HOMEWORK 1

1. Compute the volume of the parallelepiped given below. The edges of the parallelepiped are formed the three vectors which are,

$\mathbf{u} = 3\mathbf{e}_1 - \mathbf{e}_2$
$\mathbf{v} = \mathbf{e}_2 - 2\mathbf{e}_3$
$\mathbf{w} = \mathbf{e}_1 + 5\mathbf{e}_2 + 4\mathbf{e}_3$

2. Interpret $\mathbf{grad}\,\phi$, $\mathbf{div}\,\mathbf{A}$, and $\nabla^2\phi$ in both cylindrical and spherical coordinates.

3. If $\alpha \neq 0$ and $\phi(\alpha) = \int_{\alpha}^{\alpha^2} \frac{\sin \alpha x}{x} dx$ then compute $\phi'(\alpha)$ using the Leibnitz Theorem.

4. If $\mathbf{r}$ is given as $\mathbf{r} = \left(x^2 \sin y, z^2 \cos y, -xy^2\right)$ the compute $d\mathbf{r}$.

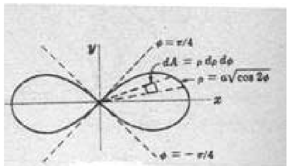
5. $\phi = x^2 y z^3, \mathbf{A} = \left(xz, -y^2, 2x^2 y\right)$ are given in Cartesian coordinate system, compute;

i. $\nabla\phi$ **ii.** $\nabla \bullet \mathbf{A}$ **iii.** $\mathbf{div}(\phi\mathbf{A})$ **iv.** $\mathbf{rot}(\phi\mathbf{A})$

6. If $\int_0^{\pi} \frac{dx}{\alpha - \cos x} = \frac{\pi}{\sqrt{\alpha^2 - 1}}$ and $\alpha > 1$ are given, using Leibnitz's theorem compute

$$\int_0^{\pi} \frac{dx}{(2 - \cos x)^2}$$ (Do not use direct integration!)

7. Compute the area of the region bounded by the lemniscate's curve $\rho = a^2 \cos(2\phi)$ on the xy plane.



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AKM 204E
FLUID MECHANICS
~ HOMEWORK I ~

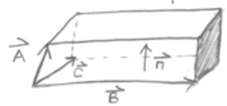
1. Compute the volume of the paralelpiped given below. The edges of the paralelpiped are formed the three vectors which are

$u = 3\vec{e_1} - \vec{e_2}$ $v = \vec{e_2} - 2\vec{e_3}$ $w = \vec{e_1} + 5\vec{e_2} + 4\vec{e_3}$

$\vec{e_1} = (1,0,0)$ $\vec{e_2} = (0,1,0)$ $\vec{e_3} = (0,0,1) \Longrightarrow$

$u = 3\vec{i} - \vec{j}$ $v = \vec{j} - 2\vec{k}$ $w = \vec{i} + 5\vec{j} + 4\vec{k}$

Volume of the paralelpiped = (h).(area of an paralellogram)
$= (\vec{A}.\vec{n})(|\vec{B} \times \vec{C}|)$
$= \vec{A}.\{|\vec{B}\times\vec{C}|\vec{n}\} = \vec{A}.(\vec{B}\times\vec{C})$



$$|\vec{A}.(\vec{B}\times\vec{C})| = \left|\begin{vmatrix} 3 & -1 & 0 \\ 0 & 1 & -2 \\ 1 & 5 & 4 \end{vmatrix}\right| = [3.(4+10)] + [0.(0-1)] - [(-1).(0+2)]$$
$$= |42+2| = |44| = 44$$

2. Interpret grad Φ, div A and $\nabla^2\Phi$ in both cylindrical and spherical coordinates.

in cyclindrical coordinates;

i) $\nabla\Phi = \text{grad}\,\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\vec{e_1} + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\vec{e_2} + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\vec{e_3}$

$u_1 = \rho$ $u_2 = \phi$ $u_3 = z$ $h_1 = 1$ $h_2 = \rho$ $h_3 = 1$

$= \frac{1}{1}\frac{\partial\Phi}{\partial\rho}\vec{e_1} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\vec{e_2} + \frac{1}{1}\frac{\partial\Phi}{\partial z}\vec{e_3}$

$= \frac{\partial\Phi}{\partial\rho}\vec{e_1} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\vec{e_2} + \frac{\partial\Phi}{\partial z}\vec{e_3}$

ii) $\nabla \cdot \vec{A} = \text{div}\,\vec{A} = \dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial u_1}(h_2h_3A_1) + \dfrac{\partial}{\partial u_2}(h_3h_1A_2) + \dfrac{\partial}{\partial u_3}(h_1h_2A_3)\right]$

$$\nabla \cdot \vec{A} = \frac{1}{(1)(\rho)(1)}\left[\frac{\partial}{\partial \rho}\left((\rho)(1)A_1\right) + \frac{\partial}{\partial \phi}\left((1)(1)A_2\right) + \frac{\partial}{\partial z}\left((1)(\rho)A_3\right)\right]$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_1) + \frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}\right]$$

