EXACT SOLUTION OF CIRCULAR VERTICAL CURVES

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ABSTRACT: In the vertical geometry of the routes, circular curves or 2nd degree parabola are used for joining two straight lines. In practice, the approximate solutions are used with the aim of simplifying the calculation of heights and kilometres of circular vertical curves. Nowadays, approximate solutions are not necessary due to the recent developments in computer technology. In addition, for the railroads and high standard roads the level of calculation precision should be in millimetres. For these reasons, the exact solutions are more suitable instead of approximate solutions. In this paper, equations of exact solution of vertical curves are evaluated and the solutions are explained. Differences of project heights and chainage (kilometres) obtained from approximate and exact solutions have been shown on a sample route.

Key words: Vertical curve, roads, high standard, exact solution, survey.

INTRODUCTION

The vertical geometry of the routes consists of straight lines and curves place between those two lines. In practice, circular curves or 2nd degree parabola are used as vertical curve (Colcord 1962, Muller 1984).

A transportation practice project covers all properties of the structure. One of the most important part named "vertical geometry of the route" represents real geometry (with no errors) and is expressed by kilometres, elevations and gradients digitally. Those digital values can be divided into the tree parts by means of error levels:

Group 1- The values chosen by designer with pre-defined criteria (usually, radiuses of vertical circular curves or the lengths of parabolas in horizontal plane, entry and exit gradients of back and forward tangents, kilometres of essential points of vertical curve (EPVC) which consist of beginning of vertical curve (BVC), end of vertical curve (EVC) and point of vertical intersection (PVI).

Group 2- The values measured graphically

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Group 3- The values determined by calculation.

The values in the second group are chosen as rounded numbers in order to simplify the computations. The numbers of the digits of the values in the second group are determined based on the scale of the longitudinal section. In most cases, the scale of the longitudinal section is 1:1000 thus the chainage values of the tangent points are obtained within the precision of 0.25 to 0.5 metres. The values in the group one are free of error while the second group values consisting small errors. In the third group, since the values in the third group are the products of a computation they have more errors according to the other two groups.

The vertical geometry of a design is presented with chainage (kilometres) and designed heights. The current demands of high precision in railway and motorway designs bring the necessity of computations which are carried out in a way to reduce the errors introduced by the calculation of chainages and designed heights due to the assumptions made during this process.

In most textbooks the computation of vertical geometry is being thought without taking the errors introduced by omissions and assumptions into account. These errors are mainly introduced by not carrying out the computations using appropriate degrees of a formula which is in a series form or assuming that slope distances can be used as plane distances (Umar and Yayla 1994; Müller 1984). The main reason of these assumptions is due to the limitations of the computations before 70s. Nowadays these limitations are overcome by means of new computation techniques and instruments. And the computations can even be executed easily by a hand calculator.

In this study, the results of the computations of circular and parabolic vertical curves are described with and without having assumptions. Also the effects of the assumptions on the results are presented.

EXACT COMPUTATIONS OF VERTICAL CURVES

In this section, the results of exact computations of circular and parabolic vertical curves are described. The basic concepts of the subject should be summarized prior to calculations.

- The vertical geometry is designed by using the longitudinal section of horizontal geometry (original surface). All vertical geometry related calculations are to be carried out on a vertical plane defined by K, H perpendicular coordinate system (see Figures 1 and 2).

- K axis shows the chainages. The points located on the same vertical line will naturally have the same chainages. All distances used in the calculations must be in horizontal plane.

- H shows the point heights. All distances used in the calculations must be in vertical plane.

- All computations are carried out in stages.

The amounts and the quality of the input values obtained for 1st, 2nd and 3rd group values should be good enough to achieve a unique solution for the calculation stages of a vertical geometry. If the amount and the quality of the values are not suitable, the calculations cannot be done. In case of having the amount of the initial data more than required number, the obtained results are different depending on the calculation method used. The contradictions to the real value concept as a result of these computations must be prevented.

