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Fluid Mechanics

for:

İstanbul Technical University

Faculty of Naval Architecture and Ocean Engineering

⮘● Scalars, Vectors and Tensors ●⮚

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SCALARS, VECTORS AND TENSORS

In fluid mechanics, it is dealt with these quantities and it is important to have a firm understanding of their meaning and of their properties. Unlike vectors and tensors, scalars are uniquely defined by a single value (e.g. temperature, density..). Scalars will generally be denoted by *italic* letters, while vectors and tensors will be denoted by **bold** letters.

1. VECTORS

1. 1 SCALARS AND VECTORS

A Scalar is a quantity having only magnitude. A vector is a quantity having both direction and magnitude. The magnitude of a vector **a** is denoted |**a**| A familiar example of a vector is the velocity **u** or the position coordinate **x**. Such a vector may be represented by an arrow whose length denotes its magnitude and whose orientation specifies its direction in space.

1. 2 VECTOR SPACE, BASIS, COMPONENTS

Let us begin with a brief review. The fundamental rules for addition of two vectors and of multiplication of a vector by a scalar form a **vector space**.

1) Commutative rule

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \tag{1.1}$$

$$a\mathbf{u} = \mathrm{u}a \tag{1.2}$$

2) Associative rule

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) = \mathbf{u} + \mathbf{v} + \mathbf{w} \tag{1.3}$$

$$a(b\mathbf{u}) = (ab)\,\mathbf{u} \tag{1.4}$$

3) Distributive rule

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v} \tag{1.5}$$

$$(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u} \tag{1.6}$$

4) Existence of a zero and negative

$$\mathbf{u} + \mathbf{0} = \mathbf{u} \tag{1.7}$$

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0} \tag{1.8}$$

5) Unit multiplication

$$1\mathbf{u} = \mathbf{u} \tag{1.9}$$

Suppose that a group of vectors { **v** } contains n independent vectors. This group { **v** } is called and n-dimensional vector space. Let us choose n such vectors $\mathbf{e}_1$, $\mathbf{e}_2$, …,$\mathbf{e}_n$ to form a vector basis. By definition of linear dependence, any vector v belonging to the group of vectors { **v** }may be written as a linear combination of $\mathbf{e}_1$, $\mathbf{e}_2$,…,$\mathbf{e}_n$:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n \tag{1.10}$$

The rules of operations in vector components are as follows:

1) Addition of two vectors. Let

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \ldots + u_n\mathbf{e}_n \text{ and } \mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$

Then

$$\mathbf{w} = \mathbf{u} + \mathbf{v} = (u_1 + v_1)\mathbf{e}_1 + (u_2 + v_2)\mathbf{e}_2 + \ldots + (u_n + v_n)\mathbf{e}_n$$

$$\mathbf{w} = (u_1 + v_1 , u_2 + v_2,\ldots,u_n + v_n)$$

$$w_i = u_i + v_i \text{ for } (i=1, 2,\ldots, n)$$

2) Multiplication of a vector by a scalar a:

$$\mathbf{v} = a\mathbf{u} = au_1\mathbf{e}_1 + au_2\mathbf{e}_2 + \quad + au_n\mathbf{e}_n$$

$$\mathbf{v} = (au_1 , au_2 ,\ldots, au_n)$$

$$v_i = au_i \text{ for } (i=1, 2, \ldots, n)$$

1. 3 SCALAR (INNER OR DOT) PRODUCT OF TWO VECTORS

The scalar product of two vectors is by definition a scalar,

$$\mathbf{u} \cdot \mathbf{v} = \text{scalar} \tag{1.11}$$

The general rules imposed on the scalar product are as follows

1) Commutative rule

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \tag{1.12}$$

2) Distributive rule

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \tag{1.13}$$

3) In general

$$\mathbf{u} \cdot (a\mathbf{v} + b\mathbf{w}) = a\mathbf{u} \cdot \mathbf{v} + b\mathbf{u} \cdot \mathbf{w} \tag{1.14}$$

Using the above rules, the scalar product of any two vectors may be determined once the scalar products between the base vectors $\mathbf{e}_1$, $\mathbf{e}_2$,…, $\mathbf{e}_n$ are specified.

Several geometric concepts such as magnitude, orthogonality and directional cosine may be defined through the use of the scalar product.

1) Magnitude of a vector

$$|\mathbf{v}| = \sqrt{\mathbf{v}\,.\,\mathbf{v}} \tag{1.15}$$

2) Orthogonality

Two vectors **u** and **v** are said to be orthogonal to each other if

$$\mathbf{u}\,.\,\mathbf{v} = 0 \tag{1.16}$$

3) Directional cosine

If θ is the angle between two vectors **u** and **v** then the directional cosine is defined by

$$\cos\theta = \frac{\mathbf{u}\,.\,\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \tag{1.17}$$

1. 4 EINSTEIN 'S SUMMATION CONVENTION

The Einstein ‘s summation convention greatly simplifies things by saying “repeated indices implies summation.” A vector

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + ... + v_n\mathbf{e}_n \;= \sum_{i=1}^{n} v_i\mathbf{e}_i \;,$$

may be written using Einstein’s convention as

$$\mathbf{v} = v_i\mathbf{e}_i$$

The summation sign with respect to i is omitted but its presence is understood. The ith component of **v** may be written formally as

$$\{\,\mathbf{v}\,\}_i = v_i \tag{1.18}$$

We must take a clear distinction between two types of indices in a system of equations. For example, let

$$u_1 = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n$$

$$u_2 = a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n$$

……

$$u_1 = a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n$$

In Einstein notation, we have

$$u_i = a_{ij}x_j$$

The index *j* is repeated implying summation and is called a **dummy index** because this index may be replaced by any other letter, say, k and the equation will be unaffected.

$$u_i = a_{ik}x_k$$

Hence, the index *i*, here a **free index**. The same free index must appear in every term in the equation.

