# Analyzing the Problem (SAW, WP, TOPSIS)

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# MADM Methods to be covered

- Elementary Methods
- Value Based Methods
  - Multi Attribute Value Theory
  - Simple Additive Weighting
  - Weighted Product
  - TOPSIS
- The Analytic Hierarchy Process (AHP)
- The Analytic Network Process (ANP)
- Outranking Methods
  - PROMETHEE methods
  - ELECTRE methods

# SAW – WP – TOPSIS

- *Ranking methods: Complete preorder*
- *Performance aggregation oriented*
- *Compensatory*

# SAW

Simple Additive Weighting

Weighted Average

Weighted Sum

- In the SAW, a global (total) score of an alternative is computed by adding the linear normalized performance values of the alternative w.r.t. each attribute
- Normalization is required to permit addition among attribute values.
- Value (global score) of an alternative can be expressed as:

$$V(a_i) = V_i = \sum_{j=1}^{n} w_j r_{ij}$$

# Example for SAW

Normalized (Linear) Decision Matrix and Global Scores

| | Price | Comfort | Perf. | Design | $V_i$ |
|---|---|---|---|---|---|
| *Norm. w* | 0.3333 | 0.2667 | 0.2 | 0.2 | |
| $a_1$ | 0.3333 | 1 | 1 | 1 | **.7778** |
| $a_2$ | 0.4 | 1 | 0.6667 | 1 | .7334 |
| $a_3$ | 0.4 | 0.6667 | 1 | 1 | .7111 |
| $a_4$ | 0.5 | 0.6667 | 1 | 0.6667 | .6778 |
| $a_5$ | 0.5 | 0.6667 | 0.6667 | 1 | .6778 |
| $a_6$ | 0.5 | 0.3333 | 1 | 1 | .6555 |
| $a_7$ | 1 | 0.3333 | 0.6667 | 0.6667 | .6889 |

a1 is recommended

# Sensitivity Analysis

- "Conducting a sensitivity analysis on attribute *i*"
- When weight of that attribute is changed, the weights of other attributes are also changed:

$$w_j = (1 - w_i)\ (w_j^0 / (1 - w_i^0)),$$

where $w^0$ is the base weight (current weight used at the evaluation)

| w1 | w2 | w3 | Total |
|---|---|---|---|
| 50% | 40% | 10% | 100% |
| 30% | 56% | 14% | 100% |
| 80% | 16% | 4% | 100% |

# WP

Weighted Product

- In the WP, a global score of an alternative is computed by multiplying the values obtained by taking the exponents (representing the attribute weights) of the performance values of the alternative w.r.t. each attribute
- Normalization is not necessary!
- Weights become exponents;
  - a positive power for benefit attributes
  - a negative power for cost attributes
- Value (global score) of an alternative can be expressed as:

$$V_i = \prod_j (x_{ij})^{w_j}$$

# Example for WP

Decision Matrix and Global Scores

| | Price | Comfort | Perf. | Design | $V_i$ |
|---|---|---|---|---|---|
| *Norm. w* | .3333 | .2667 | .2 | .2 | |
| $a_1$ | 300 | 3 | 3 | 3 | **.3108** |
| $a_2$ | 250 | 3 | 2 | 3 | .3045 |
| $a_3$ | 250 | 2 | 3 | 3 | .2964 |
| $a_4$ | 200 | 2 | 3 | 2 | .2944 |
| $a_5$ | 200 | 2 | 2 | 3 | .2944 |
| $a_6$ | 200 | 1 | 3 | 3 | .2654 |
| $a_7$ | 100 | 1 | 2 | 2 | .2843 |

a1 is recommended

# TOPSIS

Technique for Order of Preference by Similarity to Ideal Solution

Chosen alternative should have the shortest distance from the positive ideal solution and the longest distance from the negative ideal solution

- Steps:
  - Calculate normalized ratings
  - Calculate weighted normalized ratings
  - Identify positive-ideal and negative-ideal solutions
  - Calculate separation measures
  - Calculate similarities to positive-ideal solution
  - Rank preference order