Exact computations of Circular vertical curves

Where BVC, EVC and PVI are denoted as TO_n, TF_n and TF_n respectively (Figure 1). Frequently used initial data in computation of vertical curves are;

First group: entry and exit g_1 and g_2 gradients of tangents $\overline{S_{n-1}, S_n}$ and $\overline{S_n, S_{n+1}}$, radius R of circular curve, K_j kilometres of EPVC (these initial values are chosen by designer depending on some criterion).

Second group: Kilometres K_{S_n} $K_{S_{n+1}}$ of the main points S_n, S_{n+1} (these values are obtained by measuring on the profile.)

Third group: kilometre $K_{TF_{n-1}}$ and height $H_{TF_{n-1}}$ of EVC in previous group, height H_{S_n} of PVI (these values are taken into computation using the computation of previous vertical curve).

x, y perpendicular coordinate system is obtained if the origin of perpendicular axis of K and H is being moved to TO_n (Figure 1). The positive direction of x-axis always shows increasing direction of kilometres, the positive direction of y-axis also shows increasing direction of heights whatever the geometry of circular vertical curve is (sag or crest curve).

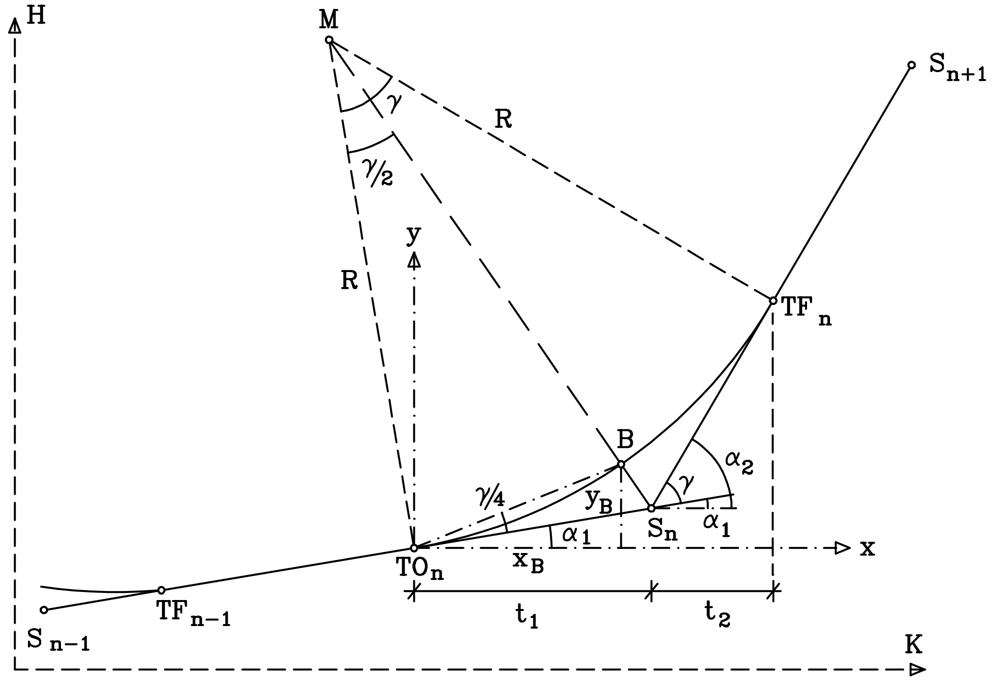

Figure 1 Circular Vertical Curve

Vertical curve equation in (x , y) perpendicular coordinate system is given by

$$(x - x_m)^2 + (y - y_m)^2 = R^2 \quad (1)$$

where x_m and y_m are centre coordinates (M) of circular vertical curve. Circular vertical curve must satisfy

$$y = y_{TO_n} = 0 \;\; for \;\; x = x_{TO_n} = 0 \quad (2)$$

$$y' = g_1 = 0 \;\; for \;\; x = x_{TO_n} = 0 \;\; and \;\; y = y_{TO_n} \quad \text{(gradient)} \quad (3)$$

$$y = y_{TF_n} = t_1 g_1 + t_2 g_2 \;\; for \;\; x = x_{TF_n} = t_1 + t_2 \quad (4)$$

$$y' = g_2 \;\; for \;\; x = x_{TF_n} = t_1 + t_2 \;\; , \;\; y = y_{TF_n} = t_1 g_1 + t_2 g_2 \quad (5)$$