1. 7 ORTHONORMAL BASIS

A basis of a n-dimensional space $\mathbf{e}_1$, $\mathbf{e}_2$,…,$\mathbf{e}_n$ is called **orthonormal** if the n base vectors fulfil the following two conditions.

(1) All the n vectors $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_n$ are unit vectors,

$$|\mathbf{e}_1| = |\mathbf{e}_2| = \ldots = |\mathbf{e}_n| = 1 \tag{1.19}$$

or

$$\mathbf{e}_i \cdot \mathbf{e}_j = 1 \text{ when } i = j \tag{1.20}$$

(2) The n base vectors are orthogonal to each other,

$$\mathbf{e}_i \cdot \mathbf{e}_j = 0 \text{ when } i \neq j \tag{1.21}$$

The above two conditions may be combined as follows

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{1.22}$$

where δ_{ij} is the **Kronecker delta**, which is defined by,

$$\delta_{ij} = 1 \text{ for } i = j \tag{1.23}$$

$$\delta_{ij} = 0 \text{ for } i \neq j \tag{1.24}$$

From here on we will be dealing with 3-dimensional orthonormal vector spaces. An example is the conventional right hand **Cartesian coordinate system**. Here the 1, 2, 3 axes correspond to the x, y, z axes, respectively, and are parallel to the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

1. 6 VECTOR (CROSS) PRODUCT

In 3-dimensional space, the cross product of two vectors **u** and **v** is defined in the form $\mathbf{w} \equiv \mathbf{u} \times \mathbf{v}$ is defined by the following rules:

(1) **w** is perpendicular to both **u** and **v,**

$$\mathbf{u} \,.\, \mathbf{w} = \mathbf{v} \,.\, \mathbf{w} = 0 \tag{1.25}$$

(2) The magnitude of **w** is by definition,

$$|\mathbf{w}| = |uv\sin\theta|, \tag{1.26}$$

where θ is the angle between **u** and **v**.

(3) **w** points in the direction so that **u**, **v**, **w** form a right hand coordinate system. The following properties of the vector product are easily proven.

$$\mathbf{u} \times \mathbf{v} = -\,(\mathbf{v} \,.\, \mathbf{u})$$

$$\mathbf{u} \times \mathbf{v} = 0 \text{ if } \mathbf{v} = a\mathbf{u}$$

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$$

1. 8 MULTIPLE PRODUCT OF VECTORS

Some useful formulae involving multiplication of multiple vectors are presented below. These formulae may be proven directly by using known vectorial identities or indirectly by working through the scalar components. One important theorem in connection with the latter approach is the following statement: “If a vector or tensor identity is proved true in one coordinate system, it will be true in all other coordinate systems.” We can therefore prove an identity using the simple coordinate system (usually the rectangular coordinate system with orthonormal base vectors), and then rewrite it in general vector form.

(1) The scalar product of three vectors:

$$\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \varepsilon_{ijk} u_i v_j w_k \tag{1.27}$$

The applying the rules of the permutation symbol, we obtain

$$\begin{aligned} &\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \,.\, (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \,.\, (\mathbf{u} \times \mathbf{v}) \\ &= -\mathbf{u} \,.\, (\mathbf{w} \times \mathbf{v}) = -\mathbf{v} \,.\, (\mathbf{u} \times \mathbf{w}) = -\mathbf{w} \,.\, (\mathbf{v} \times \mathbf{u}) \end{aligned} \tag{1.28}$$

The scalar product then follows a general cyclic rule. The sign of the determinant changes when any two rows in equation (32) are interchanged. This is because interchanging two indices in the permutation symbol causes a change in sign. Note also that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ if any two vectors are identical.

The magnitude of $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is he volume of the parellelpiped formed by the three vectors **u**, **v**, **w**. Hence if the three vectors lie in a plane, the volume vanishes.

(2) Vector product of three vectors

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{u} \cdot \mathbf{w}) - \mathbf{w} \cdot (\mathbf{u} \cdot \mathbf{v}) \qquad (1.29)$$

(3)

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) \qquad (1.30)$$

(4) Lagrange's Identity

$$|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2 \cdot |\mathbf{v}|^2 - (\mathbf{u}.\mathbf{v})^2 \qquad (1.31)$$

1.9 DIFFERENTIATION OF A VECTOR FUNCTION OF A SCALAR VARIABLE

Let $\mathbf{r}(\xi)$ be a vector function of the scalar ξ. As ξ changes, the magnitude and direction of **r** varies as shown in the figure (1). For a small change $\Delta\xi$, $\mathbf{r}(\xi)$ changes by $\Delta\mathbf{r}$ to $\mathbf{r}(\xi+\Delta\xi)$.

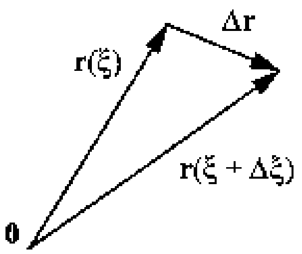

Figure 1. Differentiation of a Vector Function of a Scalar Variable

The derivative of **r** is a new vector function defined by

$$\frac{d\mathbf{r}}{d\xi} \equiv \lim_{\Delta\xi \to \infty} \frac{\mathbf{r}(\xi + \Delta\xi) - \mathbf{r}(\xi)}{\Delta\xi} \qquad (1.32)$$

1. 10 THE PRODUCT RULE

Two vector functions $\mathbf{u}(\xi)$ and $\mathbf{v}(\xi)$ may be combined in various ways such as by the inner product, the cross product $\mathbf{u}\times\mathbf{v}$. Another product called the outer product $\mathbf{uv}$ $(\mathbf{u}\otimes\mathbf{v})$ produces a higher order tensor and will be discussed later. Writing "$\bullet$" to denote "." or "×" or "$\otimes$" , the cross product rule states

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(\mathbf{u}\bullet\mathbf{v})=\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\xi}\bullet\mathbf{v}+\mathbf{u}\bullet\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\xi} \tag{1.33}$$

It is important to remember here that the operations "×" or "$\otimes$"do not commute.

