END332E QUIZ 5 (Morning Section) 22. 12. 2025

Question:

I have just purchased (at time 0) a new car for $12,000. The cost of maintaining a car during a year depends on its age at the beginning of the year, as given in Table 1. To avoid the high maintenance costs associated with an older car, I may trade in my car and purchase a new car. The price I receive on a trade-in depends on the age of the car at the time of trade-in (see Table 2). To simplify the computations, we assume that at any time, it costs $12,000 to purchase a new car. My goal is to minimize the net cost (purchasing costs + maintenance costs - money received in trade-ins) incurred during the next five years.

Solve the problem using dynamic programming. Hint: Stages: t =0,1,…,5.

Table 1. Car Maintenance cost

Age of Car (year)	Annual maintenance cost ($)
0	2,000
1	4,000
2	5,000
3	9,000
4	12,000

Table 2. Car Trade-in prices

Age of Car (year)	Trade-in Price ($)
1	7,000
2	6,000
3	2,000
4	1,000
5	0

Solution:

Stage: t = 0, 1, 2, 3, 4, 5 (time 5 is the end of the planning horizon). **(1)**

State (i) = age (years) of the current car at the beginning of stage t. **(1)**

$f_t(i)$: the minimum net cost from time t to time 5, given that the car age at time t is i. **(1)**

$M(i)$ = maintenance cost of the car at age i **(1)**

$S(i)$ T(i)= trade in price **(1)**

$P = 12000$

$f_t(i)$: min{Keep, Replace}

$Keep = M(i) + f_{t+1}(i+1)$

$Replace = P - S(i) + M(0) + f_{t+1}(1)$

$f_t(i)$: $\min\{12000 + M(i) + f_{t+1}(i+1), P - S(i) + M(0) + f_{t+1}(1)\}$

Stage t=5 **(10)**

$f_5(i) = -S(i)$

$f_5(1) =$ -7000

$f_5(2) =$ -6000

$f_5(3) =$ -2000

$f_5(4) =$ -1000

$f_5(5) =$ 0

Stage t=4 **(16)**

$f_4(1) = min\big(M(1) + f_5(2), P - S(1) + M(0) + f_5(1)\big) = min(4000 - 6000, 12000 - 7000 + 2000 - 7000) = -2000$ Keep

$f_4(2) = min(M(2) + f_5(3), P - S(2) + M(0) + f_5(1)) = min(5000 - 2000, 12000 - 6000 + 2000 - 7000) = 1000$ Replace

$f_4(3) = min(M(3) + f_5(4), P - S(3) + M(0) + f_5(1)) = min(9000 - 1000, 12000 - 2000 + 2000 - 7000) = 5000$ Replace

$f_4(4) = min(M(4) + f_5(5), P - S(4) + M(0) + f_5(1)) = min(12000 + 0, 12000 - 1000 + 2000 - 7000) = 6000$ Replace

Stage t=3 **(15)**

$f_3(1) = min(M(1) + f_4(2), P - S(1) + M(0) + f_4(1)) = min(4000 + 1000, 12000 - 7000 + 2000 - 2000) = 5000$ Keep

$f_3(2) = min(M(2) + f_4(3), P - S(2) + M(0) + f_4(1)) = min(5000 + 5000, 12000 - 6000 + 2000 - 2000) = 6000$ Replace

$f_3(3) = min(M(3) + f_4(4), P - S(3) + M(0) + f_4(1)) = min(9000 + 6000, 12000 - 2000 + 2000 - 2000) = 10000$ Replace

Stage t = 2: **(14)**

$f_2(1) = min(M(1) + f_3(2), P - S(1) + M(0) + f_3(1)) = min(4000 + 6000, 12000 - 7000 + 2000 + 5000) = 10000$ Keep

$f_2(2) = min(M(2) + f_3(3), P - S(2) + M(0) + f_3(1)) = min(5000 + 10000, 12000 - 6000 + 2000 + 2000) = 13000$ Replace

Stage t = 1: **(10)**

$f_1(1) = min(M(1) + f_2(2), P - S(1) + M(0) + f_2(1)) = min(4000 + 13000, 12000 - 7000 + 2000 + 10000) = 17000$ Keep or Replace

Stage t = 0

$f_0(0) = \min(M(0) + f_1(1), P - S(0) + m(0) + f_1(1)) = (2000 + 17000, 12000 + 2000 + 17000) = 19000$ **(10)** Keep

$12000 + 19000 = \mathbf{31000}$ **Minimum Total Cost(10)**

Alternative solutions: 0-1-3-5, 0-2-3-5, 0-2-4-5

All solutions indicate to use one car for 1 year, and two cars for 2 years. (10)

Alternative solution:

We define $g(t)$ to be the minimum net cost incurred from time t until time 5 (including the purchase cost and trade-in price for the newly purchased car) given that a new car has been purchased at time t.

