK-White has a chance of ignoring the demand of either customer 1 or customer 2. (i.e. they can choose to meet the demand of one of the two customers). How would you add this condition to the IP?

Answer

One of the following constraints should be satisfied:

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} \geq 40{,}000$$
$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} \geq 30{,}000$$

Arrange the constraints to be in the format of Either-or constraint formulation:

$$-(x_{11} + x_{21} + x_{31} + x_{41} + x_{51}) + 40{,}000 \leq 0$$
$$-(x_{12} + x_{22} + x_{32} + x_{42} + x_{52}) + 30{,}000 \leq 0$$

In this way;

$$f(x) = -(x_{11} + x_{21} + x_{31} + x_{41} + x_{51}) + 40{,}000 \text{ and}$$
$$g(x) = -(x_{12} + x_{22} + x_{32} + x_{42} + x_{52}) + 30{,}000.$$

Add the following constraints to the model:

$$-(x_{11} + x_{21} + x_{31} + x_{41} + x_{51}) + 40000 \leq M_1 y$$
$$-(x_{12} + x_{22} + x_{32} + x_{42} + x_{52}) + 30000 \leq M_2(1 - y).$$

where $y \in \left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}$. M values can be considered as 40,000 and 30,000, respectively.

The final model would be as follows:

$$Min\ Z = \sum_{i=1}^{5} f_i y_i + \sum_{i=1}^{5} \sum_{j=}^{4} c_{ij} x_{ij}$$

Subject to

$$\sum_{j=1}^{4} x_{ij} \leq K_i y_i \qquad \forall i$$

$$\sum_{i=1}^{5} x_{ij} \geq D_j \qquad j = 3,4,5.$$

$$-(x_{11} + x_{21} + x_{31} + x_{41} + x_{51}) + 40000 \leq 40000 y$$
$$-(x_{12} + x_{22} + x_{32} + x_{42} + x_{52}) + 30000 \leq 30000(1 - y).$$

$$x_{ij} \geq 0 \quad \forall i,j, \qquad y_i \in \left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\} \ \forall i, \qquad y \in \left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}.$$

Example 7. Compact Car

Suppose 1.5 tons of steel and 30 hours of labor are required for production of one compact car.

At present, 6,000 tons of steel and 60,000 hours of labor are available.

For an economically feasible production, at least 1,000 cars of compact car must be produced.

- Constraint: $x \leq 0$ or $x \geq 1000$
- Sign restriction: $x \geq 0$ and Integer

Answer

For $f(x) = x$; $g(x) = 1000 - x$

We can replace the constraint by the following pair of linear constraints:

$$x \leq M \, y$$
$$1000 - x \leq M \, (1 - y)$$
$$y = 0 \text{ or } 1$$
$$M = \min (6{,}000/1.5, \ 60{,}000/30) = 2000$$

5.1.5 If-Then Constraints

Suppose we want to ensure that

a constraint $f(x_1, x_2, \ldots, x_n) > 0$ implies

the constraint $g(x_1, x_2, \ldots, x_n) \geq 0$

Then we include the following constraints in the formulation:

$$-g(x_1, x_2, \ldots, x_n) \leq M \, y \qquad (1)$$
$$f(x_1, x_2, \ldots, x_n) \leq M \, (1 - y) \qquad (2)$$

Here y is a 0-1 variable, and M is a large positive number, chosen large enough so that $f < M$ and $-g < M$ hold for all values of decision variables that satisfy the other constraints in the problem.

If $f > 0$, then (2) can be satisfied only if $y = 0$. (1) implies $-g \leq 0$ or $g \geq 0$, which is the desired result

Example 2. Warehouse problem – If-then extension

Consider the Warehouse problem. How would you add the following condition to the model: If more than 40% of the capacity of WH-2 is used, WH-3 cannot be opened?

Answer

Initially, write a mathematical formulation for the condition:

If $x_{21} + x_{22} + x_{23} + x_{24} + x_{25} > 24{,}000$ then $y_3 = 0$.

$y_3 = 0 \rightarrow y_3 \leq 0 \rightarrow -y_3 \geq 0$

Arrange the constraints to be in the format of If-then constraint formulation

a constraint $f(x_1, x_2, \ldots, x_n) > 0$ implies
the constraint $g(x_1, x_2, \ldots, x_n) \geq 0$

$$f(x) = x_{21} + x_{22} + x_{23} + x_{24} + x_{25} - 24{,}000 > 0$$

$$g(x) = -y_3$$

Add the following constraints to the model:

$$y_3 \leq M_1 y$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} - 24000 \leq M_2(1 - y)$$

Where $y \in \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$. M values can be considered as 1 and 36,000, respectively.

The final model would be as follows:

$$Min\ Z = \sum_{i=1}^{5} f_i y_i + \sum_{i=1}^{5} \sum_{j=}^{4} c_{ij} x_{ij}$$

Subject to

$$\sum_{j=1}^{4} x_{ij} \leq K_i y_i \qquad \forall i$$

$$\sum_{i=1}^{5} x_{ij} \geq D_j \qquad \forall j$$

$$y_3 \leq y$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} - 24000 \leq 36000(1 - y)$$

$$x_{ij} \geq 0 \quad \forall i,j, \qquad y_i \in \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \ \forall i, \qquad y \in \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$

5.1.6 Membership in Specified Subsets

Set covering, set packing, and set partitioning models are a special class of IP.

Using decision variables that equal 1 if an object is part of a solution and 0 otherwise, set covering, set packing, and set partitioning models formulate problems where the core issue is membership in specified subsets.

There are many applications in areas such as location (facility, fire/police station, warehouse), scheduling (crew, airline, truck, bus), delivery, vehicle routing, political districting, capital budgeting.

- **Set covering** problems arises when each set element must appear in **at least** one subset:

$$\sum_{j \in J} x_j \geq 1$$

- **Set packing** problems arises when each set element must appear in **at most** one subset:
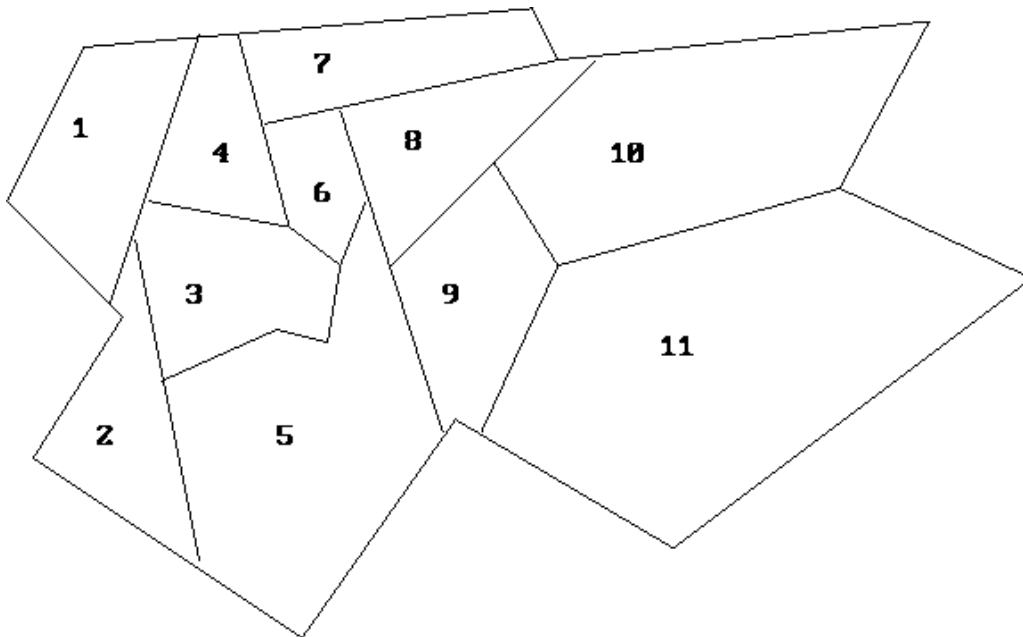
$$\sum_{j \in J} x_j \leq 1$$

- **Set partitioning** problems arises when each set element must appear in **exactly** one subset:

$$\sum_{j \in J} x_j = 1$$

Example 9. Fire Station

A county made up of 11 cities is reviewing the location of its fire stations.