Derivation of equation (1) is

$$(x - x_m) + (y - y_m)y' = 0 \tag{6}$$

When equations (2) and (3) are replaced in equations (1) and (6), respectively

$$x_m = \pm \frac{-g_1 R}{\sqrt{1+g_1^2}} \quad , \quad y_m = \frac{R}{\sqrt{1+g_1^2}} \quad \begin{Bmatrix} + \ \ for\,sag\,(open)\,curve \\ - \ \ for\ crest\,(closed)\,curve \end{Bmatrix} \tag{7}$$

and finally the equations of circular vertical curve

in sag curves $$y = -\sqrt{R^2 - (x + \frac{g_1 R}{\sqrt{1+g_1^2}})^2} + \frac{R}{\sqrt{1+g_1^2}} \tag{8}$$

in crest curves $$y = +\sqrt{R^2 - (x - \frac{g_1 R}{\sqrt{1+g_1^2}})^2} - \frac{R}{\sqrt{1+g_1^2}} \tag{9}$$

are obtained.

Length of curves t_1 and t_2 are

$$t_1 = R \tan\frac{\gamma}{2}\cos\alpha_1 \quad , \quad t_2 = R \tan\frac{\gamma}{2}\cos\alpha_2 \tag{10}$$

where

$$\alpha_1 = \arctan g_1 \quad , \quad \alpha_2 = \arctan g_2 \quad , \quad \gamma = |\alpha_1 - \alpha_2| \tag{11}$$

Middle point of circular vertical curve B

$$x_B = 2R\sin\frac{\gamma}{4}\cos(\alpha_1 \pm \frac{\gamma}{4}) \quad \begin{Bmatrix} + \ \ for\,sag\ curve \\ - \ \ forcrest\ curve \end{Bmatrix} \tag{12}$$

and the extreme points (largest height in crest curves, lowest height in sag curves) are computed by equation (12) if the first derivations of equation (8) and (9) equal to zero.

$$x_E = \pm \frac{-g_1 R}{\sqrt{1+g_1^2}} \quad \begin{Bmatrix} + \ \ for\,sag\ curve \\ - \ \ forcrest\ curve \end{Bmatrix} \tag{13}$$

y_B obtained using equations (8) and (9).

Exact computations of parabolic vertical curves

Initial values of parabolic vertical curves are the same with the initial values of circular vertical curves. However the length of curve L is chosen or computed for parabolic vertical curve instead of R.

