END332E OPERATIONS RESEARCH II
2024-2025 Fall
MIDTERM EXAM II
SOLUTIONS

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1. (25 points) Consider the following linear program.

$$\text{Min } 3x_1 + 2x_2 + x_3$$
$$\text{Subject to}$$
$$2x_1 + x_2 + x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

Apply affine scaling approach for one step starting from point (1, 1, 1) until finding a new point. Assume $\alpha = 0.8$.

Solution:

Standard Form:

Max $3x_1 + 2x_2 + x_3$
s.t.
$2x_1 + x_2 + x_3 - e_1 = 2$
$x_1, x_2, x_3, e_1 \geq 0$

Initial point $[x_1, x_2, x_3, e_1] = [1, 1, 1, 2]$

Find initial point and D - 4 points

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\tilde{x} = D^{-1}x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = [2 \quad 1 \quad 1 \quad -1]$$

$$\tilde{A} = AD = [2 \quad 1 \quad 1 \quad -1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = [2 \quad 1 \quad 1 \quad -2]$$

Find $\tilde{A}$ and $\tilde{c}$ - 3 points

$$c = [3 \quad 2 \quad 1 \quad 0]$$

$$\tilde{c} = cD = [3 \quad 2 \quad 1 \quad 0] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = [3 \quad 2 \quad 1 \quad 0]$$

$$\tilde{A}\tilde{A}^T = [2 \quad 1 \quad 1 \quad -1]\begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 10$$

$$\left(\tilde{A}\tilde{A}^T\right)^{-1} = 0.1$$

The projection matrix: $P = I - \tilde{A}^T\left(\tilde{A}\tilde{A}^T\right)^{-1}\tilde{A}$

$$P = \begin{bmatrix} 0.6 & -0.2 & -0.2 & 0.4 \\ -0.2 & 0.9 & -0.1 & 0.2 \\ -0.2 & -0.1 & 0.9 & 0.2 \\ -0.4 & 0.2 & 0.2 & 0.6 \end{bmatrix}$$

Find P - 5 points

Projected gradient: $c_p = -P\tilde{c}^T$ (for minimization)

$$c_p = -\begin{bmatrix} 0.6 & -0.2 & -0.2 & 0.4 \\ -0.2 & 0.9 & -0.1 & 0.2 \\ -0.2 & -0.1 & 0.9 & 0.2 \\ -0.4 & 0.2 & 0.2 & 0.6 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.2 \\ -1.1 \\ -0.1 \\ -1.8 \end{bmatrix}$$

Find Cp - 5 points

$$\tilde{x}_{new} = \tilde{x} + \frac{\alpha}{\mathcal{V}} c_p$$

$\mathcal{V}$ is the absolute value of the negative component of c_p having largest absolute value, $\mathcal{V} = 1.8$ and α is arbitrarily chosen as 0.8.

So:

$$\tilde{x}_{new} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{0.8}{1.8}\begin{bmatrix} -1.2 \\ -1.1 \\ -0.1 \\ -1.8 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.45 \\ 0.95 \\ 0.1 \end{bmatrix}$$

Find $\tilde{x}_{new}$ - 5 points

$$x_{new} = D\tilde{x}_{new}$$

$$x_{new} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0.4 \\ 0.45 \\ 0.95 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.45 \\ 0.95 \\ 0.2 \end{bmatrix}$$

Find x_{new} - 3 points

2. (30 points) Consider the following Non-Linear Program (NLP):

$$\begin{gathered} Min\ \ x^4 + x^2 - 2xy + y^2 \\ \text{Subject to} \\ 3x^2 + y \leq 5 \\ -x^2 + xy - 2y^2 \geq -8 \\ x + y \geq 3 \end{gathered}$$

a) Write down the KKT conditions for the given NLP.

$$\begin{gathered} Min\ \ x^4 + x^2 - 2xy + y^2 \\ \text{Subject to} \\ 3x^2 + y \leq 5 \rightarrow \lambda_1 \\ -x^2 + xy - 2y^2 \geq -8 \rightarrow \lambda_2 \\ x + y \geq 3 \rightarrow \lambda_3 \end{gathered}$$