Example: If $\mathbf{u}(\xi)$ is a vector of constant magnitude then $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\xi}$ is perpendicular to $\mathbf{u}$. By assumption, $|\mathbf{u}|$ = cons. Hence,

$$\frac{\mathrm{d}|\mathbf{u}|^2}{\mathrm{d}\xi}=2\mathbf{u}.\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\xi}=0 \tag{1.34}$$

2 TENSORS

The definition of a tensor is a natural extension of that vectors which are sometimes referred to as 1st order tensors, while scalars are zeroth order tensors. A tensor is defined by a set of scalars and a tensor basis. In 3-dimensional space, with basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, a tensor $\mathbf{A}$ may be written as

$$\mathbf{A} = A_{11}\mathbf{e}_1\mathbf{e}_1 + A_{12}\mathbf{e}_1\mathbf{e}_2 + \ldots + A_{33}\mathbf{e}_3\mathbf{e}_3 \tag{2.1}$$

or in matrix form as

$$\mathbf{A}=\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \tag{2.2}$$

or in Einstein notation

$$\mathbf{A} = A_{ij}\mathbf{e}_i\mathbf{e}_j \text{ or } \{\ \mathbf{A}\ \}_{ij} = A_{ij} \tag{2.3}$$

2. 1 TENSOR ORDER

The number of the tensor components is equal to the order (or rank) of the tensor.

Table 1. Order of tensors

Rank	No of free indices	Examples	Remarks
0	0	T, $u_i v_i$	Scalar
1	1	u_i, $A_{ij}u_i$, $\varepsilon_{ijk}u_j v_k$	Vector
2	2	$\frac{\partial u_i}{\partial x_j}, A_{ij}, \delta_{ij}$	Tensor
3	3	$A_{ij}x_k$, ε_{ijk}	3rd order tensor
n	n	$A_{ijk\ldots sj}$	

If a quantity is referred to as a tensor without reference to its order, it is assumed that the quantity is a second order tensor.

2. 2 PROPERTIES OF TENSORS

2. 2. 1 IDENTITY TENSOR

The **identity tensor I** defined by

$$\mathbf{x} = \mathbf{I} \cdot \mathbf{x} \tag{2.4}$$

In Cartesian coordinates

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{e}_i \mathbf{e}_i \tag{2.5}$$

$$\{\mathbf{I}\}_{ij} = \delta_{ij}$$

2. 2. 2 TRANSPOSE

The transpose of a tensor **A** is denoted $\mathbf{A}^T$ and is defined

$$\mathbf{A}^T = A_{ji}\mathbf{e}_j\mathbf{e}_i \text{ or } \{\mathbf{A}^T\}_{ij} = A_{ji} \tag{2.6}$$

2. 2. 3 SYMMETRY

A tensor **A** is **symmetric** if $A_{ij} = A_{ji}$

2. 2. 4 THE ADJOINT OF A TENSOR

A tensor $\tilde{\mathbf{A}}$ is called the adjoint of the tensor. **A** if for any two vectors **x** and **y**,

$$\mathbf{x} \cdot \mathbf{A} \cdot \mathbf{y} = \mathbf{y} \cdot \tilde{\mathbf{A}} \cdot \mathbf{x} \qquad (2.7)$$

A tensor is called self-adjoint if $\tilde{\mathbf{A}} = \mathbf{A}$. A symmetric tensor satisfies the condition equation (55) and is therefore self-adjoint.

2. 2. 5 The Trace of a Tensor

The trace of a tensor A is the sum of its diagonal elements, e.

$$\mathrm{tr}(\mathbf{A}) = A_{ii} = A_{11} + A_{22} + A_{33} \qquad (2.8)$$

As will be verified later by inspection, the trace of a tensor is independent of the orientation of the coordinate system. Note also that the trace of the Kronecker delta is

$$\delta_{ii} = 3 \qquad (2.9)$$

2. 2. 6 The Determinant of a Tensor

The determinant of a tensor **A** is defined by:

$$\det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \qquad (2.10)$$

$$= A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31})$$

$$= \det(\mathbf{A}) = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$$

2. 2. 7 The Inverse of A Tensor

The inverse of a tensor **A** is the tensor $\mathbf{A}^{-1}$ if

$$\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I} \qquad (2.11)$$

In Cartesian coordinates,

$$A_{ij} A_{jk}^{-1} = \delta_{jk} \qquad (2.12)$$

2. 2.8 Eigenvalues and Eigenvectors

For a tensor **A** there exists a particular set of **eigenvectors u** and **eigenvalues** λ for which

$$\mathbf{A}\,\mathbf{u} = \lambda\,\mathbf{u} \tag{2.13}$$

The **u** ’s define the directions of **principal axes** of A. If all the eigenvalues of **A** are positive then **A** is said to be **positive definite**.

Theorem (without proof): The eigenvalues of a real, self-adjoint (symmetric) 2nd order tensor are real, and the eigenvectors are mutually orthogonal.

From equation (64), for **u**≠0, we must have

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{2.14}$$

$$\begin{vmatrix} A_{11}-\lambda & A_{12} & A_{13} \\ A_{21} & A_{22}-\lambda & A_{23} \\ A_{31} & A_{32} & A_{33}-\lambda \end{vmatrix} = 0 \tag{2.15}$$

Hence we obtain **a characteristic equation** that is cubic in λ,

$$\lambda^3 - I_1\,\lambda^2 + I_2\,\lambda - I_3 = 0 \tag{2.16}$$

The solution is the three eigenvalues λ_1, λ_2, λ_3, we note that the three equations

$$(A_{ij} - \lambda\delta_{ij})\,u_j = 0 \tag{2.17}$$

And

$$\begin{aligned} (A_{11} - \lambda)u_1 + A_{12}u_2 + A_{13}u_3 &= 0 \\ A_{21}u_1 + (A_{22} - \lambda)u_2 + A_{23}u_3 &= 0 \\ A_{31}u_1 + A_{32}u_2 + (A_{33} - \lambda)u_3 &= 0 \end{aligned} \tag{2.18}$$