A fire station can be placed in any city.

It is able to handle the fires for both its city and any adjacent city (any city with a non-zero border with its home city).

How many fire stations should be built and where?

Answer

We can create one variable x_j for each city j (1 if we place a station in the city, 0 otherwise):

Each constraint should state that there must be a station either in city j or in some adjacent city.

The jth column of the constraint matrix represents the set of cities that can be served by a fire station in city *j*.

We are asked to find a set of such subsets *j* that *covers* the set of all cities in the sense that every city appears in the service subset associated with *at least* one fire station. There must be at least one fire station either in city j or in some adjacent city (set covering constraints).

$$\begin{aligned}
\min \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} \\
\text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 && \geq 1 \ (\text{city } 1) \\
& x_1 + x_2 + x_3 + x_5 && \geq 1 \ (\text{city } 2) \\
& x_1 + x_2 + x_3 + x_4 + x_5 + x_6 && \geq 1 \ (\text{city } 3) \\
& x_1 + x_3 + x_4 + x_6 + x_7 && \geq 1 \ (\text{city } 4) \\
& x_2 + x_3 + x_5 + x_6 + x_8 + x_9 && \geq 1 \ (\text{city } 5) \\
& x_3 + x_4 + x_5 + x_6 + x_7 + x_8 && \geq 1 \ (\text{city } 6) \\
& x_4 + x_6 + x_7 + x_8 && \geq 1 \ (\text{city } 7) \\
& x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} && \geq 1 \ (\text{city } 8) \\
& x_5 + x_8 + x_9 + x_{10} + x_{11} && \geq 1 \ (\text{city } 9) \\
& x_8 + x_9 + x_{10} + x_{11} && \geq 1 \ (\text{city } 10) \\
& x_9 + x_{10} + x_{11} && \geq 1 \ (\text{city } 11)
\end{aligned}$$

All x_j = 0 or 1

Example 10. Timetable Scheduling

In an IE department, on Monday, 4 classes have to be assigned to classrooms. There are 3 classrooms devoted to the IE department. Each day is divided into 2 periods: morning and afternoon. Schedule a class timetable for the department.

Answer

x_{ijk} = 1 if class *i* is assigned to classroom *j* at time period *k*, 0 otherwise

   i = 1, 2, 3, 4 (classes), j = 1, 2, 3 (rooms), k = 1, 2 (periods)

Each class should be assigned to exactly one classroom at a specific time period (set partitioning constraints):

$$\begin{aligned}
& x_{111} + x_{112} + x_{121} + x_{122} + x_{131} + x_{132} = 1 && (\text{class } 1) \\
& x_{211} + x_{212} + x_{221} + x_{222} + x_{231} + x_{232} = 1 && (\text{class } 2) \\
& x_{311} + x_{312} + x_{321} + x_{322} + x_{331} + x_{332} = 1 && (\text{class } 3) \\
& x_{411} + x_{412} + x_{421} + x_{422} + x_{431} + x_{432} = 1 && (\text{class } 4)
\end{aligned}$$

At most one class can be held in each classroom at a specific time period (set packing constraints):

$x_{111} + x_{211} + x_{311} + x_{411} \leq 1$ (room 1 – morning)

$x_{112} + x_{212} + x_{312} + x_{412} \leq 1$ (room 1 – afternoon)

$x_{121} + x_{221} + x_{321} + x_{421} \leq 1$ (room 2 – morning)

$x_{122} + x_{222} + x_{322} + x_{422} \leq 1$ (room 2 – afternoon)

$x_{131} + x_{231} + x_{331} + x_{431} \leq 1$ (room 3 – morning)

$x_{132} + x_{232} + x_{332} + x_{432} \leq 1$ (room 3 – afternoon)

Therefore the IP model that would be formulated to schedule the timetable will be:

max f (or any other objective function)

s.t. $x_{111} + x_{112} + x_{121} + x_{122} + x_{131} + x_{132} = 1$ (class 1)

    $x_{211} + x_{212} + x_{221} + x_{222} + x_{231} + x_{232} = 1$ (class 2)

    $x_{311} + x_{312} + x_{321} + x_{322} + x_{331} + x_{332} = 1$ (class 3)

    $x_{411} + x_{412} + x_{421} + x_{422} + x_{431} + x_{432} = 1$ (class 4)

    $x_{111} + x_{211} + x_{311} + x_{411} \leq 1$ (room 1 – morning)

    $x_{112} + x_{212} + x_{312} + x_{412} \leq 1$ (room 1 – afternoon)

    $x_{121} + x_{221} + x_{321} + x_{421} \leq 1$ (room 2 – morning)

    $x_{122} + x_{222} + x_{322} + x_{422} \leq 1$ (room 2 – afternoon)

    $x_{131} + x_{231} + x_{331} + x_{431} \leq 1$ (room 3 – morning)

    $x_{132} + x_{232} + x_{332} + x_{432} \leq 1$ (room 3 – afternoon)

All x_{ijk} = 0 or 1

5.1.7 Traveling Salesperson Problems

"Given a number of cities and the costs of traveling from any city to any other city, what is the cheapest round-trip route (tour) that visits each city once and then returns to the starting city?"

This problem is called the traveling salesperson problem (TSP), not surprisingly.

An itinerary that begins and ends at the same city and visits each city once is called a *tour*.

Suppose there are *N* cities.

Let c_{ij} = Distance from city *i* to city *j* (for $i \neq j$) and

Let c_{ii} = M (a very large number relative to actual distances)

Also define x_{ij} as a 0-1 variable as follows:

    x_{ij} = 1 if s/he goes from city *i* to city *j*;

$x_{ij} = 0$ otherwise

The formulation of the TSP is:

$$\min \quad \sum_i \sum_j c_{ij}\, x_{ij}$$

$$\text{s.t.} \quad \sum_i x_{ij} = 1 \qquad \text{for all } j$$

$$\sum_j x_{ij} = 1 \qquad \text{for all } i$$

$$u_i - u_j + N\, x_{ij} \le N - 1 \qquad \text{for } i \ne j;\ i, j > 1$$

All $x_{ij} = 0$ or 1, All $u_i \ge 0$

The first set of constraints ensures that s/he arrives once at each city.

The second set of constraints ensures that s/he leaves each city once.

The third set of constraints ensure the following:

Any set of x_{ij}'s containing a subtour will be infeasible

Any set of x_{ij}'s that forms a tour will be feasible

$u_i - u_j + N\, x_{ij} \le N - 1 \qquad$ for $i \ne j;\ i, j > 1$

Assume $N = 5$

Subtours: 1-5-2-1, 3-4-3 ???

Choose the subtour that does not contain city 1:

$$u_3 - u_4 + 5\, x_{34} \le 4$$

$$\underline{u_4 - u_3 + 5\, x_{43} \le 4}$$

$$5\,(x_{34} + x_{43}) \le 8$$

This rules out the possibility that $x_{34} = x_{43} = 1$

The formulation of an IP whose solution will solve a TSP becomes unwieldy and inefficient for large TSPs.

When using branch and bound methods to solve TSPs with many cities, large amounts of computer time may be required. For this reason, heuristics, which quickly lead to a good (but not necessarily optimal) solution to a TSP, are often used.

5.2 SOLVING IP

We have gone through a number of examples of IPs at the "Formulating IP Problems" section.

"How can we get solutions to these models?" There are two common approaches:

The technique based on dividing the problem into a number of smaller problems in a *tree search* method called **branch and bound**.

The method based on **cutting planes** (adding constraints to force integrality).

Solving IP

Actually, all these approaches involve solving a series of LP.

For solving LP's we have *general purpose* (independent of the LP being solved) and *computationally effective* (able to solve large LP's) algorithms (simplex or interior point).