$(\vec{A} = A_1\vec{e_1} + A_2\vec{e_2} + A_3\vec{e_3})$

iii) $\nabla^2 \Phi = \dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial u_1}\left(\dfrac{h_2h_3}{h_1}\dfrac{\partial \Phi}{\partial u_1}\right) + \dfrac{\partial}{\partial u_2}\left(\dfrac{h_3h_1}{h_2}\dfrac{\partial \Phi}{\partial u_2}\right) + \dfrac{\partial}{\partial u_3}\left(\dfrac{h_1h_2}{h_3}\dfrac{\partial \Phi}{\partial u_3}\right)\right]$

$$= \frac{1}{(1)(\rho)(1)}\left[\frac{\partial}{\partial \rho}\left(\frac{(\rho)(1)}{(1)}\frac{\partial \Phi}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{(1)(1)}{\rho}\frac{\partial \Phi}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\frac{(1)(\rho)}{(1)}\frac{\partial \Phi}{\partial z}\right)\right]$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

in spherical coordinates;

i) $\text{grad}\,\Phi = \dfrac{\partial \Phi}{\partial r}\vec{e_1} + \dfrac{1}{r}\dfrac{\partial \Phi}{\partial \theta}\vec{e_2} + \dfrac{1}{r\sin\theta}\dfrac{\partial \Phi}{\partial \phi}\vec{e_3}$

ii) $\text{div}\,\vec{A} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2A_1) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(\sin\theta A_2) + \dfrac{1}{r\sin\theta}\dfrac{\partial A_3}{\partial \phi}$

$(\vec{A} = A_1\vec{e_1} + A_2\vec{e_2} + A_3\vec{e_3})$

iii) $\nabla^2 \Phi = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial \Phi}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\theta}{\partial \theta}\left(\sin\theta\dfrac{\partial \Phi}{\partial \theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 \Phi}{\partial \phi^2}$

3. If $\alpha \neq 0$ and $\phi(\alpha) = \displaystyle\int_{\alpha}^{\alpha^2} \frac{\sin \alpha x}{x}\,dx$ then compute $\phi'(\alpha)$ using the Leibnitz Theorem.

Using Leibnitz's Theorem;

$$\phi'(\alpha) = \int_{\alpha}^{\alpha^2} \frac{\partial}{\partial \alpha}\left(\frac{\sin \alpha x}{x}\right)dx + \frac{\sin(\alpha \cdot \alpha^2)}{\alpha^2}\frac{d}{d\alpha}(\alpha^2) - \frac{\sin(\alpha \cdot \alpha)}{\alpha}\frac{d}{d\alpha}(\alpha)$$

$$= \int_{\alpha}^{\alpha^2} \cos \alpha x\,dx + \frac{2\sin \alpha^3}{\alpha} - \frac{\sin \alpha^2}{\alpha}$$

$$= \frac{\sin \alpha x}{\alpha}\Bigg|_{\alpha}^{\alpha^2} + \frac{2\sin \alpha^3}{\alpha} - \frac{\sin \alpha^2}{\alpha}$$

$$= \frac{\sin \alpha^3}{\alpha} - \frac{\sin \alpha^2}{\alpha} + \frac{2\sin \alpha^3}{\alpha} - \frac{\sin \alpha^2}{\alpha} = \frac{3\sin \alpha^3}{\alpha} - \frac{2\sin \alpha^2}{\alpha}$$

$$= \frac{3\sin \alpha^3 - 2\sin \alpha^2}{\alpha}$$

4. If $\vec{r}$ is given as $\vec{r} = (x^2\sin y, z^2\cos y, -xy^2)$ the compute $d\vec{r}$.

$$\frac{\partial \vec{r}}{\partial x} = (2x\sin y)\vec{i} - y^2\vec{k} \qquad \frac{\partial \vec{r}}{\partial y} = (x^2\cos y)\vec{i} - (z^2\sin y)\vec{j} - (2xy)\vec{k}$$

$$\frac{\partial \vec{r}}{\partial z} = (2z\cos y)\vec{j}$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial x}dx + \frac{\partial \vec{r}}{\partial y}dy + \frac{\partial \vec{r}}{\partial z}dz$$

$$= \left[(2x\sin y)\vec{i} - y^2\vec{k}\right]dx + \left[(x^2\cos y)\vec{i} - (z^2\sin y)\vec{j} - (2xy)\vec{k}\right]dy + \left[(2z\cos y)\vec{j}\right]dz$$

$$= (2x\sin y\,dx + x^2\cos y\,dy)\vec{i} + (2z\cos y\,dz - z^2\sin y\,dy)\vec{j} - (y^2dx + 2xy\,dy)\vec{k}$$