We also define c_{tx} to be the net cost (including purchase cost and trade-in price) of purchasing a machine at time t and operating it until time x. Then the appropriate recursion is

$$g(t) = \min_{x}\{c_{tx} + g(x)\} \qquad (t = 0,1,2,3,4,5)$$

$$c_{01}=c_{12}=c_{23}=c_{34}=c_{45}= 12+2 \quad -7= 7$$
$$c_{02}=c_{13}=c_{24}= \quad c_{35} = 12+2+4 \quad -6=12$$
$$c_{03}=c_{14}=c_{25}= 12+2+4+5-2=21$$
$$c_{04}=c_{15}= 12+2+4+5+9 \quad -1=31$$
$$c_{05}= 12+2+4+5+9+12-0=44$$

Stage 5:

$g(5) = 0$

Stage 4:

$g(4) = c_{45} + g(5) = 7$

Stage 3:

$$g(3) = min\begin{cases} c_{34} + g(4) = 7+7 = 14 \\ c_{35} + g(5) = 12 * \end{cases} = 12$$

Stage 2:

$$g(2) = min\begin{cases} c_{23} + g(3) = 7+12 = 19 * \\ c_{24} + g(4) = 12+7 = 19 * \\ c_{25} + g(5) = 21+0 = 21 \end{cases} = 19$$

Stage 1:

$$g(1) = min\begin{cases} c_{12} + g(2) = 7+19 = 26 \\ c_{13} + g(3) = 12+12 = 24 * \\ c_{14} + g(4) = 21+7 = 28 \\ c_{15} + g(5) = 31+0 = 31 \end{cases} = 24$$

Stage 0:

$$g(0) = min\begin{cases} c_{01} + g(1) = 7+24 = 31 * \\ c_{02} + g(2) = 12+19 = 31 * \\ c_{03} + g(3) = 21+12 = 33 \\ c_{04} + g(4) = 31+7 = 38 \\ c_{05} + g(5) = 44+0 = 44 \end{cases} = 31$$

Alternative solutions: 0-1-3-5, 0-2-3-5, 0-2-4-5

All solutions indicate to use one car for 1 year, and two cars for 2 years.

END332E QUIZ 5 (Afternoon Section) 22. 12. 2025

Consider an electronic system consisting of three components in series; all three must function for the entire system to work. The system's reliability can be improved by adding extra protection units to the components. The table below provides the probability of each component functioning based on the number of protection units (0, 1, or 2) assigned to it.

Protection Units	Probabilities that component will work		
	Component 1	Component 2	Component 3
0	0.6	0.7	0.4
1	0.7	0.75	0.6
2	0.8	0.8	0.9

A total of three protection units are available. Each component can be assigned a maximum of two units. Using dynamic programming, determine the optimal allocation of these three protection units across the components to maximize the overall probability of the system functioning.

Solution:

Stage (t): Component t (t = 1, 2, 3). **(1)**
State (b): Number of protection units available at the beginning of stage t. **(1)**

$$f_t(b) = \max_{x_t}\{0 \leq x \leq 2\}\{\, p_t(x) \cdot f_{t+1}(b - x_t)\,\}$$

$$f_3(b) = p_3(b), \quad b = 0, 1, 2$$

Stage 3: **(18)**

$$f_3(b) = \max_{x_3}\{0 \leq x \leq 2\}\{\, p_3(x_3) \cdot f_4(b - x_3)\,\}$$

$$b - x_3 \leq 2$$

$$f_3(0) = 0.4, \quad f_3(1) = 0.6, \quad f_3(2) = 0.9$$

Stage 2: **(40)**

$$f_2(b) = \max_{x_2}\{0 \leq x \leq 2\}\{\, p_2(x_2) \cdot f_3(b - x_2)\,\}$$

b = 0:

$$x_2 = 0 \;\Rightarrow\; f_2(0) = p_2(0) \cdot f_3(0) = 0.7 \cdot 0.4 = 0.28$$

$$Thus\colon\; f_2(0) = 0.28$$

b = 1:

$$x_2 = 0 \;\Rightarrow\; p_2(0) \cdot f_3(1) = 0.7 \cdot 0.6 = 0.42$$

$$x_2 = 1 \;\Rightarrow\; p_2(1) \cdot f_3(0) = 0.75 \cdot 0.4 = 0.30$$

$$Max = 0.42 \;(at\; x_2 = 0) \;\Rightarrow\; f_2(1) = 0.42$$

b = 2:

$x_2 = 0 \;\Rightarrow\; 0.7 \cdot f_3(2) = 0.7 \cdot 0.9 = 0.63$

$x_2 = 1 \;\Rightarrow\; 0.75 \cdot f_3(1) = 0.75 \cdot 0.6 = 0.45$

$x_2 = 2 \;\Rightarrow\; 0.8 \cdot f_3(0) = 0.8 \cdot 0.4 = 0.32$

$Max \;=\; 0.63 \;(at\; x_2 = 0) \;\Rightarrow\; f_2(2) = 0.63$

b = 3:

$x_2 = 0 \;\Rightarrow\; 0.7 \cdot f_3(2) = 0.7 \cdot 0.9 = 0.63$

$x_2 = 1 \;\Rightarrow\; 0.75 \cdot f_3(1) = 0.75 \cdot 0.6 = 0.45$

$Max \;=\; 0.63 \;(at\; x_2 = 0) \;\Rightarrow\; f_2(2) = 0.63$

Stage 1: (20)

$x_1 = 0 \;\Rightarrow\; p_1(0) \cdot f_2(3) = 0.6 \cdot 0.675 = 0.405$

$x_1 = 1 \;\Rightarrow\; p_1(1) \cdot f_2(2) = 0.7 \cdot 0.63 = 0.441$

$x_1 = 2 \;\Rightarrow\; p_1(2) \cdot f_2(1) = 0.8 \cdot 0.42 = 0.336$

$f_1(3) \;=\; max\,\{\,0.405, 0.441, 0.336\,\} \;=\; 0.441$

$(x_1, x_2, x_3) \;=\; (1, 0, 2)$

$R* = \; p_1(1) \cdot p_2(0) \cdot p_3(2) \;=\; 0.7 \cdot 0.7 \cdot 0.9 \;=\; 0.441$

Allocating 1 unit to component 1, 0 units to component 2, and 2 units to component 3 maximizes the series-system reliability, achieving R* = 0.441. **(20)**