For solving IP's *no* similar general purpose and computationally effective algorithms exist

5.2.1 Categorization

Categorization (w.r.t. Purpose)

- General purpose methods will solve any IP but potentially computationally ineffective (will only solve relatively small problems); or

- Special purpose methods are designed for one particular type of IP problem but potentially computationally more effective.

Categorization (w.r.t. Algorithm)

- Optimal algorithms mathematically guarantee to find the optimal solution

- Heuristic algorithms are used to solve a problem by trial and error when an optimal algorithm approach is impractical. They hopefully find a good feasible solution that, in objective function terms, is close to the optimal solution.

Why Heuristics?

Because the size of problem that we want to solve is beyond the computational limit of known optimal algorithms within the computer time we have available.

We could solve optimally but feel that this is not worth the effort (time, money, etc) we would expend in finding the optimal solution.

In fact it is often the case that a well-designed heuristic algorithm can give good quality (near-optimal) results.

Solution Algorithms Categories

- General Purpose, Optimal

  Enumeration, branch and bound, cutting plane

- General Purpose, Heuristic

  Running a general purpose optimal algorithm and terminating after a specified time

- Special Purpose, Optimal

  Tree search approaches based upon generating bounds via dual ascent, lagrangean relaxation

- Special Purpose, Heuristic

  Bound based heuristics, tabu search, simulated annealing, population heuristics (e.g. genetic algorithms), interchange

5.2.2 LP Relaxation

For any IP we can generate an LP by taking the same objective function and same constraints but with the requirement that variables are integer replaced by appropriate continuous constraints:

"$x_i = 0$ or 1" $\rightarrow$ $x_i \geq 0$ and $x_i \leq 1$

"$x_i \geq 0$ and integer" $\rightarrow$ $x_i \geq 0$

The LP obtained by omitting all integer and 0-1 constraints on variables is called the **LP Relaxation of the IP** (**LR**). We can then solve this LR of the original IP.

Example 1. IP

Write the LR of the following IP:

Maximize $\quad$ $8\,x_1 + 5\,x_2$

Subject to $\qquad$ $x_1 + \quad x_2 \leq 6$

$\qquad\qquad$ $9\,x_1 + 5\,x_2 \leq 45$

$\qquad\qquad$ $x_1, x_2 \geq 0$ and integer

Answer

Maximize $\quad$ $8\,x_1 + 5\,x_2$

Subject to $\qquad$ $x_1 + \quad x_2 \leq 6$

$\qquad\qquad$ $9\,x_1 + 5\,x_2 \leq 45$

$\qquad\qquad$ $x_1, x_2 \geq 0$

Example 2. Binary IP

Write the LR of the following IP:

Maximize $\quad$ $0.2\,x_1 + 0.3\,x_2 + 0.5\,x_3 + 0.1\,x_4$

Subject to $\quad$ $0.5\,x_1 + 1\quad x_2 + 1.5\,x_3 + 0.1\,x_4 \leq 3.1$

$\qquad\qquad$ $0.3\,x_1 + 0.8\,x_2 + 1.5\,x_3 + 0.4\,x_4 \leq 2.5$

$\qquad\qquad$ $0.2\,x_1 + 0.2\,x_2 + 0.3\,x_3 + 0.1\,x_4 \leq 0.4$

$\qquad\qquad$ $x_j = 0$ or $1 \quad j = 1, \dots 4$

Answer

Maximize $\quad$ $0.2\,x_1 + 0.3\,x_2 + 0.5\,x_3 + 0.1\,x_4$

Subject to $\quad$ $0.5\,x_1 + 1\quad x_2 + 1.5\,x_3 + 0.1\,x_4 \leq 3.1$

$$0.3\,x_1 + 0.8\,x_2 + 1.5\,x_3 + 0.4\,x_4 \leq 2.5$$

$$0.2\,x_1 + 0.2\,x_2 + 0.3\,x_3 + 0.1\,x_4 \leq 0.4$$

$$x_1 \leq 1$$

$$x_2 \leq 1$$

$$x_3 \leq 1$$

$$x_4 \leq 1$$

$$\text{All } x_i \geq 0$$

Example 3. Mixed IP

Write the LR of the following IP:

max $z = x_1 + x_2 + x_3$

s.t. $x_1 + 6x_2 + x_3 \leq 8$

       $x_1 + 2x_2 + 1{,}5x_3 \leq 2$

      $2x_1 + x_2 + 5x_3 \leq 8$

      $x_1 \geq 0$, $x_2 \geq 0$ and integer, x_3 binary

Answer

max $z = x_1 + x_2 + x_3$

s.t. $x_1 + 6x_2 + x_3 \leq 8$

       $x_1 + 2x_2 + 1{,}5x_3 \leq 2$

      $2x_1 + x_2 + 5x_3 \leq 8$

           $x_3 \leq 1$

     $x_1, x_2, x_3 \geq 0$

Naturally Integer LP

If LR is optimized by integer variables then that solution is feasible and optimal for IP. In other words, if the solution is turned out to have all variables taking integer values at the optimal solution, it is also optimal solution for IP:

LR – IP Relation

Since LR is less constrained than IP:

- If IP is a maximization problem, the optimal objective value for LR is greater than or equal to that of IP.

- If IP is a minimization problem, the optimal objective value for LR is less than or equal to that of IP.

- If LR is infeasible, then so is IP.

So solving LR does give some information. It gives a bound on the optimal value, and, if we are lucky, may give the optimal solution to IP.

5.2.3 Enumeration

Unlike LP (where variables took continuous values) in IP's (where all variables are integers) each variable can only take a finite number of discrete (integer) values. Hence the obvious solution approach is simply to *enumerate* all these possibilities - calculating the value of the objective function at each one and choosing the (feasible) one with the optimal value.

Example 4. Multi-period Capital Budgeting

Maximize $0.2 x_1 + 0.3 x_2 + 0.5 x_3 + 0.1 x_4$

Subject to $0.5 x_1 + 1\ \ x_2 + 1.5 x_3 + 0.1 x_4 \leq 3.1$

$0.3 x_1 + 0.8 x_2 + 1.5 x_3 + 0.4 x_4 \leq 2.5$

$0.2 x_1 + 0.2 x_2 + 0.3 x_3 + 0.1 x_4 \leq 0.4$

$x_j = 0$ or 1 $j = 1, \dots 4$

Possible Solutions

There are $2^4 = 16$ possible solutions:

0 0 0 0 do no projects	1 1 1 0 do three projects
	1 1 0 1
0 0 0 1 do one project	1 0 1 1
0 0 1 0	0 1 1 1
0 1 0 0	
1 0 0 0	1 1 1 1 do four projects
0 0 1 1 do two projects	
0 1 0 1	
1 0 0 1	
0 1 1 0	
1 0 1 0	
1 1 0 0	

Review

Hence for our example, we merely have to examine 16 possibilities before we know precisely what the best possible solution is. This example illustrates a general truth about integer programming.

What makes solving the problem easy when it is small is precisely what makes it hard very quickly as the problem size increases.

This is simply illustrated: suppose we have 100 integer variables each with two possible integer values then there are 2x2x2x ... x2 = 2^{100} (approximately 10^{30}) possibilities which we have to enumerate (obviously many of these possibilities will be infeasible, but until we generate one we cannot check it against the constraints to see if it is feasible).

For 100 integer variable - **conceptually** there is not a problem - simply enumerate all possibilities and choose the best one. But **computationally** (numerically) this is just impossible.

5.2.4 The Branch-and-Bound Method

The most effective general purpose optimal algorithm is an LP-based tree search approach called as **branch and bound** (B&B).

The method was first put forward in the early 1960's by Land and Doig.

This is *a way of systematically (implicitly) enumerating* feasible solutions such that the optimal integer solution is found.

Where this method differs from the enumeration method is that *not all* the feasible solutions are enumerated but only a part (hopefully a small part) of them. However we can still *guarantee* that we will find the optimal integer solution.