Are linearly dependent since the characteristic determinant is zero ($|\mathbf{A} - \lambda\mathbf{I}| = 0$). The three unknown components of **u** (u_1, u_2, u_3) cannot be uniquely determined. We can however determine the ratios u_1/u_2 and u_2/u_3 by selecting equation (69):

$$\frac{u_1}{u_2} = \begin{vmatrix} -A_{13} & A_{12} \\ -A_{23} & A_{22}-\lambda \end{vmatrix} \cdot \begin{vmatrix} A_{11}-\lambda & A_{12} \\ A_{21} & A_{22}-\lambda \end{vmatrix}^{-1}$$

$$\frac{u_2}{u_3} = \begin{vmatrix} -A_{11}-\lambda & -A_{13} \\ A_{21} & A_{23} \end{vmatrix} \cdot \begin{vmatrix} A_{11}-\lambda & A_{12} \\ A_{21} & A_{22}-\lambda \end{vmatrix}^{-1} \qquad (2.19)$$

The procedure is applied for all three eigenvalues. We need two additional conditions to determine (a) the magnitude and (b) the sense of **u**, **v**, **w**. By convention, the magnitudes are normalized to unity and the sense of the three eigenvectors are chosen to form a right hand orthonormal basis set.

$$|\,\mathbf{u}\,| = |\,\mathbf{v}\,| = |\,\mathbf{w}\,| = 1$$

$$\mathbf{u} \times \mathbf{v} = \mathbf{w}\,,\ \mathbf{v} \times \mathbf{w} = \mathbf{u}\,,\ \mathbf{w} \times \mathbf{u} = \mathbf{v} \qquad (2.20)$$

2. 2. 9 THE INVERSE OF A TENSOR

Although the operations described below apply to vectors spaces of arbitrary dimension, we will use a 3-dimensional space. Let x_1, x_2, x_3 and x'_1, x'_2, x'_3 be two Cartesian coordinate systems. The frame x_1, x_2, x_3 is in arbitrary direction and has base vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, while the frame x'_1, x'_2, x'_3 is called the **principal frame** since its base vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$,are in the principal directions. Suppose that **A** is a Cartesian tensor in the arbitrary frame x_1, x_2, x_3, and has the general component form,

$$\mathbf{A} = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \qquad (2.21)$$

The tensor **A** can be transformed by a rotation to the principal frame. In the principal frame the tensor will be denoted **A**’. We shall find that **A**’ is a diagonal matrix and the three diagonal elements are the eigenvalues of **A**. It is said that **A**’is in the **canonical form** which is,

$$\mathbf{A}' = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} \qquad (2.22)$$

3. TRANSFORMATION UNDER ROTATION AND TRANSLATION

The directional cosines of $\mathbf{v}$ for the $\mathbf{e}_i$ axis are defined as

$$\cos(\mathbf{e}_i) = \frac{v_i}{|\mathbf{v}|} \tag{3.1}$$

Suppose that we introduce two right hand Cartesian coordinate systems K and K' with orthonormal basis $\mathbf{e}_i$ and $\mathbf{e}'_I$ we wish to express the coordinates x_i 's of an arbitrary point in the system K in terms of its coordinates x'_i Let the origin O' of the system K' have radius vector $\mathbf{r}'_0$ and coordinates x'_{0i} in the system K, while the origin O of the system K has radius vector $\mathbf{r}_0 = -\mathbf{r}'_0$ and coordinates x_{0i} in the system K' . Finally, let $\alpha_{j'k}$ be the cosine of the angle between the jth axis of the system K' and kth axis of system K,

$$\alpha_{j'k} = \cos(x'_j , x_k) = \mathbf{e}'_j , \mathbf{e}_k \tag{3.2}$$

Then

$$x'_i \ \mathbf{e}'_i = x_i \ \mathbf{e}_i + x_{0i} \ \mathbf{e}_i \tag{3.3}$$

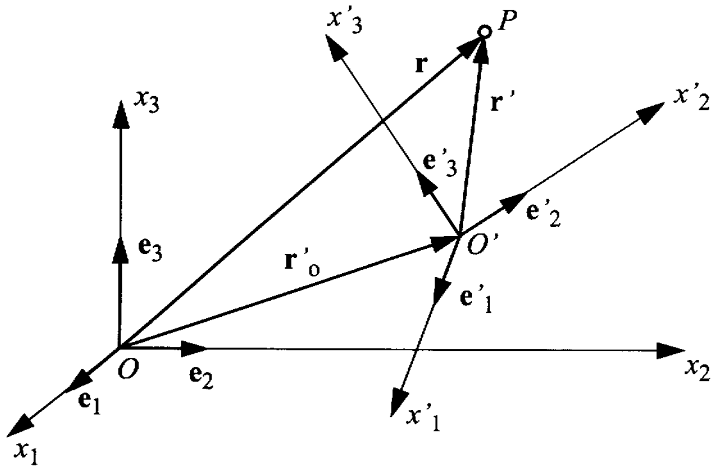

Figure 2. Definition of two coordinate systems. (O' and O)

Taking the scalar product of equation (100) with $\mathbf{e}'_j$, we obtain

$$x'_j = \alpha_{j'i} \, x_i + x_{0j} \tag{3.4}$$

Similarly

$$x_j = \alpha_{i'j} \, x'_i + x'_{0j} \tag{3.5}$$

The basis vectors are also related

$$\begin{aligned} \mathbf{e}'_i &= \alpha_{i'j} \, \mathbf{e}_j \\ \mathbf{e}_i &= \alpha_{j'i} \, \mathbf{e}'_j \end{aligned} \tag{3.6}$$

From orthogonality condition

$$\mathbf{e}'_i\ \mathbf{e}'_k = \alpha_{i'j}\,\alpha_{k'j} = \delta_{ik}$$
$$\mathbf{e}_i\ \mathbf{e}_k = \alpha_{j'i}\,\alpha_{j'k} = \delta_{ik} \qquad (3.7)$$