Lagrangian function:

$$L(x, y, \lambda) = x^4 + x^2 - 2xy + y^2 + \lambda_1\big(5 - (3x^2 + y)\big) + \lambda_2\big(-8 - (-x^2 + xy - 2y^2)\big) + \lambda_3(3 - (x + y))$$

Gradient equations:

$$4x^3 + 2x - 2y - 8x\lambda_1 - \lambda_2(-2x + y) - \lambda_3 = 0$$

$$-2x + 2y - \lambda_1 - \lambda_2(x - 4y) - \lambda_3 = 0$$

Complementary slackness:

$$\lambda_1(5 - (3x^2 + y)) = 0$$

$$\lambda_2\big(-8 - (-x^2 + xy - 2y^2)\big) = 0$$

$$\lambda_3(3 - (x + y)) = 0$$

Sign restrictions:

$$\lambda_1 \leq 0$$

$$\lambda_2 \geq 0$$

$$\lambda_3 \geq 0$$

Write
Gradient equations: 4 points
Complementary slackness: 3 points
Sign restrictions: 3 points
Primal Constraints: 2 points

Primal Constraints:

$$\begin{gathered} 3x^2 + y \leq 5 \\ -x^2 + xy - 2y^2 \geq -8 \\ x + y \geq 3 \end{gathered}$$

b) Is there a KKT point where 1st and 3rd constraints are binding?

$3x^2 + y = 5$

$x + y = 3 \rightarrow y = 3 - x$

Substitute $y = 3 - x$ into the first constraint:

$3x^2 + 3 - x = 5 \rightarrow 3x^2 - x - 2 = 0$

$Using\ the\ quadratic\ formula:\ x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The two solutions are: $x_1 = 1\ and\ x_2 = -2/3$

Using $y = 3 - x$

$If\ x = 1 \rightarrow\ \ y = 3 - 1 = 2$

$If\ x = -\frac{2}{3} \rightarrow\ y = 3 - \left(-\frac{2}{3}\right) = \frac{11}{3}$

Find two points: 3 points
Analyze point (-2/3, 11/3): 2 points
Analyze point (1, 2): 3 points

Case 1: $x = -\frac{2}{3}, y = \frac{11}{3}$

The second constraint:

$-\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)\left(\frac{11}{3}\right) - 2\left(\frac{11}{3}\right)^2 = -268/9$

$-\frac{268}{9} \geq -8$ does not hold

Case 2: $x = 1, y = 2$

The second constraint:

$-x^2 + xy - 2y^2 \geq -6 \rightarrow\ -(1)^2 + (1)(2) - 2(2)^2 = -7$

$-7 \geq -8$ holds.

from the complementary slackness, $\lambda_2 = 0$

Put $x = 1, y = 2, \lambda_2 = 0$ in gradient equations:

$$4 + 2 - 4 - 8\lambda_1 - \lambda_3 = 0$$
$$-2 + 4 - \lambda_1 - \lambda_3 = 0$$

$\lambda_1 = 0, \lambda_3 = 2$ sign restrictions are satisfied.

So, the point (1,2) is a KKT point.

c) Suppose a KKT point is found for the problem, is this point can be a global optimum?

For the KKT point to be global optimum, the objective function must be convex for a minimization problem, and constraints must satisfy the required convexity conditions.

- Hessian of the objective function is as follows:

Find objective function convex: 2 points
Find first constraint convex: 2 points
Find second constraint concave: 2 points
Find third constraint linear (concave/convex): 2 points
Indicate NLP is convex program and KKT point is global min.: 2 points

$$H = \begin{bmatrix} 12x^2 + 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\Delta_{H_1} = 12x^2 + 2 \geq 0$$

$$\Delta_{H_2} = 24x^2 \geq 0$$

All principal minors of H are nonnegative, so objective function is a convex function.

- Hessian of the first constraint $3x^2 + y \leq 5$ is:

$$H = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Delta_{H_1} = 6 \geq 0, \Delta_{H_2} = 0 \geq 0$$

All principal minors of H are nonnegative, so the first constraint is a convex function. As it is $\leq$ function, it should be convex function.

- Hessian of the the second constraint $-x^2 + xy - 2y^2 \geq -8$ is:

$$H = \begin{bmatrix} -2 & 1 \\ 1 & -4 \end{bmatrix}$$

$$\Delta_{H_1} = -2 \leq 0, \Delta_{H_2} = 7 \geq 0$$

The second constraint is a concave function. As it is $\geq$ function, it should be convex function.