# Steps

- Calculate normalized ratings
  - Vector normalization (Euclidean distance) is used
  - Do not take the inverse rating for cost attributes!
- Calculate weighted normalized ratings
  - $v_{ij} = w_j * r_{ij}$
- Identify positive-ideal and negative-ideal solutions

$$a^* = \{v_1^*, v_2^*, \ldots, v_j^*, \ldots, v_n^*\} = \left\{ \left( \max_i v_{ij} \middle| j \in J_1 \right), \left( \min_i v_{ij} \middle| j \in J_2 \right) \middle| i = 1, \ldots, m \right\}$$

$$a^- = \{v_1^-, v_2^-, \ldots, v_j^-, \ldots, v_n^-\} = \left\{ \left( \min_i v_{ij} \middle| j \in J_1 \right), \left( \max_i v_{ij} \middle| j \in J_2 \right) \middle| i = 1, \ldots, m \right\}$$

where $J_1$ is a set of benefit attributes

and $J_2$ is a set of cost attributes

# Steps

- Calculate separation measures
  - Euclidean distance (separation) of each alternative from the ideal solutions are measured:

$$S_i^* = \sqrt{\sum_j (v_{ij} - v_j^*)^2} \qquad S_i^- = \sqrt{\sum_j (v_{ij} - v_j^-)^2}$$

- Calculate similarities to positive-ideal solution

$$C_i^* = S_i^- / (S_i^* + S_i^-)$$

- Rank preference order
  - Rank the alternatives according to similarities in descending order.
  - Recommend the alternative with the maximum similarity

# Example for TOPSIS

- Normalized (Vector) Decision Matrix

| | Price | Comfort | Perf. | Design |
|---|---|---|---|---|
| *Norm. w* | 0.3333 | 0.2667 | 0.2 | 0.2 |
| $a_1$ | 0.5108 | 0.5303 | 0.433 | 0.4121 |
| $a_2$ | 0.4256 | 0.5303 | 0.2887 | 0.4121 |
| $a_3$ | 0.4256 | 0.3536 | 0.433 | 0.4121 |
| $a_4$ | 0.3405 | 0.3536 | 0.433 | 0.2747 |
| $a_5$ | 0.3405 | 0.3536 | 0.2887 | 0.4121 |
| $a_6$ | 0.3405 | 0.1768 | 0.433 | 0.4121 |
| $a_7$ | 0.1703 | 0.1768 | 0.2887 | 0.2747 |

No inverse rating!

# Example for TOPSIS

- Weighted Normalized Decision Matrix

| | Price | Comfort | Perf. | Design |
|---|---|---|---|---|
| $a_1$ | 0.1703 | 0.1414 | 0.0866 | 0.0824 |
| $a_2$ | 0.1419 | 0.1414 | 0.0577 | 0.0824 |
| $a_3$ | 0.1419 | 0.0943 | 0.0866 | 0.0824 |
| $a_4$ | 0.1135 | 0.0943 | 0.0866 | 0.0549 |
| $a_5$ | 0.1135 | 0.0943 | 0.0577 | 0.0824 |
| $a_6$ | 0.1135 | 0.0471 | 0.0866 | 0.0824 |
| $a_7$ | 0.0568 | 0.0471 | 0.0577 | 0.0549 |

# Example for TOPSIS

- Positive-Ideal and Negative-Ideal Solutions

| | Price | Comfort | Perf. | Design |
|---|---|---|---|---|
| $a_1$ | 0.1703 | 0.1414 | 0.0866 | 0.0824 |
| $a_2$ | 0.1419 | 0.1414 | 0.0577 | 0.0824 |
| $a_3$ | 0.1419 | 0.0943 | 0.0866 | 0.0824 |
| $a_4$ | 0.1135 | 0.0943 | 0.0866 | 0.0549 |
| $a_5$ | 0.1135 | 0.0943 | 0.0577 | 0.0824 |
| $a_6$ | 0.1135 | 0.0471 | 0.0866 | 0.0824 |
| $a_7$ | 0.0568 | 0.0471 | 0.0577 | 0.0549 |
| $a^*$ | .0568 | .1414 | .0866 | .0824 |
| $a^-$ | .1703 | .0471 | .0577 | .0549 |

# Example for TOPSIS

- Separation from Positive-Ideal

$$S_i^* = \sqrt{\sum_j (v_{ij} - v_j^*)^2}$$

| | Price | Comfort | Perf. | Design | | |
|---|---|---|---|---|---|---|
| $a_1$ | 0.1703 | 0.1414 | 0.0866 | 0.0824 | | |
| $a^*$ | 0.0568 | 0.1414 | 0.0866 | 0.0824 | | |
| $(a_1 - a^*)^2$ | 0.0129 | 0 | 0 | 0 | SUM | 0.0129 |
| | | | | | SQRT | 0.1135 |

# Example for TOPSIS

- Separation from Negative-Ideal

$$S_i^- = \sqrt{\sum_j (v_{ij} - v_j^-)^2}$$

| | Price | Comfort | Perf. | Design | | |
|---|---|---|---|---|---|---|
| $a_1$ | 0.1703 | 0.1414 | 0.0866 | 0.0824 | | |
| $a^-$ | 0.1703 | 0.0471 | 0.0577 | 0.0549 | | |
| $(a_1 - a^-)^2$ | 0.0000 | 0.00889 | 0.00084 | 0.00076 | SUM | 0.0105 |
| | | | | | SQRT | 0.1024 |