For the transformation of a tensor **A**, we have

$$A'_{ij} = \alpha_{i'k}\,\alpha_{j'l}\,A_{kl}$$
$$A_{ij} = \alpha_{k'i}\,\alpha_{l'j}\,A'_{kl} \qquad (3.8)$$

Example: Suppose that we are interested in rotation of a Cartesian coordinate system about the z axis (here we write x,y,z instead of x_1, x_2, x_3). Let φ be the angle between the new x'–axis and the old x-axis. Then the general formula,

$$x'_i = T_{ij}\,x_j + x_{0i} \qquad (3.9)$$

Becomes

$$x = x\cos\varphi + y\sin\varphi$$
$$y' = -x\sin\varphi + y\cos\varphi \qquad (3.10)$$
$$z' = -z$$

Hence **T** becomes,

$$\mathbf{T} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (3.11)$$

This special form of the transformation matrix is often called the **rotation matrix**, denoted by **R**.

4 VECTOR CALCULUS

In vector calculus, we encounter the vector operator “del” or ∇. Del may be written in index notation as $\mathbf{e}_i \frac{\partial}{\partial x_i}$. It appears in the gradient, divergence, curl and laplacian of a scalar, vector or tensor.

4.1 GRADIENT

A vector often encountered in fluid mechanics is the **gradient of a scalar**, $\text{grad}\,\phi = \nabla\phi$.

Consider a scalar field ϕ, where $\phi(\mathbf{x})=\phi(x_1,x_2,x_3)$ is a function of the position vector from the origin **x**. The differential of ϕ is:

$$d\phi=\frac{\partial\phi}{\partial x_1}dx_1+\frac{\partial\phi}{\partial x_2}dx_2+\frac{\partial\phi}{\partial x_3}dx_3 \tag{4.1}$$

$$\frac{\partial\phi}{\partial x_i}dx_i=\delta_{ij}\frac{\partial\phi}{\partial x_j}dx_i=\frac{\partial\phi}{\partial x_j}\mathbf{e}_j\,\mathbf{e}_i\,dx_i=\nabla\phi.d\mathbf{x} \tag{4.2}$$

The gradient of ϕ is a vector,

$$\nabla\phi=\frac{\partial\phi}{\partial x_1}\mathbf{e}_1+\frac{\partial\phi}{\partial x_2}\mathbf{e}_2+\frac{\partial\phi}{\partial x_3}\mathbf{e}_3 \tag{4.3}$$

Or in index notation,

$$\{\text{grad}\,\phi\}_i=\{\nabla\phi\}_i\equiv\frac{\partial\phi}{\partial x_i} \tag{4.4}$$

Let us denote $ds = |d\mathbf{x}|$ as the differential distance along the unit vector **n**; then $dx = \mathbf{n}ds$ and,

$$d\phi=\nabla\phi.d\mathbf{x}=\mathbf{n}.\nabla\phi.ds\ ,$$

so that we obtain:

$$\frac{d\phi}{ds}=\mathbf{n}.\nabla\phi\ , \tag{4.5}$$

which is defined as the **directional derivative**. So, the directional derivative of ϕ in the direction characterized by the unit vector **n** is the projection of $\nabla\phi$ on **n**.

Example: If $r=\sqrt{x_k x_k}$ is the distance from the origin to some point, evaluate $\nabla\left(\frac{1}{r}\right)$. We use index notation to find the ith component.

$$\left\{\nabla\left(\frac{1}{r}\right)\right\}_i=\frac{\partial}{\partial x_i}\left(\frac{1}{\sqrt{x_k x_k}}\right)=-\frac{1}{2}\frac{\left(\frac{\partial x_k}{\partial x_i}\right)x_k}{\left(x_k x_k\right)^{3/2}}-\frac{1}{2}\frac{x_k\left(\frac{\partial x_k}{\partial x_i}\right)}{\left(x_k x_k\right)^{3/2}}=-\frac{\delta_{ik}x_k}{r^3}=-\frac{x_i}{r^3} \tag{114}$$

4. 2 THE PHYSICAL SIGNIFICANCE OF $\nabla\phi$

(1) Suppose $\phi(\mathbf{r})=C=consant$ represents a surface in space. Let $d\mathbf{r}$ denote any tangent

vector of the surface at point P. Because ϕ ia constant on the surface, we must have

$$d\phi = \nabla\phi . d\mathbf{r} = 0. \qquad (4.6)$$

Thus $\nabla\phi$ is a vector normal to $d\mathbf{r}$ and therefore perpendicular to the surface. Hence, the unit normal to the surface is given by $\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|}$ and points in the direction of increasing ϕ.

Example: Find the normal to the surface of the sphere of radius R centered on the origin,

$$\phi = x_i x_i = R^2 \qquad (4.6)$$

Hence,

$$\mathbf{n} = \frac{2x_i\mathbf{e}_i}{\sqrt{4x_i x_i}} = \frac{x_i\mathbf{e}_i}{R} \qquad (4.7)$$

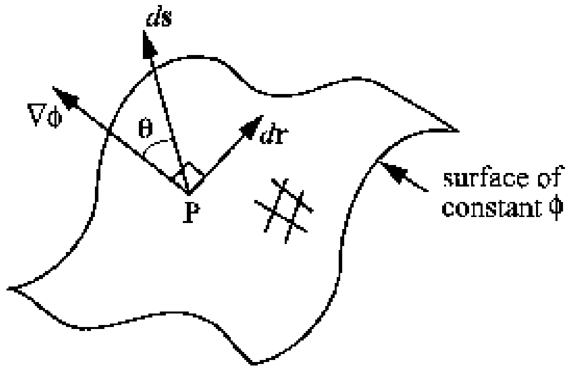

Figure 3. Physical significance of $\nabla\phi$.