By solving a single sub-problem, many possible solutions may be eliminated from consideration.

Sub-problems are generated by branching on an appropriately chosen fractional-valued variable.

Suppose that in a given sub-problem (call it subp.1), assumes a fractional value between the integers *i* and *i*+1. Then the two newly generated sub-problems:

Subp.2 = Subp.1 + Constraint "$x_i \geq i+1$"

Subp.3 = Subp.1 + Constraint "$x_i \leq I$"

If all variables have integer values in the optimal solution to the sub-problem then the solution is a **feasible** solution for the original IP.

If the current feasible solution for the IP has a better *z*-value than any previously obtained feasible solution, then it becomes a **candidate solution**, and its *z*-value becomes the current **Lower Bound** (**LB**) on the optimal *z*-value (for a max problem).

If it is unnecessary to branch on a sub-problem, we say that it is fathomed (inactive):

- The sub-problem is infeasible

- The sub-problem yields an optimal solution in which all variables have integer values
- The optimal z-value for the sub-problem does not exceed the current LB, so it cannot yield the optimal solution of the IP

Two general approaches are used to determine which sub-problem should be solved next:

- Backtracking (LIFO)

  Leads us down one side of the B&B tree and finds a candidate solution. Then we backtrack our way up to the top of the other side of the tree.

- Jumptracking

  Solves all the problems created by branching. Then it branches again on the node with the best z-value. Often jumps from one side of the tree to the other.

A display of the sub-problems that have been created is called a ***tree***.

Each sub-problem is referred to as a ***node*** of the tree.

Each additional constraint is referred to as a line (***arc***) connecting two nodes (old sub-problem and one of the new sub-problems) of the tree.

5.2.4.1   B&B for Solving Pure IP Problems

Example 5. Pure IP (*Winston 9.3., p. 513*)

max $z = 8 x_1 + 5 x_2$

s.t. $x_1 + x_2 \leq 6$

$9 x_1 + 5 x_2 \leq 45$

$x_1, x_2 \geq 0$ and integer

Answer

Suppose that we were to solve the LR of the problem [replace "$x_1, x_2 \geq 0$ and integer" by "$x_1, x_2 \geq 0$"], then using any LP package or utilizing simplex method we get

$z = 165/4, x_1 = 15/4, x_2 = 9/4$

As a result of this we now know something about the optimal integer solution, i.e. this value of 165/4 is an ***Upper Bound*** on the optimal integer solution. This is because when we relax the integrality constraint we (as we are maximizing) end up with a solution value at least that of the optimal integer solution (and maybe better).

We arbitrarily choose a variable that is fractional in the optimal solution to the LR (subp.1): say x_1.

We need x_1 to be integer. We branch on x_1 and create two new sub-problems:

Subp.2: LR + "$x_1 \geq 4$"

Subp.3: LR + "$x_1 \leq 3$"

Observe that neither subp.2 nor subp.3 includes any points with $x_1 = 15/4$. This means that the optimal solution to LR can not recur when we solve these new sub-problems.

We now arbitrarily choose to solve subp.2.

We see that the optimal solution to subp.2 is

$z = 41$, $x_1 = 4$, $x_2 = 9/5$

We choose x_2 that is fractional in the optimal solution to subp.2.

We need x_2 to be integer. We branch on x_2 and create two new sub-problems:

Subp.4: LR + $x_1 \geq 4$ and $x_2 \geq 2$ = Subp.2 + $x_2 \geq 2$

Subp.5: LR + $x_1 \geq 4$ and $x_2 \leq 1$ = Subp.2 + $x_2 \leq 1$

The set of unsolved sub-problems are consists of subp.3, 4, and 5.

We choose to solve the most recently created sub-problem (This is called LIFO): The LIFO rule implies that we should next solve subp.4 or 5.

We now arbitrarily choose to solve subp.4.

We see that subp.4 is infeasible. Thus subp.4 can not yield the optimal solution to the IP.

Because any branches emanating from subp.4 will yield no useful information, it is fruitless to create them.

LIFO rule implies that we should next solve subp.5.

The optimal solution to subp.5 is

$z = 365/9$, $x_1 = 40/9$, $x_2 = 1$

We branch on fractional-valued x_1:

Subp.6: Subp.5 + $x_1 \geq 5$

Subp.7: Subp.5 + $x_1 \leq 4$

Subp.3, 6, and 7 are now unsolved.

The LIFO rule implies that we next solve subp.6 or 7.

We now arbitrarily choose to solve subp.7.

The optimal solution to subp.7 is

$z = 37$, $x_1 = 4$, $x_2 = 1$

As both variables assume integer values, this solution is *feasible* for the original IP → this solution is a *candidate solution*

We must keep this candidate solution until a better feasible solution to the IP (if any exists) is found.

We may conclude that the optimal z-value for the IP ≥ 37 → **Lower Bound** (LB)

LIFO rule implies that we should next solve subp.6.

The optimal solution to subp.6 is

$z = 40$, $x_1 = 5$, $x_2 = 0$

Its z-value of 40 is larger than LB.

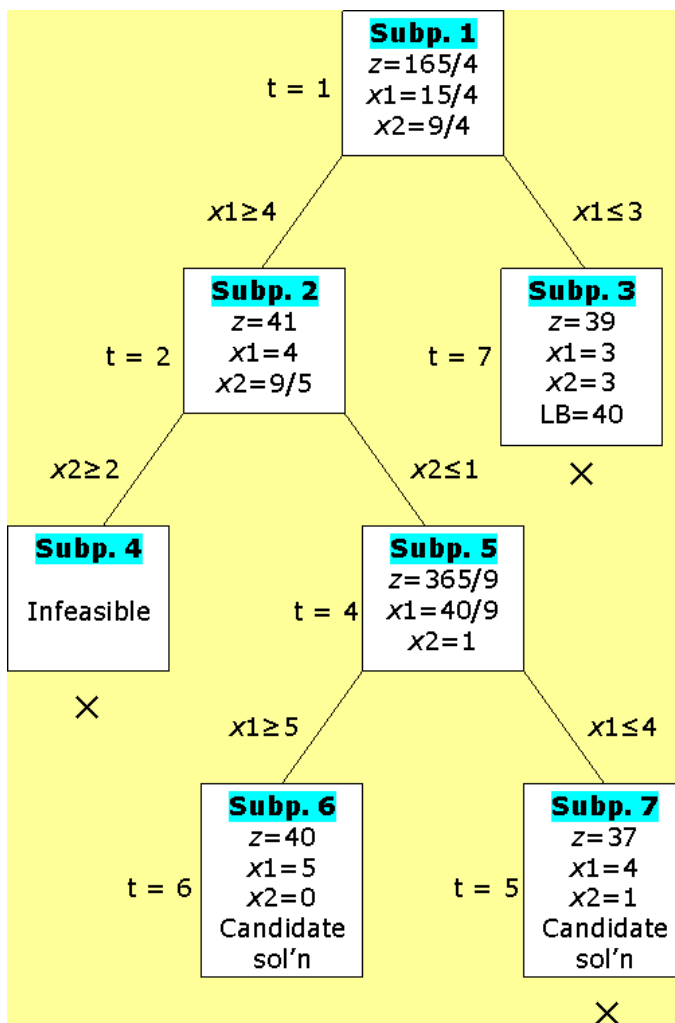Thus subp.7 cannot yield the optimal solution of the IP.

We update our LB to 40.

Subp.3 is the only remaining unsolved problem.

The optimal solution to subp.3 is

$z = 39$, $x_1 = 3$, $x_2 = 3$

Subp.3 cannot yield a z-value exceeding the current LB, so it cannot yield the optimal solution to the IP.

Final B&B Tree

Optimal Sol'n

Thus, the optimal solution to the IP

$z = 40$, $x_1 = 5$, $x_2 = 0$

5.2.4.2   B&B for Solving Mixed IP Problems

In MIP, some variables are required to be integers and others are allowed to be either integer or nonintegers.

To solve a MIP by B&B method, modify the method by branching only on variables that are required to be integers.