# Example for TOPSIS

- Separations from Positive Ideal and Negative Ideal Solutions

| | $S^*$ | $S^-$ | | |
|---|---|---|---|---|
| $a_1$ | 0.1135 | 0.1024 | | |
| $a_2$ | 0.0899 | 0.1022 | | |
| $a_3$ | 0.0973 | 0.0679 | | |
| $a_4$ | 0.0787 | 0.0792 | | |
| $a_5$ | 0.0792 | 0.0787 | | |
| $a_6$ | 0.11 | 0.0693 | | |
| $a_7$ | 0.1024 | 0.1135 | | |

# Example for TOPSIS

- Similarities to Positive Ideal Solution

| | $S^*$ | $S^-$ | $C^*$ | |
|---|---|---|---|---|
| $a_1$ | 0.1135 | 0.1024 | 0.4742 | |
| $a_2$ | 0.0899 | 0.1022 | 0.5321 | |
| $a_3$ | 0.0973 | 0.0679 | 0.4111 | |
| $a_4$ | 0.0787 | 0.0792 | 0.5016 | |
| $a_5$ | 0.0792 | 0.0787 | 0.4984 | |
| $a_6$ | 0.11 | 0.0693 | 0.3866 | |
| $a_7$ | 0.1024 | 0.1135 | 0.5258 | |

$$C_i^* = S_i^- /(S_i^* + S_i^-)$$

# Example for TOPSIS

- Separations, Similarities, and Ranking

| | $S^*$ | $S^-$ | $C^*$ | Rank |
|---|---|---|---|---|
| $a_1$ | 0.1135 | 0.1024 | 0.4742 | 5 |
| **$a_2$** | 0.0899 | 0.1022 | **0.5321** | 1 |
| $a_3$ | 0.0973 | 0.0679 | 0.4111 | 6 |
| $a_4$ | 0.0787 | 0.0792 | 0.5016 | 3 |
| $a_5$ | 0.0792 | 0.0787 | 0.4984 | 4 |
| $a_6$ | 0.11 | 0.0693 | 0.3866 | 7 |
| $a_7$ | 0.1024 | 0.1135 | 0.5258 | 2 |

a2 is recommended

# Additional Example I
# Fighter Aircraft Selection

| Attributes | w | Alternatives A1 | A2 | A3 | A4 | A5 | ideal |
|---|---|---|---|---|---|---|---|
| Performance | | | | | | | |
| Maximum speed (mach) | 20% | 2 | 2 | 2.5 | 2 | 1.8 | 2.5 |
| Service ceiling (1000ft) | 4% | 60 | 50 | 60 | 50 | 50 | 60 |
| Maximum payload (Lbs) | 4% | 23 | 20 | 18 | 20 | 21 | 23 |
| Ferry range (NM) | 12% | 1900 | 2000 | 3500 | 2400 | 2300 | 3500 |
| Usability | | | | | | | |
| Maneuverability* | 9% | 7 | 8 | 8 | 9 | 9 | 9 |
| Safety* | 21% | 8 | 9 | 7 | 8 | 8 | 9 |
| Service | | | | | | | |
| Reliability* | 12% | 8 | 7 | 9 | 8 | 8 | 9 |
| Maintainability* | 8% | 9 | 7 | 8 | 7 | 7 | 9 |
| Cost | | | | | | | |
| Acquisition cost ($M) | 6% | 4.5 | 5 | 6.5 | 5.5 | 5 | 4.5 |
| Operation cost ($1000/year) | 4% | 90 | 90 | 100 | 80 | 70 | 70 |

*(1: very bad; 10: very good)