(2) At any point P, the change of ϕ with position is the fastest along the direction of $\nabla\phi$. So the rate of change with distance s is its maximum.

$$\left|\frac{d\phi}{ds}\right|_{\max} = |\nabla\phi| \qquad (4.8)$$

in the direction normal to the surface ϕ = const at P. To prove this, let $\mathbf{n}$ be the unit normal vector at P, then

$$d\phi = \nabla\phi . d\mathbf{s} = |\nabla\phi|\mathbf{n}.d\mathbf{s} = |\nabla\phi|\cos\theta\, ds \qquad (4.9)$$

Where $|d\mathbf{s}| = ds$ and θ is the angle between $d\mathbf{s}$ and $\mathbf{n}$. Hence, $\left|\frac{d\phi}{ds}\right|$ is its maximum when $\mathbf{s}$

coincides with n and $\theta = 0$ or π. Then

$$\left|\frac{d\phi}{ds}\right|_{\max} = |\nabla\phi| \tag{4.10}$$

(1) The variation of ϕ in multidimensional space may be depicted as a series of equi-value surfaces. The gradient $\nabla\phi$ defines the direction along which the change of ϕ with position $\frac{d\phi}{ds}$ is greatest. The position at which $\nabla\phi = 0$, ϕ is maximum or minimum and is called **stationary point**. A curve in space which is everywhere tangent to the gradient $\nabla\phi$ is known **path of steepest ascent (descent).**

4. 3 THE GRADIENT OF VECTORS AND TENSORS

The differential for a vector v = $v_k\mathbf{e}_k$ is:

$$d\mathbf{v} = \frac{\partial\mathbf{v}}{\partial x_1}dx_1 + \frac{\partial\mathbf{v}}{\partial x_2}dx_2 + \frac{\partial\mathbf{v}}{\partial x_3}dx_3 = \frac{\partial\mathbf{v}}{\partial x_i}dx_i = \frac{\partial\mathbf{v}}{\partial x_j}\left(\mathbf{e}_j\,.\mathbf{e}_i\right)dx_i = \nabla\mathbf{v}\,.\,d\mathbf{x} \tag{4.11}$$

In Cartesian coordinate system,

$$\nabla\mathbf{v} = \frac{\partial}{\partial x_j}\left(v_i\,\mathbf{e}_i\right)\mathbf{e}_j = \frac{\partial v_i}{\partial x_j}\mathbf{e}_i\mathbf{e}_j \text{ or } \{\nabla\mathbf{v}\}_{ij} = \frac{\partial v_i}{\partial x_j} \tag{4.12}$$

The nine partial derivatives in the gradient of a vector are:

$$\nabla\mathbf{v} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} \tag{4.13}$$

The directional derivative of a vector is written

$$\frac{d\mathbf{v}}{ds} = \left(\mathbf{n}\,.\nabla\right)\,\mathbf{v} \tag{4.14}$$

$$\frac{dv_i}{ds} = n_k\frac{dv_i}{dx_k} \tag{4.15}$$

Note that the gradient of the position vector is the Kronecker delta

$$\delta_{ij} = \frac{\partial x_i}{\partial x_j} \tag{4.16}$$

The differential of a tensor $\mathbf{A} = A_{ij}\mathbf{e}_i\mathbf{e}_j$ is

$$dA_{ij} = \frac{\partial A_{ij}}{\partial x_k} dx_k \text{ or } d\mathbf{A} = \nabla\mathbf{A}.d\mathbf{x} \tag{4.17}$$

The gradient of **A** is

$$d\mathbf{A} = \frac{\partial}{\partial x_k}\left(A_{ij}\mathbf{e}_i\mathbf{e}_j\right)\mathbf{e}_k = \frac{\partial A_{ij}}{\partial x_k}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k \text{ or } \{\nabla\mathbf{A}\}_{ijk} = \frac{\partial \mathbf{A}_{ij}}{\partial x_k} \tag{4.18}$$

Notice that the gradient operator always raises the rank of the tensor by one (the gradient of a tensor is third rank tensor).

4. 4 DIVERGENCE

The divergence of a vector **v** and a tensor **A** is

$$\text{div } \mathbf{v} = \nabla.\mathbf{v} = \frac{\partial}{\partial x_j}\left(v_i\mathbf{e}_i\right).\mathbf{e}_j = \frac{\partial v_i}{\partial x_j}\left(\mathbf{e}_i.\mathbf{e}_j\right) = \frac{\partial v_i}{\partial x_j}\delta_{ij} = \frac{\partial v_i}{\partial x_i} \tag{4.19}$$

$$\nabla.\mathbf{v} = \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \tag{4.20}$$

If $\nabla.\mathbf{v} = 0$ then **v** is **solenoid.**

$$\text{div } \mathbf{A} = \nabla.\mathbf{A} = \frac{\partial}{\partial x_k}\left(A_{ij}\mathbf{e}_i\mathbf{e}_j\right).\mathbf{e}_k = \frac{\partial A_{ij}}{\partial x_k}\mathbf{e}_i\,\mathbf{e}_j.\mathbf{e}_k = \frac{\partial A_{ij}}{\partial x_k}\delta_{jk}\mathbf{e}_i = \frac{\partial A_{ij}}{\partial x_j}\mathbf{e}_i \tag{4.21}$$

$$\{\text{div } \mathbf{A}\}_j = \{\nabla.A\}_j \equiv \frac{\partial A_{ij}}{\partial x_j}\begin{bmatrix} \frac{\partial A_{11}}{\partial x_1} & \frac{\partial A_{12}}{\partial x_2} & \frac{\partial A_{13}}{\partial x_3} \\ \frac{\partial A_{21}}{\partial x_1} & \frac{\partial A_{22}}{\partial x_2} & \frac{\partial A_{23}}{\partial x_3} \\ \frac{\partial A_{31}}{\partial x_1} & \frac{\partial A_{32}}{\partial x_2} & \frac{\partial A_{33}}{\partial x_3} \end{bmatrix} \tag{4.22}$$

Notice that the contraction process performed by the divergence operator reduces the rank of the tensor by one (a vector becomes a scalar).