The last constraint is linear, so it also satisfies the conditions for convex program.

Therefore, as the given NLP is a convex program, the KKT point is the global minimum and the optimal solution.

3. (28 points) There are three routes from the city center to the village Mountyard. Since the routes pass through a mountainous area, there is a risk of avalanches and landslides. Currently, the probabilities of each route being open on any given day during the winter season is as follows: **0.5** for Route-1, **0.6** for Route-2, and **0.4** for Route-3.

The local government aims to enhance the probability of reaching Mountyard from the city center by investing in these routes. This probability is determined by at least one of the routes being open.

To achieve this objective:

- Investing **2 million TL** in Route-1 increases its probability of being open to **0.7**, while investing **5 million TL** raises it to **0.9**.
- Investing **3 million TL** in Route-2 increases its probability of being open to **0.75**, while investing **5 million TL** raises it to **0.85**.
- For Route-3, an investment of **3 million TL** increases the probability to **0.5**, **5 million TL** raises it to **0.7**, and **6 million TL** boosts it to **0.9**.

Given a total budget of **10 million TL**, determine the optimal investment plan that maximizes the probability of reaching Mountyard from the city center using dynamic programming (solutions with other methods will not be accepted).

Define stages, states and recursion formula and apply the dynamic programming algorithm to solve the problem.

Solution:

Notice that the probability of reaching Mountyard from the city center by is determined by at least one of the routes being open. It is equivalent to minimum probability of all routes are closed.

Let $P_t(x_t)$ is probability of route t is closed when x_t million TL is invested, these probabilities are calculated as follows:

$P_1(0) = 0.5$	$P_2(0) = 0.4$	$P_3(0) = 0.6$
$P_1(2) = 0.3$	$P_2(3) = 0.25$	$P_3(3) = 0.5$
$P_1(5) = 0.1$	$P_2(5) = 0.15$	$P_3(5) = 0.3$
		$P_3(6) = 0.1$

Find probabilities: 5 points

Define stages, states and recursion formula: 5 points

Stage: routes $t = 1,2,3$

State (i): amount of money left to routes t, t+1,…t+3.

Decision (x_t): amount of money invested to route t.

$f_t(i)$: Minimum probability of NOT reaching Mountyard with budget i from routes t,t+1,…3.

$$f_t(i) = \min_{0 \le x_t \le i} \{P_t(x_t) \times f_{t+1}(i - x_t)\}, \qquad t = 1, 2$$

$$f_3(i) = P_3(i)$$

Apply dynamic programming:
Stage 3: 4 points
Stage 2: 5 points
Stage 1: 4 points

Stage 3:

$$f_3(i) = 0.6, \quad i = 0,1,2 \qquad x_3(i) = 0$$
$$f_3(i) = 0.5, \quad i = 3,4 \qquad x_3(i) = 3$$
$$f_3(i) = 0.3, \quad i = 5 \qquad x_3(i) = 5$$
$$f_3(i) = 0.1, \quad i \geq 6 \qquad x_3(i) = 6$$

Stage 2:

$$f_2(10) = min \begin{cases} x_2 = 0 \to P_2(0) \times f_3(10-0) = 0.4 \times 0.1 = 0.04 \\ x_2 = 3 \to P_2(3) \times f_3(10-3) = 0.25 \times 0.1 = 0.025 \\ x_2 = 5 \to P_2(5) \times f_3(10-5) = 0.15 \times 0.3 = 0.045 \end{cases} = 0.025 \quad x_2(10) = 3$$

$$f_2(8) = min \begin{cases} x_2 = 0 \to P_2(0) \times f_3(8-0) = 0.4 \times 0.1 = 0.04 \\ x_2 = 3 \to P_2(3) \times f_3(8-3) = 0.25 \times 0.3 = 0.075 \\ x_2 = 5 \to P_2(5) \times f_3(8-5) = 0.15 \times 0.5 = 0.075 \end{cases} = 0.04 \quad x_2(8) = 0$$

$$f_2(5) = min \begin{cases} x_2 = 0 \to P_2(0) \times f_3(5-0) = 0.4 \times 0.3 = 0.12 \\ x_2 = 3 \to P_2(3) \times f_3(5-3) = 0.25 \times 0.6 = 0.15 \\ x_2 = 5 \to P_2(5) \times f_3(5-5) = 0.25 \times 0.6 = 0.15 \end{cases} = 0.12 \quad x_2(5) = 0$$