For a solution to a sub-problem to be a candidate solution, it need only assign integer values to those variables that are required to be integers

Example 6. Mixed IP (Winston 9.4., p. 523)

max $z = 2 x_1 + x_2$

s.t. $5 x_1 + 2 x_2 \leq 8$

$x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$; $x1$ integer

Answer

We solve the LR (subp.1) of the problem

[replace "$x_1 \geq 0$ and integer" by "$x_1 \geq 0$"]

Then using any LP package or utilizing simplex or graphical solution method we get

$z = 11/3$, $x_1 = 2/3$, $x_2 = 7/3$

Because x_2 is allowed to be fractional, we do not branch on x_2.

We branch on x_1 and create two new sub-problems:

Subp.2: LR + $x_1 \geq 1$

Subp.3: LR + $x_1 \leq 0$

We see that the optimal solution to subp.2 is

$z = 7/2$, $x_1 = 1$, $x_2 = 3/2$

As only x_1 assume integer value, this solution is feasible for the original MIP → Candidate solution; LB = 7/2

The optimal solution to subp.3 is

$z = 3$, $x_1 = 0$, $x_2 = 3$

Subp.3 cannot yield a *z*-value exceeding the current LB, so it cannot yield the optimal solution to the MIP.

Optimal Sol'n

Thus, the optimal solution to the MIP

$z = 7/2$, $x_1 = 1$, $x_2 = 3/2$

5.2.4.3   B&B for Solving Binary IP Problems

One aspect of the B&B method greatly simplify:

Due to each variable equaling 0 or 1, branching on x_i will yield in

$x_i = 0$ and $x_i = 1$

Example 7. Binary IP

max z = $0.2 x_1 + 0.3 x_2 + 0.5 x_3 + 0.1 x_4$

s.t. $0.5 x_1 + 1\ \ \ x_2 + 1.5 x_3 + 0.1 x_4 \leq 3.1$

$0.3 x_1 + 0.8 x_2 + 1.5 x_3 + 0.4 x_4 \leq 2.5$

$0.2 x_1 + 0.2 x_2 + 0.3 x_3 + 0.1 x_4 \leq 0.4$

$x_j = 0$ or 1 $j = 1, \dots 4$

Answer

Replace "$x_j = 0$ or 1 (j=1,...,4)" by "$0 \leq x_j \leq 1$ (j=1,...,4)" → LR of the problem

Optimal solution to the LR:

z=0.65, x_2=0.5, x_3=1, x_1=x_4=0

The variable x_2 is fractional. To resolve this we can generate two new problems:

P1: LR + x_2=0

P2: LR + x_2=1

We now have two new sub-problem to solve (*jumptracking*).

If we do this we get

P1 solution: z=0.6, x_1=0.5, x_3=1, x_2=x_4=0

P2 solution: z=0.63, x_2=1, x_3=0.67, x_1=x_4=0

Choosing sub-problem P2 (the best z–value), we branch on x_3 and get

P3 (P2 + x_3=0) sol'n: z=0.5, x_1=x_2=1, x_3=x_4=0

P4 (P2 + x_3=1) sol'n: infeasible

P3 solution is feasible for the original binary IP → Candidate solution; LB = 0.5

Choosing the only remaining sub-problem P1, we branch on x_1 and get

P5 (P1 + x_1=0) sol'n: z=0.6, x_3=x_4=1, x_1=x_2=0
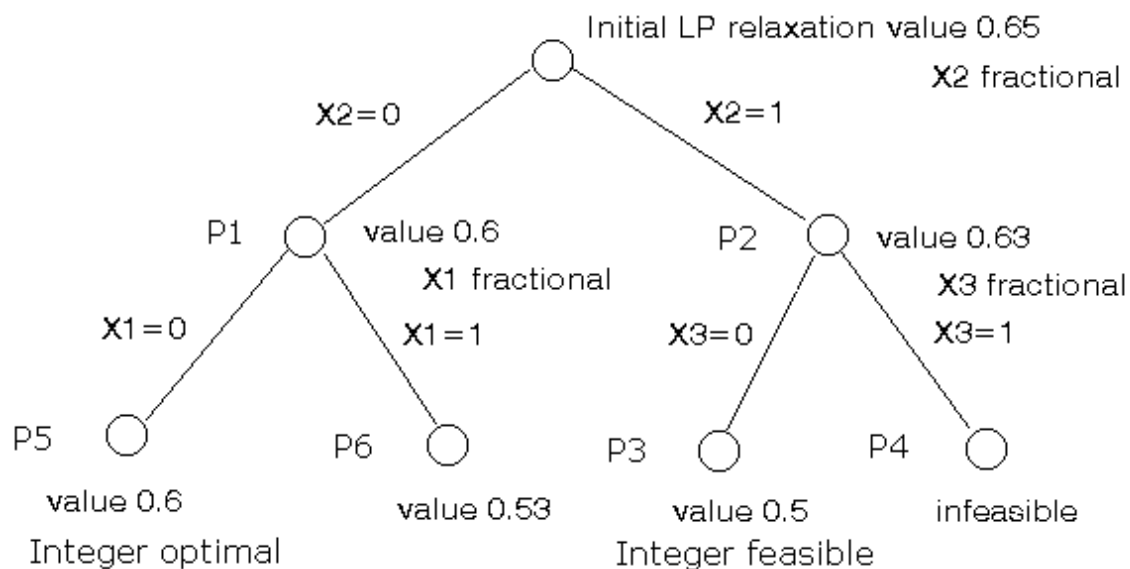
P6 (P1 + x_1=1) sol'n: z=0.53, x_1=1, x_3=0.67, x_2=x_4=0

P5 solution is feasible for the original binary IP → New candidate solution; updated LB = 0.6

P6 cannot yield a z-value exceeding the current LB, so it cannot yield the optimal solution to the binary IP.

Thus, the optimal solution to the binary IP

$z = 0.6$, $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = 1$

Review

Note here that B&B, like complete enumeration, also involves powers of 2 as we progress down the (binary) tree.

However also note that we did not enumerate all possible integer solutions (of which there are 16). Instead here we solved 7 LP's.

This is an important point, and indeed why tree search works at all. We do not need to examine as many LP's as there are possible solutions.

While the computational efficiency of tree search differs for different problems, it is this basic fact that enables us to solve problems that would be completely beyond us where we to try complete enumeration

5.2.4.4   B&B for Solving Knapsack Problems

Please recall that a knapsack problem is an IP, in which each variable must be equal to 0 or 1, with a single constraint:

max $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t. $a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$

$x_i = 0$ or 1   ($i = 1, 2, \ldots, n$)

Two aspects of the B&B method greatly simplify:

- Due to each variable equaling 0 or 1, branching on x_i will yield in $x_i = 0$ and $x_i = 1$
- The LP relaxation may be solved by *inspection* instead of using any LP package or utilizing simplex or graphical solution method

Inspection

Recall that

c_i is the benefit obtained if item i is chosen

b is the total amount of an available resource

a_i is the amount of the available resource used by item i

Observe that ratio $r_i (c_i/a_i)$ may be interpreted as the benefit item i earns for each unit of the resource used by item i.

Thus, the best items have the largest value of r and the worst items have the smallest values of r.

To solve any sub-problem resulting from a knapsack problem, compute all the ratios.

Then put the best item in the knapsack.

Then put the second best item in the knapsack.

Continue in this fashion until the best remaining item will overfill the knapsack.

Then fill the knapsack with as much of this item as possible.