# Fighter Aircraft Selection

| Normalized values | | Alternatives | | | | | |
|---|---|---|---|---|---|---|---|
| Attributes | w | A1 | A2 | A3 | A4 | A5 | ideal |
| Performance | | | | | | | |
| Maximum speed (mach) | 20% | 0.80 | 0.80 | 1.00 | 0.80 | 0.72 | 1 |
| Service ceiling (1000ft) | 4% | 1.00 | 0.83 | 1.00 | 0.83 | 0.83 | 1 |
| Maximum payload (Lbs) | 4% | 1.00 | 0.87 | 0.78 | 0.87 | 0.91 | 1 |
| Ferry range (NM) | 12% | 0.54 | 0.57 | 1.00 | 0.69 | 0.66 | 1 |
| Usability | | | | | | | |
| Maneuverability* | 9% | 0.78 | 0.89 | 0.89 | 1.00 | 1.00 | 1 |
| Safety* | 21% | 0.89 | 1.00 | 0.78 | 0.89 | 0.89 | 1 |
| Service | | | | | | | |
| Reliability* | 12% | 0.89 | 0.78 | 1.00 | 0.89 | 0.89 | 1 |
| Maintainability* | 8% | 1.00 | 0.78 | 0.89 | 0.78 | 0.78 | 1 |
| Cost | | | | | | | |
| Acquisition cost ($M) | 6% | 1.00 | 0.90 | 0.69 | 0.82 | 0.90 | 1 |
| Operation cost ($1000/year) | 4% | 0.78 | 0.78 | 0.70 | 0.88 | 1.00 | 1 |

# Fighter Aircraft Selection

SAW

| | A1 | A2 | A3 | A4 | A5 | ideal |
|---|---|---|---|---|---|---|
| Vi | 0.8396 | 0.8274 | **0.8953** | 0.8400 | 0.8323 | 1 |

WP

| | A1 | A2 | A3 | A4 | A5 | ideal |
|---|---|---|---|---|---|---|
| Vi | 8.1716 | 8.0707 | 8.7689 | 8.2565 | 8.1485 | 9.8763 |
| r | 0.8274 | 0.8172 | **0.8879** | 0.8360 | 0.8251 | 1 |

# Additional Example II
# Scholar Selection

| Candidates | GRE | GPA | Interview* | Recom.* | SoP* |
|---|---|---|---|---|---|
| | 30% | 20% | 20% | 15% | 15% |
| Aynur | 690 | 3.1 | 9 | 7 | 4 |
| Batuhan | 590 | 3.9 | 7 | 6 | 10 |
| Canan | 600 | 3.6 | 8 | 8 | 7 |
| Deniz | 620 | 3.8 | 7 | 10 | 6 |
| Eda | 700 | 2.8 | 10 | 4 | 6 |
| Fatih | 650 | 4 | 6 | 9 | 8 |
| Ideal | 700 | 4 | 10 | 10 | 10 |

*(1: very bad; 10: very good); SoP: Statement of Purpose

# Scholar Selection

TOPSIS

| $r_{ij}$ | GRE | GPA | Interview | Recom. | SoP |
|---|---|---|---|---|---|
| Aynur | 0.4381 | 0.3555 | 0.4623 | 0.3763 | 0.2306 |
| Batuhan | 0.3746 | 0.4472 | 0.3596 | 0.3226 | 0.5764 |
| Canan | 0.3809 | 0.4128 | 0.4109 | 0.4301 | 0.4035 |
| Deniz | 0.3936 | 0.4357 | 0.3596 | 0.5376 | 0.3458 |
| Eda | 0.4444 | 0.3211 | 0.5137 | 0.2150 | 0.3458 |
| Fatih | 0.4127 | 0.4587 | 0.3082 | 0.4838 | 0.4611 |

| $v_{ij}$ | GRE | GPA | Interview | Recom. | SoP |
|---|---|---|---|---|---|
| Aynur | 0.1314 | 0.0711 | 0.0925 | 0.0564 | 0.0346 |
| Batuhan | 0.1124 | 0.0894 | 0.0719 | 0.0484 | 0.0865 |
| Canan | 0.1143 | 0.0826 | 0.0822 | 0.0645 | 0.0605 |
| Deniz | 0.1181 | 0.0871 | 0.0719 | 0.0806 | 0.0519 |
| Eda | 0.1333 | 0.0642 | 0.1027 | 0.0323 | 0.0519 |
| Fatih | 0.1238 | 0.0917 | 0.0616 | 0.0726 | 0.0692 |
| +ve ideal | 0.1333 | 0.0917 | 0.1027 | 0.0806 | 0.0865 |
| -ve ideal | 0.1124 | 0.0642 | 0.0616 | 0.0323 | 0.0346 |

# Scholar Selection

TOPSIS

| | $S^*$ | $S^-$ | $C^*$ | rank |
|---|---|---|---|---|
| Aynur | 0.0617 | 0.0441 | 0.4167 | 6 |
| Batuhan | 0.0493 | 0.0608 | 0.5519 | 2 |
| Canan | **0.0424** | 0.0498 | 0.5396 | 4 |
| Deniz | 0.0490 | 0.0575 | 0.5399 | 3 |
| Eda | 0.0655 | 0.0493 | 0.4291 | 5 |
| Fatih | 0.0463 | **0.0609** | **0.5681** | 1 |