4.5 CURL

The curl of vector **v** is defined by

$$\mathbf{s} = \text{curl}\,\mathbf{v} = \nabla\times\mathbf{v} = \mathbf{e}_j \frac{\partial}{\partial x_j}\times v_k\mathbf{e}_k = \mathbf{e}_j\times\mathbf{e}_k \frac{\partial v_k}{\partial x_j} = \varepsilon_{ijk}\frac{\partial v_k}{\partial x_j}\mathbf{e}_i \tag{4.23}$$

$$\{\nabla\times\mathbf{v}\}_i \equiv \varepsilon_{ijk}\frac{\partial v_k}{\partial x_j} \tag{4.24}$$

Hence,

$$s_1 = \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\,,\; s_2 = \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\,,\; s_3 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \tag{4.25}$$

If $\nabla\times\mathbf{v}=0$ then **v** is **irrotational**. If **v** denotes the velocity vector, then $\mathbf{w}=\nabla\times\mathbf{v}$ is the **vorticity**, a pseudo vector that will encounter later.

4.6 LAPLACIAN

The **Laplacian**, $\nabla^2\phi$, of a scalar, ϕ, is defined as:

$$\nabla^2\phi = \nabla.\nabla\phi = \mathbf{e}_i \frac{\partial}{\partial x_i}.\mathbf{e}_j\frac{\partial\phi}{\partial x_j} = \mathbf{e}_i\mathbf{e}_j\frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial x_j}\right) = \frac{\partial^2\phi}{\partial x_i^2} = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} \tag{4.26}$$

In the above definition, ϕ can be replaced by a vector or tensor of any order.

Note: The expressions for the operations defined above apply only to Cartesian coordinate system. In curvilinear coordinate systems (e.g. cylindrical, spherical, etc…) the expressions for the operators div, curl, etc. are more complicated and out of scope of this chapter.

4.7 SOME VECTOR IDENTITIES

The following identities may be proved in rectilinear coordinates. According to a general theorem, once the result is true in one coordinate system, it is true in other coordinates.

$$
\begin{aligned}
&\nabla.(\phi\mathbf{u})=\phi(\nabla.\mathbf{u})+\mathbf{u}.(\nabla\phi)\\
&\nabla\times\phi\mathbf{u}=\phi(\nabla\times\mathbf{u})+\nabla\phi\times\mathbf{u}\\
&\nabla.(\mathbf{u}\times\mathbf{v})=\mathbf{v}.(\nabla\times\mathbf{u})-\mathbf{u}.(\nabla\times\mathbf{v})\\
&\nabla\times(\mathbf{u}\times\mathbf{v})=\mathbf{v}.\nabla\mathbf{u}-\mathbf{u}.\nabla\mathbf{v}+\mathbf{u}.(\nabla.\mathbf{v})-\mathbf{v}.(\nabla.\mathbf{u})\\
&\nabla.(\mathbf{u}.\mathbf{v})=\mathbf{v}.\nabla\mathbf{u}+\mathbf{u}.\nabla\mathbf{v}+\mathbf{u}\times(\nabla.\mathbf{v})+\mathbf{v}\times(\nabla.\mathbf{u})\\
&\nabla\times\nabla\phi=0\\
&\nabla\times(\nabla\times\mathbf{u})=\nabla(\nabla.\mathbf{u})-\nabla^2\mathbf{u}\\
&\nabla.(\nabla\psi_1\times\nabla\psi_2)=0
\end{aligned}
\tag{4.27}
$$

4. 8 LAMINAR AND SOLENOIDAL VECTOR FIELDS

A vector field $\mathbf{u}(\mathbf{r})$ is called lamellar (irrotational) if $\nabla\times\mathbf{u}=0$ and is called solenoidal (incompressible) if $\nabla.\mathbf{u}=0$. By the identity $\nabla\times\nabla\phi=0$ (from equation (137)) we may express an irrotational vector field $\mathbf{u}$ by

$$\mathbf{u}=\nabla\phi \tag{4.28}$$

The the condition $\nabla\times\mathbf{u}=0$ is satisfies and multiple component vector field is given by a single scalar function ϕ. If $\mathbf{u}$ is solenoidal, $\mathbf{u}$ may be written as

$$\mathbf{u}=\nabla\times\mathbf{a} \tag{4.29}$$

By the identity $\nabla.\nabla\times\mathbf{a}=0$, the condition $\nabla.\mathbf{u}=0$ is satisfied. The functions ϕ and $\mathbf{a}$ are called the scalar and vector potentials, respectively.

Theorem (without proof): Any vector field $\mathbf{u}(\mathbf{r})$ may be written as the sum of a lamellar field and a solenoidal field,

$$\mathbf{u}=\nabla\phi+\nabla\times\mathbf{a} \tag{4.30}$$

5 INTEGRAL THEOREMS

5. 1 THE DIVERGENCE THEOREM

The divergence theorem is sometimes referred to as Gauss' theorem and is an extension of the familiar result,

$$\int_a^b \frac{df}{dx}dx=f(a)-f(b) \tag{5.1}$$

Consider a closed piece-wise smooth surface ∂D, which encloses a domain D. Let **n** be the unit outer normal to ∂D and let **u** be any vector whose components and their first derivatives are finite and continuous everywhere within D and on ∂D.

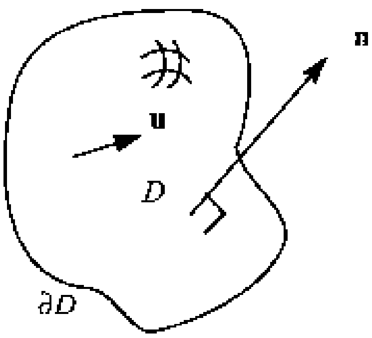

Figure 4. Definition of the divergence theorem

Then, the divergence theorem states that

$$\int_D \mathrm{div}\,\mathbf{u}\,dV = \int_{\partial D} (\mathbf{n}\,.\mathbf{u})\,dA \qquad (5.2)$$

or

$$\int_D \frac{\partial u_i}{\partial x_i}\,dV = \int_{\partial D} n_i u_i\,dA \qquad (5.3)$$

This gives a relation between a volume integral and a surface integral and states, surprisingly enough, that the volume integral of div**u** depends only on the values of **u** on the surface enclosing the domain D. The divergence theorem is essentially a statement of conservation. Sources and sinks in the volume (on the left hand side of equation (5.3)) are balanced by the flux through the surface (on the right hand side of equation (5.3)).