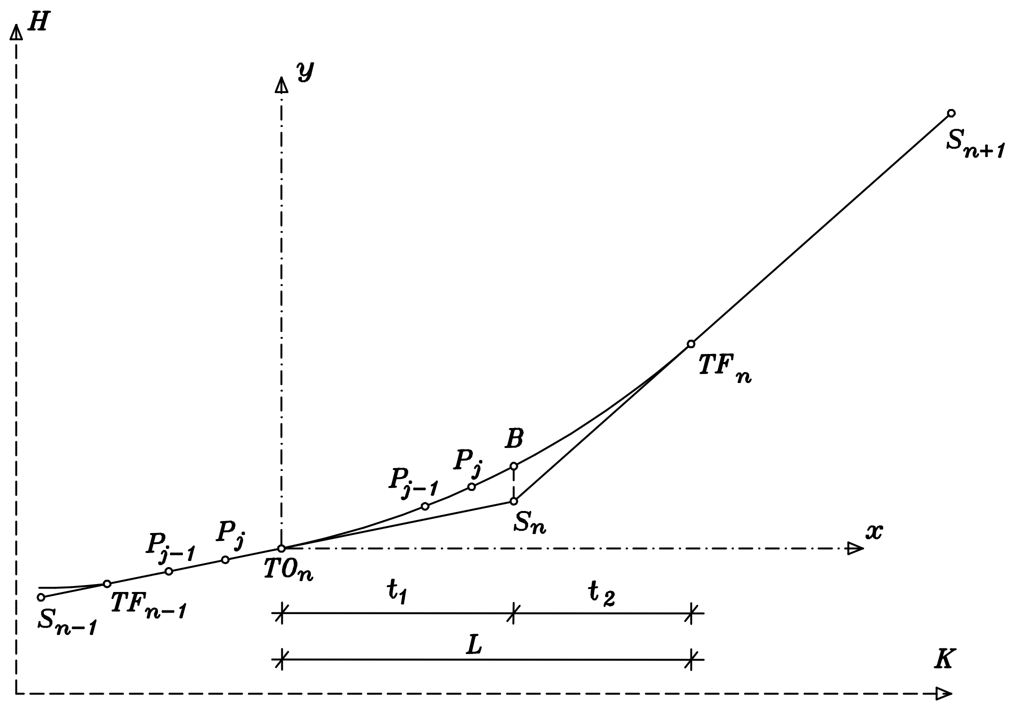

Figure 2 Parabolic Vertical Curve

Parabolic vertical curve equation in (x, y) perpendicular coordinate system is given by

$$y = ax^2 + bx + c \quad , \quad y' = 2a + b \quad (14)$$

and

$$y = y_{TO_n} = 0 \;\; for \;\; x = x_{TO_n} = 0 \qquad (15)$$

$$y' = g_1 \quad forx = x_{TO_n} = 0 \qquad (16)$$

$$y = y_{TF_n} = t_1 g_1 + t_2 g_2 \;\; for \;\; x = x_{TF_n} = L \qquad (17)$$

$$y' = g_2 \;\; for \;\; x = x_{TF_n} = L \qquad (18)$$

Coefficients of parabolic curve equation can be formed using equation (15), (16) and (17) in (14)

$$c = 0 \quad , \quad b = g_1 \quad , \quad a = \frac{g_2 - g_1}{2L} = \frac{G}{L} \tag{19}$$

and the equation of parabola

$$y = \frac{G}{2L} x^2 + g_1 x \quad , \quad y' = \frac{G}{L} x + g_1 \tag{20}$$

with $t_2 = L - t_1$ and equations (11), (17)

$$t_1 = t_2 = \frac{L}{2} \tag{21}$$

Middle point of parabolic vertical curve B

$$x_B = t_1 = \frac{L}{2} \quad , \quad y_B = -\frac{GL}{8} + \frac{g_1 L}{2} \tag{22}$$

and the extreme point

$$y' = -\frac{g_1^2 L}{2G} \tag{23}$$

There is one extreme point in sag curves if the gradients are $g_1 < 0$ and $g_2 > 0$, and in crest curves if the gradients are $g_1 > 0$ and $g_2 < 0$.

Computation Phase

Computation phase covers the portion of the vertical curve between TF_{n-1} and TF_n . First chainages of all main points (TO_n , B , E , TF_n) then grade (project) heights of the all main and intermediate points are calculated (Figure 1 and Figure 2).

Various methods may be used in the calculation of grade heights. One of the methods is using the calculated height for the calculation of the next height. This method will be taken into account in the following calculations because it is suitable with computer programing and provides exact calculation check

1.) Main point chainages:

$$K_{TO_n} = K_{S_n} - t_1 \ , \ K_{TF_n} = K_{S_n} + t_2 \ , \ K_B = K_{TO_n} + x_B \ , \ K_E = K_{TO_n} + x_E \qquad (24)$$

Values for t_1 , t_2 , x_B and x_E can be calculated using equations (11) and (10), (12), (13) for the circular curves, equations (21), (22), (23) for parabolic curve.