Example 8. Knapsack

max $z =$ $8 x_1 + 11 x_2 + 6 x_3 + 4 x_4$

s.t. $5 x_1 + 7 x_2 + 4 x_3 + 3 x_4 \leq 14$

          $x_j = 0$ or 1 $j = 1, \ldots 4$

Answer

We compute the ratios:

$r_1 = 8 / 5 = 1.6$

$r_2 = 11 / 7 = 1.57$

$r_3 = 6 / 4 = 1.5$

$r_4 = 4 / 3 = 1.33$

Using the ratios, LR solution is

$x_1 = 1, x_2 = 1, x_3 = 0.5, x_4 = 0, z = 22$

We branch on x_3 and get

P1 (LR + x_3=0) sol'n: x_3=0, x_1=x_2=1, x_4=2/3, z=21.67

P2 (LR + x_3=1) sol'n: x_3=x_1=1, x_2=5/7, x_4=0, z=21.85

Choosing sub-problem P2 (the best z–value), we branch on x_2 and get

P3 (P2 + x_2=0) sol'n: x_3=1, x_2=0, x_1=1, x_4=1, z=18

P4 (P2 + x_2=1) sol'n: x_3=x_2=1, x_1=3/5, x_4=0, z=21.8

P3 solution is feasible for the original knapsack problem → Candidate solution; LB = 18

Choosing sub-problem P4, we branch on x_1 and get

P5 (P4 + x_1=0) sol'n: x_3=x_2=1, x_1=0, x_4=1, z=21

P6 (P4 + x_1=1) sol'n: Infeasible (x_3=x_2=x_1=1: LHS=16)

P5 solution is feasible for the original knapsack problem → New candidate solution; updated LB = 21

The only remaining sub-problem is P1 with solution value 21.67

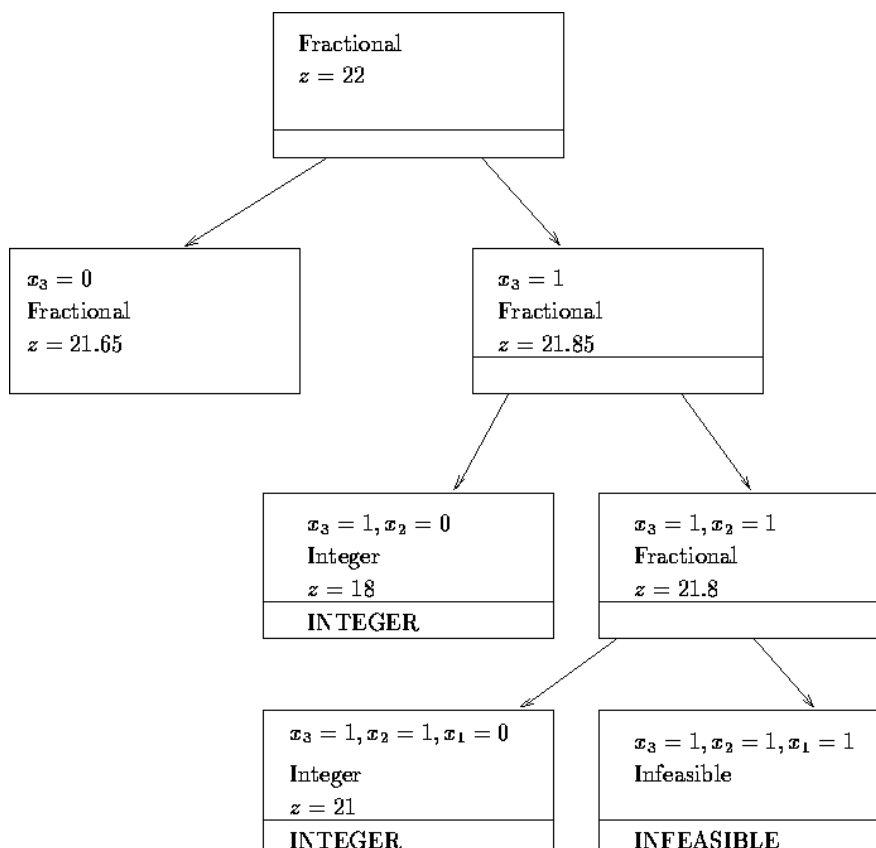There is no better solution for this sub-problem than 21. But we already have a solution with value 21.

It is not useful to search for another such solution. We can fathom P1 based on this bounding argument and mark P1 as inactive.

Optimal sol'n and Report

Thus, the optimal solution is z=21, x_1=0, x_2=1, x_3=1, x_4=1

Items 2, 3, and 4 should be put in the knapsack; the total value would be 21.

5.2.4.5　　B&B for Solving Combinatorial Optimization Problems

A ***combinatorial (discrete) optimization problem*** is any optimization problem that has a finite number of feasible solutions.

A B&B approach is often an efficient way to solve them.

Examples of combinatorial optimization problems:

- Ten jobs must be processed on a single machine. It is known how long it takes to complete each job and the time at which each job must be completed (the job's due date). What ordering of the jobs minimizes the total delay of the 10 jobs?

- A salesperson must visit each of the 10 cities before returning to her/his home. What ordering of the cities minimizes the total distance the salesperson must travel before returning home? (TSP).

In each of these problems, many possible solutions must be considered.

Example 9: Machine Scheduling
Please refer to Winston 9.6. p. 528

TSP

Please recall that

We define x_{ij} as a 0-1 variable:

x_{ij} = 1 if TS goes from city i to city j;

x_{ij} = 0 otherwise

c_{ij} = distance form city i to city j (for $i{\neq}j$)

c_{ii} = M (a very large number relative to actual distances)

An itinerary that begins and ends at the same city and visits each city once is called a tour.

It seems reasonable that we might be able to find the answer to TSP by solving an assignment problem having a cost matrix whose ijth is c_{ij}.

If the optimal solution to the assignment problem yields a tour, it is the optimal solution to the TSP.

Unfortunately, the optimal solution to the assignment problem need not be a tour (may yield subtours).

If we could exclude all feasible solutions that contain subtours and then solve the assignment problem, we would obtain the optimal solution to TSP → Not easy to do...

Several B&B approaches have been developed for solving TSPs.

One approach start with solving the preceding assignment problem (sub-problem 1).

Because this sub-problem contains no provisions to prevent subtours, it is a relaxation of the original TSP.

Thus, if the optimal solution to the subp.1 is feasible for the TSP (no subtours), then it is also optimal for the TSP.

If it is infeasible (contain subtours), we branch on the subp.1 in a way that will prevent one of subp.1's subtours from recurring in solutions to subsequent sub-problems.

Example 10. TSP *(Winston 9.6., p. 530)*

Joe State lives in Gary, Indiana and owns insurance agencies in Gary, Fort Wayne, Evansville, Terre Haute, and South Bend.

Each December, he visits each of his insurance agencies.

The distance between each agency:

miles	G	FW	E	TH	SB
G	0	132	217	164	58
FW	132	0	290	201	79
E	217	290	0	113	303
TH	164	201	113	0	196
SB	58	79	303	196	0

What order of visiting his agencies will minimize the total distance traveled?

Answer

We first solve the assignment problem (subp.1) applying the Hungarian method to the cost matrix shown:

COSTS	G	FW	E	TH	SB
G	1000	132	217	164	58
FW	132	1000	290	201	79
E	217	290	1000	113	303
TH	164	201	113	1000	196
SB	58	79	303	196	1000

The optimal solution will be:

$x_{15}=x_{21}=x_{34}=x_{43}=x_{52}=1$, $z=495$

The optimal solution to subp.1 contains two subtours:

- recommends going from Gary (1) to South Bend (5), then to Fort Wayne (2), and then back to Gary (1–5–2–1).
- also suggests that if Joe is in Evansville (3), he should go to Terre Haute (4) and then to Evansville (3–4–3).

Thus, the optimal solution can not be the optimal solution to Joe's problem.

We arbitrarily choose to exclude the subtour 3-4-3.

Observe that the optimal solution to Joe's problem must have either $x_{34}=0$ or $x_{43}=0$.

Thus, we can branch on subp.1 by creating two new sub-problems.

Subp.2: Subp.1 + ($x_{34}=0$, or $c_{34}=M$)

Subp.3: Subp.1 + ($x_{43}=0$, or $c_{43}=M$)

Now arbitrarily choose subp.2 to solve.