There are two extensions to this theorem:

(a) If $B_{jkl\ldots}$ is a tensor of arbitrary order, then provided its components and their first derivatives are finite and continuous,

$$\int_D \frac{\partial}{\partial x_i} B_{jkl\ldots}\,dV = \int_{\partial D} (n_i B_{jkl\ldots})\,dA \qquad (5.4)$$

(b) If the domain D is contained between two closed surfaces ∂D and $\partial D'$:

Then

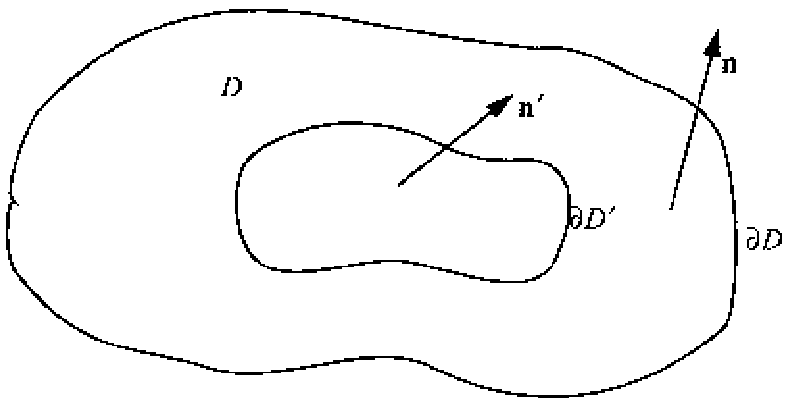

Figure 5 Extension of the divergence theorem.

$$\int_D \frac{\partial}{\partial x_i} B_{jkl\ldots} dV = \int_{\partial D} \left(n_i B_{jkl\ldots} \right) dA - \int_{\partial D'} \left(n'_i \, B_{jkl\ldots} \right) dA \qquad (5.5)$$

Where n' is the outer normal to $\partial D'$. This latter result can readily be extended to the case where the domain D is contained in the region between several closed surfaces.

5. 2 GREEN'S THEOREM

If f and ϕ are two scalar functions which are continuous with continuous first and second derivatives, then

$$\int_D \left\{ f \nabla^2 \phi - \phi \nabla^2 f \right\} dV = \int_{\partial D} \left\{ f \frac{\partial \phi}{\partial x_i} - \phi \frac{\partial f}{\partial x_i} \right\} n_i dA \qquad (5.6)$$

5. 3 STOKES' THEOREM

Let C be closed curve which coincides with the edge of the surface S. Then:

$$\oint_C u_i dx_i = \varepsilon_{ijk} \int_S n_i \left(\frac{\partial u_k}{\partial x_j} \right) dA = \int_S \left(\left(\nabla \times \mathbf{u} \right) dA \right) \qquad (5.7)$$

Where $d\mathbf{x}$ is everywhere tangent to C and the integration is in the counter-clockwise direction. Thus Stokes' theorem provides us with a relation between a surface and a line integral. When **u** is the velocity, then the line integral is known as the **circulation**.

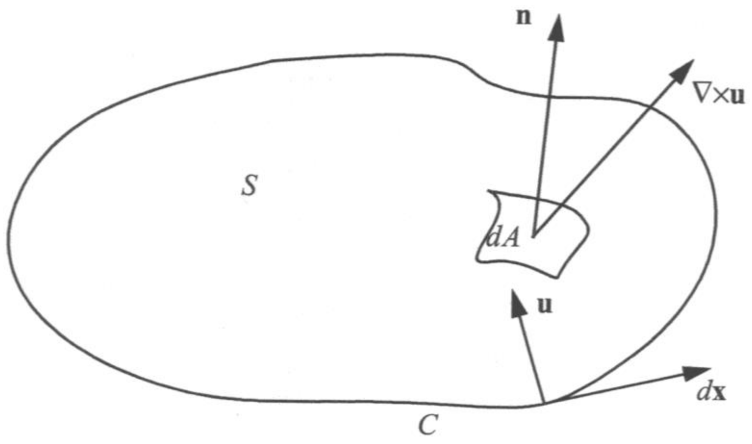

Figure 6 Application of Stokes' theorem

In 2-dimensions, equation (5.7) reduces to Green's theorem in a plane. In rectangular coordinates, Green's theorem reads,

$$\int_A \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) dx_1 dx_2 = \oint_C \left(u_1 dx_1 - u_2 dx_2 \right) \qquad (5.8)$$

5. 4 LEIBNITZ'S THEOREM

Let $B_{ij\ldots}(x_i, t)$ be any time be any time dependent scalar, vector or tensor field. Suppose the volume integral

$$I_{ij\ldots}(t) = \int_{D(t)} B_{ij\ldots}(x_i, t) dV \qquad (5.9)$$

Is over domain of integration that is a function of time $D(t)$. Let v_i be th velocity of the surface $\partial D(t)$. The Leibnitz theorem then allows us to find $\frac{dI_{ij\ldots}}{dt}$ as follows

$$\frac{d}{dt}\int_{D(t)} B_{ij\ldots}(x_i,t)dV = \int_{D(t)} \frac{\partial}{\partial t} B_{ij\ldots}(x_i,t)dV + \int_{\partial D(t)} n_k v_k B_{ij\ldots}(x_i,t)dS \qquad (5.10)$$

The surface integral takes into account the motion of the boundary, e. how fast $B_{ij\ldots}$is coming into $D(t)$ because of the surface velocity v The one dimensional version of Leibnitz theorem is given by

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx + \frac{db}{dt} f(b,t) - \frac{da}{dt} f(a,t) \qquad (5.11)$$