2.) The hight of the S_{n+1}

$$H_{S_{n+1}} = H_{S_n} + (K_{S_{n+1}} - K_{S_n})\, g_2 \qquad (25)$$

3.) The grade heights of intermediate points between TF_{n-1} , TO_n straight line (Figure 1 and Figure 2): The chainage values of these points satisfy the following condition

$$K_{TF_{n-1}} < K_j \le K_{TO_n} \ , \quad j \equiv P_1 \ , \ P_2 \ , \ldots , \ TO_n \qquad (26)$$

Calculation of grade heights carried out using:

$$\begin{aligned} H_{P_1} &= H_{TF_{n-1}} + (K_{P_1} - K_{TF_{n-1}})\, g_1 \\ H_{P_2} &= H_{P_1} + (K_{P_2} - K_{P_1})\, g_1 \\ &\ldots \\ H_j &= H_{j-1} + (K_j - K_{j-1})\, g_1 \end{aligned} \qquad (27)$$

At the end of the calculation the grade height of point TO_n is obtained.

4.) First calculation check: The height of the point TO_n is calculated once more using

$$H_{TO_n} = H_{S_n} - g_1 t_1 \qquad (28)$$

Two height values for TO_n obtained by (27) and (28) should be equal to each other with a reasonable difference (generally in millimeter).

5.) The grade height of the intermediate points between points TO_n ve TF_n (vertical curve): The chainage values of these points satisfy the following condition

$$K_{TO_n} < K_j \le K_{TF_n} \ , \quad j \equiv P_1 \ , P_2 \ , \ldots , \ B \ , \ldots , \ E \ , \ldots , \ TF_n \qquad (29)$$

The grade heights are calculated using following equations:

$$H_{P_1} = H_{TO_n} + y_1 \ , \ H_{P_2} = H_{P_1} + (y_2 - y_1) \ , \ldots , \ H_j = H_{j-1} + (y_j - y_{j-1}) \qquad (30)$$

Equation (8) or equation (9) for circular curve and equations (20) and (19) for parabolic curve are used to calculate the values of y_j. For the calculation of x abscissae equation

$$x_j = K_j - K_{TO_n} \tag{31}$$

is valid. The calculation process is terminated by the calculation of the grade height of point TF_n.

7.) Second calculation check: The height of the point TF_n is calculated once more using

$$H_{TF_n} = H_{S_n} + g_2 t_2 \tag{32}$$

Two height values obtained by (31) and (32) should be equal to each other with a reasonable difference (generally in millimeter).

Numerical Example

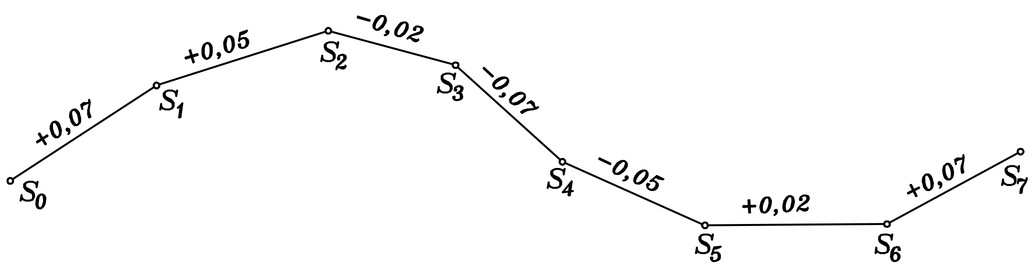

Figure 3 Numerical Example

Both methods, approximate solution and the exact solution, were applied to the profile data of sag and crest vertical curves to point out differences between two methods (Figure 3) and the initial data were given in Table 1.