COSTS	G	FW	E	TH	SB
G	1000	132	217	164	58
FW	132	1000	290	201	79
E	217	290	1000	1000	303
TH	164	201	113	1000	196
SB	58	79	303	196	1000

The optimal solution will be:

$x_{14}=x_{25}=x_{31}=x_{43}=x_{52}=1$, $z=652$

This solution includes the subtours 1–4–3–1 and 2–5–2.

Thus, it can not be the optimal solution to Joe's problem.

Following the LIFO approach, now branch sub-problem 2 in an effort to exclude the subtour 2-5-2. Thus we add two additional sub-problems.

Subp.4: Subp.2 + ($x_{25}=0$, or $c_{25}=M$)

Subp.5: Subp.2 + ($x_{52}=0$, or $c_{52}=M$)

By using the Hungarian method on subp.4, we obtain the optimal solution

$x_{15}=x_{24}=x_{31}=x_{43}=x_{52}=1$, $z=668$

This solution contains no subtours and yields the tour 1–5–2–4–3–1

It is a candidate solution and any node that **cannot** yield a z-value < 668 may be eliminated from consideration.

We next solve subp.5.

$x_{14}=x_{43}=x_{32}=x_{25}=x_{51}=1$, $z=704$

This solution also yields a tour 1–4–3–2–5–1

But $z=704$ is not as good as the subp.4 candidate's $z=668$

Thus this subp.5 may be eliminated from consideration.

Only subp.3 remains.

The optimal solution

$x_{13}=x_{25}=x_{34}=x_{41}=x_{52}=1$, $z=652$.

This solution includes the subtours 1–3–4–1 and 2–5–2.

However, it is still possible for this sub-problem to yield a solution with no subtours that beats $z=668$.

Next branch on sub-problem 3 creating new sub-problems.

Subp.6: Subp.3 + ($x_{25}=0$, or $c_{25}=M$)

Subp.7: Subp.3 + ($x_{52}=0$, or $c_{52}=M$)

Both of these sub-problems have a z-value that is larger than 668.

Optimal sol'n and Report

Subp.4 thus yields the optimal solution:

$x_{15}=x_{24}=x_{31}=x_{43}=x_{52}=1$, $z=668$

Joe should travel from Gary (1) to South Bend (5), from South Bend to Fort Wayne (2), from Fort Wayne to Terre Haute (4), from Terre Haute to Evansville (3), and then back to Gary.

He will travel a total distance of 668 miles.

5.2.4.6   Heuristics for TSPs

An IP formulation can be used to solve a TSP but can become unwieldy and inefficient for large TSPs.

When using B&B methods to solve TSPs with many cities, large amounts of computer time is needed.

Heuristic methods, or heuristics, can be used to quickly lead to a good (but not necessarily optimal) solution.

Two types of heuristic methods can be used to solve TSP:

1. The Nearest-Neighbor
2. The Cheapest-Insertion

The Nearest-Neighbor Heuristic

1. Begin at any city and then "visit" the nearest city.
2. Then go to the unvisited city closest to the city we have most recently visited.
3. Continue in this fashion until a tour is obtained.
4. After applying this procedure beginning at each city, take the best tour found.

Example 11. Applying the NNH to TSP

We arbitrarily choose to begin at city 1.

Of the cities 2, 3, 4, and 5, city 5 is the closest city to city 1 $\rightarrow$ Generate the arc 1–5

Of the cities 2, 3, and 4, city 2 is the closest city to city 5 → 1–5–2

Of the cities 3 and 4, city 4 is the closest city to city 2 → 1–5–2–4

Joe must next visit city 3 and then return to city 1 → 1–5–2–4–3–1 (668 miles).

In this case, the NNH yields the optimal tour.

If we had begun at city 3, however, NNH yields the tour 3–4–1–5–2–3 (704 miles).

Thus, the NNH need not yield an optimal tour.

This procedure should be applied beginning at each city, and then the best tour found should be taken as solution.

The Cheapest-Insertion Heuristic

1. Begin at any city and find its closest neighbor.

2. Then create a subtour joining those two cities.

3. Next, replace an arc in the subtour (say, arc (i, j)) by the combinations of two arcs (i, k) and (k, j), where k is not in the current subtour that will increase the length of the subtour by the smallest (or cheapest) amount.

4. Continue with this procedure until a tour is obtained.

5. After applying this procedure beginning with each city, we take the best tour found.

Example 12. Applying the CIH to TSP

We arbitrarily choose to begin at city 1.

Of the cities 2, 3, 4, and 5, city 5 is the closest city to city 1 → Generate the arc 1–5

We create a subtour (1, 5)–(5, 1)

We could replace arc (1, 5) by (1, 2)–(2, 5), (1, 3)–(3, 5), or (1, 4)–(4, 5)

We could also replace (5, 1) by (5, 2)–(2, 1), (5, 3)–(3, 1), or (5, 4)–(4, 1)

The computations used to determine which arc of (1, 5)–(5, 1) should be replaced are given in the Table:

Arc replaced	Arcs added	Added length
(1, 5)*	(1, 2)–(2, 5)	$c_{12}+c_{25}-c_{15}=153$
(1, 5)	(1, 3)–(3, 5)	$c_{13}+c_{35}-c_{15}=462$
(1, 5)	(1, 4)–(4, 5)	$c_{14}+c_{45}-c_{15}=302$
(5, 1)*	(5, 2)–(2, 1)	$c_{52}+c_{21}-c_{51}=153$
(5, 1)	(5, 3)–(3, 1)	$c_{53}+c_{31}-c_{51}=462$
(5, 1)	(5, 4)–(4, 1)	$c_{54}+c_{41}-c_{51}=302$

* indicates the correct replacement: either (1, 5) or (5, 1)

We arbitrarily choose to replace arc (1, 5) by arcs (1, 2) and (2, 5) → New subtour: (1, 2)–(2, 5)–(5, 1)

We then determine which arc should be replaced

Arc replaced	Arcs added	Added length
(1, 2)	(1, 3)–(3, 2)	375
(1, 2)*	(1, 4)–(4, 2)	233
(2, 5)	(2, 3)–(3, 5)	514
(2, 5)	(2, 4)–(4, 5)	318
(5, 1)	(5, 3)–(3, 1)	462
(5, 1)	(5, 4)–(4, 1)	302

We now replace arc (1, 2) by arcs (1, 4) and (4, 2) → New subtour: (1, 4)–(4, 2)–(2, 5)–(5, 1)

Which arc should be replaced?

Arc replaced	Arcs added	Added length
(1, 4)*	(1, 3)–(3, 4)	166
(4, 2)	(4, 3)–(3, 2)	202
(2, 5)	(2, 3)–(3, 5)	514
(5, 1)	(5, 3)–(3, 1)	462

We now replace arc (1, 4) by arcs (1, 3) and (3, 4)

This yields the tour (1, 3)–(3, 4)–(4, 2)–(2, 5)–(5, 1)

In this case, the CIH yields the optimal tour.

But, in general, the CIH does not necessarily do so.

This procedure should be applied beginning at each city, and then the best tour found should be taken as solution.