Stage 1:

$$f_1(10) = min \begin{cases} x_1 = 0 \to P_2(0) \times f_2(10-0) = 0.5 \times 0.025 = 0.125 \\ x_1 = 2 \to P_1(2) \times f_2(10-2) = 0.3 \times 0.04 = 0.012 \\ x_1 = 5 \to P_1(5) \times f_2(10-5) = 0.1 \times 0.12 = 0.012 \end{cases} = 0.012 \quad \begin{matrix} x_1(10) = 2 \\ or = 5 \end{matrix}$$

The optimal solution:

Two alternative solutions: $x_1 = 2, x_2 = 0, x_3 = 6 \quad or\ x_1 = 5, x_2 = 0, x_3 = 5$

Probability of all routes are closed is 0.012

Probability of at least one of the routes being open is 1- 0.012 = 0.988

Find optimal solution
Write two alternative solutions: 3 points
Find probabilities: 2 points

4. (27 points) In a textile plant, cloth is manufactured in rolls of length L. Defects sometimes occur along the length of the cloth. Consider a specific roll with $(N-1)$ defects appearing at distances $y_1, y_2 \ldots, y_{N-1}$ from the start of the roll ($y_{i+1} > y_i$ for all i). Denote the start of the roll by y_0, the end by y_N.

   The roll is cut into pieces for sale. The value of a piece depends on its length and the number of defects.

   Let $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{m})$, the value of a piece of length $\boldsymbol{x}$ having $\boldsymbol{m}$ defects.

   Assume that all cuts are made through defects and that such cutting removes the defect.

   Formulate a dynamic programming recursion to specify how to determine where to cut the cloth to maximize total value. Define stages, states, decision, recursion formula, and solution procedure.

Stages (i): (Stages and states 5 points)

Defects and edges (Possible cutting points and edges)

$i = 0, 1, \ldots, N-1, N$. $i = 0$ start of the roll, $i = N$ end of the roll.

State: No state is defined for this problem. Stages will show the points to cut.

Decision (j): (4 points)

j is the next cutting point if the cloth cut at defect i.

Recursion formula definition: (3 points)

f_i: maximum total value of cloth from defect i to end of roll, if the cloth is cut at defect i.

Recursion formula: (11 points)

For i = 0,1,…, N-1;

$$f_i = \max_{i<j\le N}\{\boldsymbol{v}(y_j - y_i, j-i-1) + f_j\}$$

Solution procedure: (4 points)

Initialization: $f_N = 0$

Termination: Find f_0 =?

ALTERNATIVE SOLUTION:

Stages: i =0, 1, ..., $N-1, N$. $i=0$ start of the roll, $N=$ end of the roll.	
State: (i, j) the roll segment being considered is $[y_i, y_j]$, and it contains $j-i-1$ defects. i is the starting defect index of the current segment. j is the ending defect index of the segment.	
Decision: The decision is whether or not to make a cut at defect y_k for $i<k<j$ which effectively partitions the segment $[y_i, y_j]$ into two smaller segments.	
Recursion formula definition: $f(i, j)$ represent the maximum total value that can be obtained for the segment of cloth between defects i and j.	
Recursion formula: The DP recursion can be defined as: $$f(i,j)=\begin{cases} v(x_{ij}, m_{ij}), & if\ j=i+1\ (no\ interior\ defects) \\ \max\limits_{i<k<j}\left(f(i,k)+f(k,j)\right), & \forall k\ , otherwise \end{cases}$$ $x_{ij}=y_j-y_i$ is the length of the segment. $m_{ij}=j-i-1$ is the number of defects in the segment $[y_j, y_i]$. For segments with no interior defects, the value is directly given by $v(x_{ij}, 0)$: $$f(i, i+1)=\ v(y_{i+1}-y_i, 0)$$	
Solution procedure: Initialization: Compute $f(i, i+1)=v(x_{i,i+1},\ 0)$ for all i. Termination: The optimal value is $f(0, N)$, which represents the maximum value for entire roll.	