5. $\phi = x^2yz^3$ $\vec{A} = (xz, -y^2, 2x^2y)$ are given in Cartesian coordinate system, compute;

i. $\nabla\phi$ ii. $\nabla.\vec{A}$ iii. $div(\phi A)$ iv. $rot(\phi\vec{A})$

i) $$\nabla\phi = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$= \frac{\partial(x^2yz^3)}{\partial x}\vec{i} + \frac{\partial(x^2yz^3)}{\partial y}\vec{j} + \frac{\partial(x^2yz^3)}{\partial z}\vec{k}$$

$$= 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$$

ii) $$\nabla.\vec{A} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right).\left(xz\vec{i} - y^2\vec{j} + 2x^2y\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y) = z - 2y$$

iii) $$div(\phi\vec{A}) = \nabla.(\phi\vec{A}) = \nabla.\left(x^3yz^4\vec{i} - x^2y^3z^3\vec{j} + 2x^4y^2z^3\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^3yz^4) + \frac{\partial}{\partial y}(-x^2y^3z^3) + \frac{\partial}{\partial z}(2x^4y^2z^3)$$

$$= 3x^2yz^4 - 3x^2y^2z^3 + 6x^4y^2z^2$$

iv) $$rot(\phi\vec{A}) = \nabla\times(\phi\vec{A}) = \nabla\times\left(x^3yz^4\vec{i} - x^2y^3z^3\vec{j} + 2x^4y^2z^3\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3yz^4 & -x^2y^3z^3 & 2x^4y^2z^3 \end{vmatrix}$$

$$\left[\phi\vec{A} = (x^2yz^3)\left(xz\vec{i} - y^2\vec{j} + 2x^2y\vec{k}\right) = (x^3yz^4)\vec{i} - (x^2y^3z^3)\vec{j} + (2x^4y^2z^3)\vec{k}\right]$$

$$= \left(\frac{\partial}{\partial y}(2x^4y^2z^3) - \frac{\partial}{\partial z}(-x^2y^3z^3)\right)\vec{i} + \left(\frac{\partial}{\partial z}(x^3yz^4) - \frac{\partial}{\partial x}(2x^4y^2z^3)\right)\vec{j}$$

$$+ \left(\frac{\partial}{\partial x}(-x^2y^3z^3) - \frac{\partial}{\partial y}(x^3yz^4)\right)\vec{k}$$

$$= \left(4x^4yz^3 + 3x^2y^3z^2\right)\vec{i} + \left(4x^3yz^3 - 8x^3y^2z^3\right)\vec{j} + \left(-2xy^3z^3 - x^3z^4\right)\vec{k}$$

6. If $\int_0^{\pi} \frac{dx}{\alpha - \cos x} = \frac{\pi}{\sqrt{\alpha^2 - 1}}$ and $\alpha > 1$ are given, using Leibnitz's theorem compute

$$\int_0^{\pi} \frac{dx}{(2 - \cos x)^2}$$

$$\phi(\alpha) = \int_0^{\pi} \frac{dx}{\alpha - \cos x} = \pi(\alpha^2 - 1)^{-1/2}$$

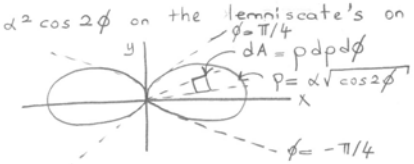
$$\phi'(\alpha) = -\int_0^{\pi} \frac{dx}{(\alpha - \cos x)^2} = \frac{-1}{2}\pi(\alpha^2 - 1)^{-3/2}.2\alpha$$

$$= \frac{-\pi\alpha}{(\alpha^2 - 1)^{3/2}}$$

$$\int_0^{\pi} \frac{dx}{(\alpha - \cos x)^2} = \frac{\pi\alpha}{(\alpha^2 - 1)^{3/2}} \Longrightarrow \int_0^{\pi} \frac{dx}{(2 - \cos x)^2} = \frac{2\pi}{(2^2 - 1)^{3/2}}$$

$$\int_0^{\pi} \frac{dx}{(2 - \cos x)^2} = \frac{2\pi}{3^{3/2}} = \frac{2\pi}{3\sqrt{3}}$$

7. Compute the area of the region bounded by the curve $\rho = \alpha^2 \cos 2\phi$ on the lemniscate's on the xy plane.



$$4\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{\alpha\sqrt{\cos 2\phi}} \rho\, d\rho\, d\phi = 4\int_{\phi=0}^{\pi/4} \left.\frac{\rho^2}{2}\right|_{\rho=0}^{\alpha\sqrt{\cos 2\phi}} d\phi$$

$$= 2\int_{\phi=0}^{\pi/4} \alpha^2 \cos 2\phi\, d\phi$$

$$= \alpha^2 \sin 2\phi \Big|_{\phi=0}^{\pi/4} = \alpha^2 \overbrace{\sin\frac{\pi}{2}}^{1} - \alpha^2 \overbrace{\sin 0}^{0}$$

$$= \alpha^2$$