Evaluation of Heuristics

- Performance guarantees

  Gives a worse-case bound on how far away from optimality a tour constructed by the heuristic can be

- Probabilistic analysis

  A heuristic is evaluated by assuming that the location of cities follows some known probability distribution

- Empirical analysis

  Heuristics are compared to the optimal solution for a number of problems for which the optimal tour is known

5.2.5 Cutting Planes

A linear inequality is a *valid inequality* for a given IP problem if it holds for all integer feasible solutions to the model.
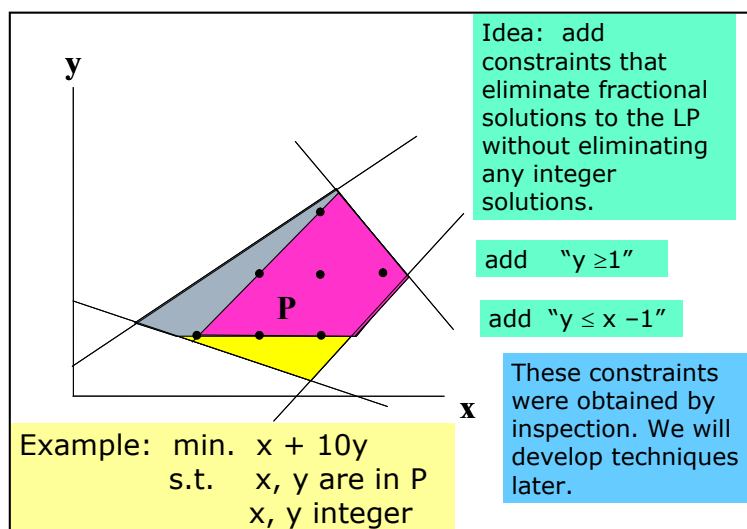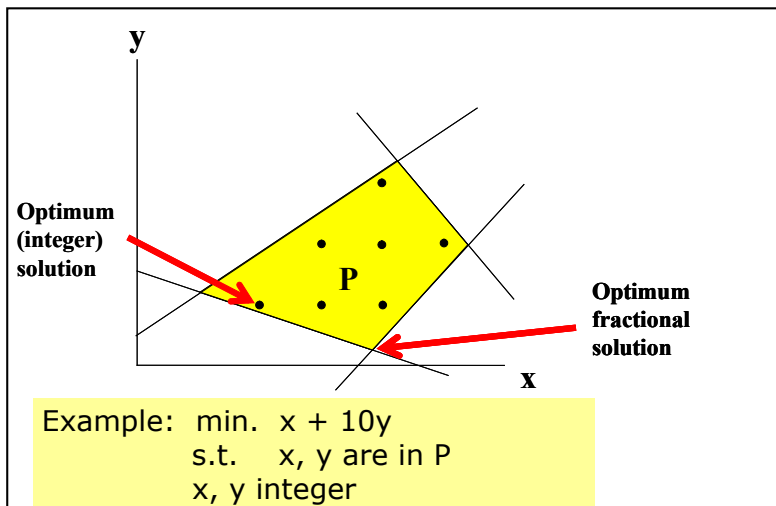
Relaxations can often be strengthened dramatically by including valid inequalities that are not needed by a correct discrete model.

To strengthen a relaxation, a valid inequality must cut off (render infeasible) some feasible solutions to current LR that are not feasible in the IP model.

This need to cut off non-integer relaxation solutions is why valid inequalities are sometimes called *cutting planes*.

Example 13. Cutting Plane - Conceptualization
Consider the following problem in the figure.

Example: min. x + 10y
     s.t. x, y are in P
     x, y integer

Idea: add constraints that eliminate fractional solutions to the LP without eliminating any integer solutions.

add "y ≥1"

add "y ≤ x −1"

These constraints were obtained by inspection. We will develop techniques later.

Example: min. x + 10y
     s.t. x, y are in P
     x, y integer

Example: min. x + 10y
s.t.   x, y are in P
       x, y integer

If we add exactly the right inequalities, then every corner point of the LP will be integer, and the IP can be solved by solving the LP

We call this minimal LP, the *convex hull* of the IP solutions. For large problems, these constraints are hard to find.

The tightest possible constraints are very useful, and are called *facets*

Cut Classification

- General purpose

  A fractional extreme point can always be separated (LP-based approach, that works for IP)

  o Disjunctive cuts

  o Gomory cutting planes

- Problem specific

  Derived from problem structure, generally facets. (Capital Budgeting (Knapsack), Set Packing...)

Cutting Plane Algorithm (Gomory cut)

Find the optimal tableau for the IP's LR.

If all variables in the optimal solution assume integer values, we have found an optimal solution! Otherwise proceed to next step

Pick a constraint in the optimal tableau whose RHS has the fractional part closest to ½.

For the constraint identified, put all of the integer parts on the left side (round down), and all the fractional parts on the right

Generate the cut as:

"RHS of the modified constraint" ≤ 0

Use the dual simplex to find the optimal solution to the LR, with the cut as an additional constraint.

- If all variables assume integer values in the optimal solution, we have found an optimal solution to the IP.

- Otherwise, pick the constraint with the most fractional right-hand side and use it to generate another cut, which is added to the tableau.

We continue this process until we obtain a solution in which all variables are integers. This will be an optimal solution to the IP.

Dual Simplex Method

Please recall that at dual simplex:

- o We choose the most negative RHS.

- o BV of this pivot row leaves the basis.

- o For the variables that have a negative coefficient in the pivot row, we calculate the ratios (coefficient in R0 / coefficient in pivot row).

- o Variable with the smallest ratio (absolute value) enters basis.

Example 14. Telfa Co. *(Winston 9.8., p. 546)*

max $z =$ $8 x_1 + 5 x_2$

s.t. $x_1 + x_2 \leq 6$

              $9 x_1 + 5 x_2 \leq 45$

$x_1, x_2 \geq 0$ and integer

Answer

If we ignore integrality, we get the following optimal tableau:

z	x_1	x_2	s_1	s_2	RHS
1	0	0	1.25	0.75	41.25
0	0	1	2.25	-0.25	2.25
0	1	0	-1.25	0.25	3.75

Let's choose the constraint whose RHS has the fractional part closest to ½ (Arbitrarily choose the second constraint):

$x_1 - 1.25 s_1 + 0.25 s_2 = 3.75$

We can manipulate this to put all of the integer parts on the left side (round down), and all the fractional parts on the right to get:

$x_1 - 2 s_1 + 0 s_2 - 3 = 0.75 - 0.75 s_1 - 0.25 s_2$

Now, note that the LHS consists only of integers, so the right hand side must add up to an integer. It consists of some positive fraction minus a series of positive values.

Therefore, the right hand side cannot be a positive value. Therefore, we have derived the following constraint:

$0.75 - 0.75\ s_1 - 0.25\ s_2 \le 0$

This constraint is satisfied by every feasible integer solution to our original problem. But, in our current solution, s_1 and s_2 both equal 0, which is infeasible to the above constraint. This means the above constraint is a cut, called the **Gomory cut** after its discoverer.

We can now add this constraint to the linear program and be guaranteed to find a different solution, one that might be integer.

z	$x\,1$	$x\,2$	$s\,1$	$s\,2$	$s\,3$	RHS
1	0	0	1.25	0.75	0	41.25
0	0	1	2.25	-0.25	0	2.25
0	1	0	-1.25	0.25	0	3.75
0	0	0	-0.75	-0.25	1	-0.75

The dual simplex ratio test indicates that s_1 should enter the basis instead of s_3.

The optimal solution is an IP solution:

$z = 40$, $x_1 = 5$, $x_2 = 0$

Example 15. Supplementary Problem

min $z = 6\ x_1 + 8\ x_2$

s.t. $3\ x_1 + x_2 \ge 4$

$x_1 + 2\ x_2 \ge 4$

$x_1, x_2 \ge 0$ and integer

Answer

Optimal tableau for LR

z	$x\,1$	$x\,2$	$e\,1$	$e\,2$	RHS
1	0	0	-0.80	-3.60	17.60
0	1	0	-0.40	0.20	0.80
0	0	1	0.20	-0.60	1.60

Choose the second constraint

$x_2 + 0.2\ e_1 - 0.6\ e_2 = 1.6$

Manipulate this:

$x_2 - e_2 - 1 = 0.6 - 0.2\ e_1 - 0.4\ e_2$

Cut:

$0.6 - 0.2\ e_1 - 0.4\ e_2 \le 0$

New LP tableau

z	x1	x2	e1	e2	s3	RHS
1	0	0	-0.8	-3.6	0	17.6
0	1	0	-0.4	0.2	0	0.8
0	0	1	0.2	-0.6	0	1.6
0	0	0	-0.2	-0.4	1	-0.6

The dual simplex ratio test indicates that e_1 should enter the basis instead of s_3.

The optimal solution is an IP solution:

$z = 20$, $x_1 = 2$, $x_2 = 1$