SOLUTIONS TO QUIZ 3

Answer

Consider the following function.

$$f(x,y,z) = x^3 + y^3 + x^2 + 5y^2 + 2xy + z^2$$

a) Calculate Gradient vector and the Hessian Matrix.

Gradient vector: $\nabla f(x,y,z) = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}]$

$$\nabla f(x,y,z) = \begin{bmatrix} 3x^2 + 2x + 2y \\ 3y^2 + 10y + 2x \\ 2z \end{bmatrix}$$

Hessian Matrix: $H(x,y,z) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$

$$H(x,y,z) = \begin{bmatrix} 6x+2 & 2 & 0 \\ 2 & 6y+10 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

b) Determine whether the function is convex, concave, or indefinite over the entire $\mathbb{R}^3$ domain.

The first principal minors of **H** are $6x + 2,\ 6y + 10\ \ and\ 2 \rightarrow$ can be positive or negative depending on the value of x and y.

The second principal minors are $(6x + 2)(6y + 10) - (2)(2) = 36xy + 60x + 12y + 16$

$$(6x + 2)(2) - 0 = 12x + 4$$

$$(6y + 10)(2) - 0 = 12y + 20$$

The third principal minor is $\det(\mathrm{H}(x,y,z)) = 72xy + 120x + 24y + 32$

Because the first principal minors $6x + 2$ and $6y + 10$, are not guaranteed to be positive and not guaranteed to be negative for all possible values of x and y in the domain, the function is ***indefinite*** over the entire domain.

c) Determine whether the function is convex, concave, or indefinite for the region where $x > 0, y > 0\ and\ z > 0$.

The first principal minors of **H** are $6x + 2,\ 6y + 10\ \ and\ 2 > 0$ for $x > 0, y > 0$

The second principal minors are $36xy + 60x + 12y + 16 > 0$ for $x > 0, y > 0$

$12x + 4 > 0$ for $x > 0$

$12y + 20 > 0$ for $y > 0$

The third principal minor is $72xy + 120x + 24y + 32 > 0$ for $x > 0, y > 0$

As all principal minors of **H** are positive, $f(x,y,z)$ is a strict convex function for the region where $x > 0, y > 0\ and\ z > 0$.

Answer

Consider the single variable function $f(x) = x^2 - 6x + 15$.

We want to find the minimum of this function within the interval [0, 5] using the Golden Section Search method. Take the Golden Ratio constant r = 0.618.

a) How many iterations do we need to find the interval of uncertainty having a length less than 0.3?

$$r^k(b-a) < \epsilon$$

$$0.618^k(5-0) < 0.3$$

$$0.618^k < 0.06 \rightarrow k\ln(0.618) < \ln(0.06)$$

$$k(-0.4812) < (-2.8134) \rightarrow k > 5.846$$

Thus, ***6 iterations*** of Golden Section Search must be performed.

b) Perform Golden Section Search method for 2 iterations. Explicitly state the reduced interval [a, b] after each step. Calculate the function values at the test points (Note: Keep 3 decimal places in your calculations).

$x_1 = b - r(b-a) \rightarrow x_1 = 5 - 0.618(5-0) = 1.910$

$x_2 = a + r(b-a) \rightarrow x_2 = 0 + 0.618(5-0) = 3.090$

$f(x_1) = f(1.910) = 7.188, \;\; f(x_2) = f(3.090) = 6.008 \rightarrow f(x_1) > f(x_2)$

The minimum must lie to the right of x_1, the new interval is $(a_1, b] = (1.910, 5]$.

$x_2 = x_3 = 3.090$

$x_4 = 1.910 + 0.618(5 - 1.910) = 3.820$

$f(x_3) = f(3.090) = 6.008, \;\; f(x_4) = f(3.820) = 6.672 \rightarrow f(x_4) > f(x_3)$

We discard the interval $[x_4, b]$, which is [3.820,5].

Reduced Interval after Iteration 2: $[a_1, b_1) = [1.910, 3.820)$.

Alternative solution:

Consider the single variable function $f(x) = -x^2 + 6x - 15$.

the maximum of this function within the interval [0, 5]

$x_1 = b - r(b-a) \rightarrow x_1 = 5 - 0.618(5-0) = 1.910$

$x_2 = a + r(b-a) \rightarrow x_2 = 0 + 0.618(5-0) = 3.090$

$f(x_1) = f(1.910) = -7.910, \;\; f(x_2) = f(3.090) = -6.008 \rightarrow f(x_2) > f(x_1)$

The maximum must lie to the right of x_1.

Reduced Interval after Iteration 1: $(a_1, b] = (1.910, 5]$.

$x_2 = x_3 = 3.090$

$x_4 = 1.910 + 0.618(5 - 1.910) = 3.820$

$f(x_3) = f(3.090) = -6.008, \;\; f(x_4) = f(3.820) = -6.672 \rightarrow f(x_3) > f(x_4)$

We discard the interval $[x_4, b]$, which is [3.820,5].

Reduced Interval after Iteration 2: $[a_1, b_1) = [1.910, 3.820)$.