Table 1. Initial Data for Numerical Example

Point No	Chainage K (Km+m)	Raidus R (m)	Height H (m)	Grade *g*
S_O	0+000	-	500	
				+0.07
S_1	0+500	10000	(535)	
				+0.05
S_2	1+500	10000	(585)	
				-0.02
S_3	2+500	10000	(565)	
				-0.07
S_4	3+500	10000	(495)	
				-0.05
S_5	4+500	10000	(445)	
				+0.02
S_6	5+500	10000	(465)	
				+0.07
S_7	6+000	-	(500)	

The height of S_0 is taken as an initial datum and the other heights in parenthesis were calculated (Table 1). In addition, chainages of two intermediate points at each curve and straight line are taken as initial data.

The differences are from –1114 millimeters to +602 millimeters in kilometers and from –78 millimeters to +78 millimeters in heights (Table 1).

CONCLUSION

In this study, exact solutions and solution stages for computer programming of circular and parabolic vertical curve are examined. The calculations method for circular curve given in literature as approximate solution and exact solution are compared using a numerical example. As seen in Table 2, there are differences in calculations for both methods reaching up to 100 cm in chainages and up to 8 cm in project heights. These differences are obtained for the radius of R=10000m. It is clear that the differences become larger as the radius increases.

If the current requirement of computations for chainage and height at millimeter level is taken into consideration for railways and highways, the differences given in Table 2 cannot be neglected. If the easiness of computation for time being is considered, the use of exact solution equations given in this study will be extremely appropriate.

Table 2. Comparison of Approximate and Exact Calculations of Circular Vertical Curves

	Approx.	**Calculation**	**Exact**	**Calcula.**	**Differences**	
Nokta No	K (km+m)	H (m)	K (km+m)	H (m)	ΔK (mm)	ΔL (mm)
	0+000	500.000	0+000	500.000	-	-
	0+300	521.000	0+300	521.000	-	-
	0+400	528.000	0+400.602	528.042	602	42
	0+450	531.375	0+450	531.377	-	2
	0+500	534.500	0+500.030	534.504	30	4
	0+550	537.375	0+550	537.377	-	2
	0+600	540.000	0+599.517	539.976	-483	-24
	0+700	545.000	0+700	545.000	-	-
	1+000	560.000	1+000	560.000	-	-
	1+150	567.500	1+150.515	567.526	515	26
	1+300	573.875	1+300	573.880	-	5
	1+500	578.875	1+500.092	578.881	92	6
	-	-	1+649.891	580.003	-	-
	1+700	579.875	1+700	579.877	-	2
	1+850	578.000	1+849.851	578.003	-149	3
	2+000	575.000	2+000	575.000	-	-
	2+150	572.000	2+150	572.000	-	-
-	2+250	570.000	2+250.555	569.989	555	-11
	2+350	567.500	2+350	567.505	-	5
	2+500	561.875	2+499.860	561.891	-140	16
	2+650	554.000	2+650	554.008	-	8
	2+750	547.500	2+748.886	547.578	-1114	78
	2+900	537.000	2+900	537.000	-	-
	3+200	516.000	3+200	516.000	-	-
	3+400	502.000	3+400.602	501.958	602	-42
	3+450	498.625	3+450	498.623	-	-2
	3+500	495.500	3+500.030	495.496	30	-4
	3+550	492.625	3+550	492.623	-	-2
	3+600	490.000	3+599.517	490.024	-483	24
	3+750	482.500	3+750	482.500	-	-
	4+000	470.000	4+000	470.000	-	-
	4+150	462.500	4+150.515	462.474	515	-26
	4+300	456.125	4+300	456.121	-	-4
	4+500	451.125	4+500.092	451.120	92	-5
	-	-	4+669.891	449.997	-	-
	4+750	450.500	4+750	450.499	-	-1
	4+850	452.500	4+849.852	451.997	-148	-3
	5+000	455.000	5+000	455.000	-	-
	5+150	458.000	5+150	458.000	-	-
	5+250	460.000	5+250.555	460.011	555	11
	5+350	462.500	5+350	462.495	-	-5
	5+500	468.125	5+499.860	468.109	-140	-16
	5+650	476.000	5+650	475.992	-	-8
	5+750	482.500	5+750.886	482.422	-1114	-78