Akışkanlar Mekaniği: Temelleri ve Uygulamaları, 2nd Edition
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Bölüm 9

DIFERANSİYEL AKIŞ ANALİZİ

Bu bölümde akışkan hareketinin temel denklemleri türetilir ve basit bazı akışlar için analitik olarak nasıl çözüleceği gösterilir. Daha karmaşık akışlar örneğin resimde gösterilen bir tornado tarafından tetiklenen hava akışının kesin çözümü yoktur.

Amaçlar

- Kütle ve momentum korumununa ait diferansiyel denklemlerin nasıl türetildiğini anlayabilmelisiniz.
- Akım fonksiyonu ve basınç alanını hesaplayabilmeli ve verilen bir hız alanı için akım çizgilerini çizebilmelisiniz
- Basit akışlar için hareket denklemlerinin analitik çözümlerini elde edebilmelisiniz.

9–1 ■ GİRİŞ

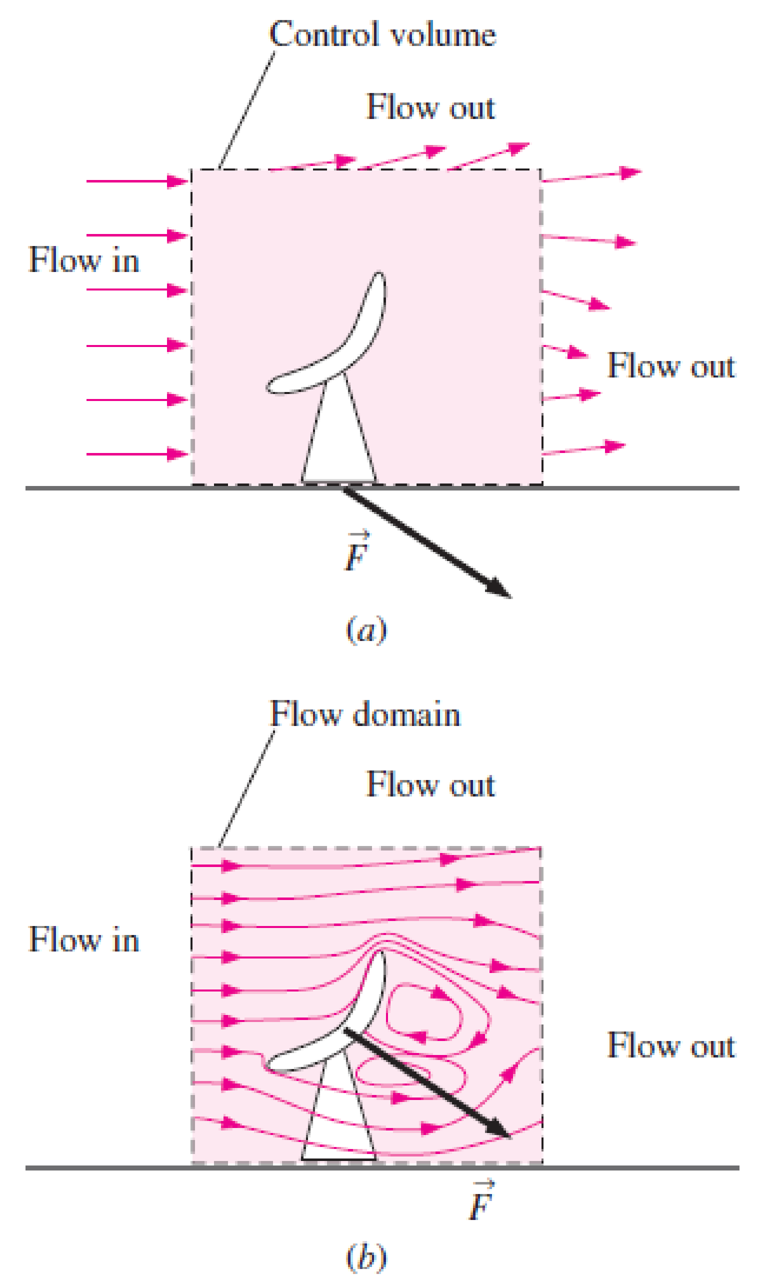
Kontrol hacmi tekniği, kontrol hacmine giren ve kontrol hacminden çıkan kütlesel debiler veya cisimler üzerine uygulanan kuvvetler gibi bir akışın genel özellikleriyle ilgilendiğimizde yararlıdır.

Diferansiyel analiz, akışkan hareketinin diferansiyel denklemlerinin **akış bölgesi** olarak adlandırılan bir bölge boyunca akış alanındaki *her* noktaya uygulanmasını gerektirir.

Değişkenlerin **sınır şartları** da, girişleri, cıkışları ve katı çeperleri de içine alacak şekilde *akış bölgesinin tüm sınırlarında belirtilmelidir.*

Öte yandan eğer akış daimi değilse, akış alanının değişmeye devam ettiği zaman boyunca çözümümüzü yürütmek gereklidir.

(a) Kontrol hacmi analizinde kontrol hacminin içi bir kara kutu gibi ele alınır, fakat (b) diferansiyel analizde akışın *film* detayları akış bölgesindeki *her bir* noktada çizilir.



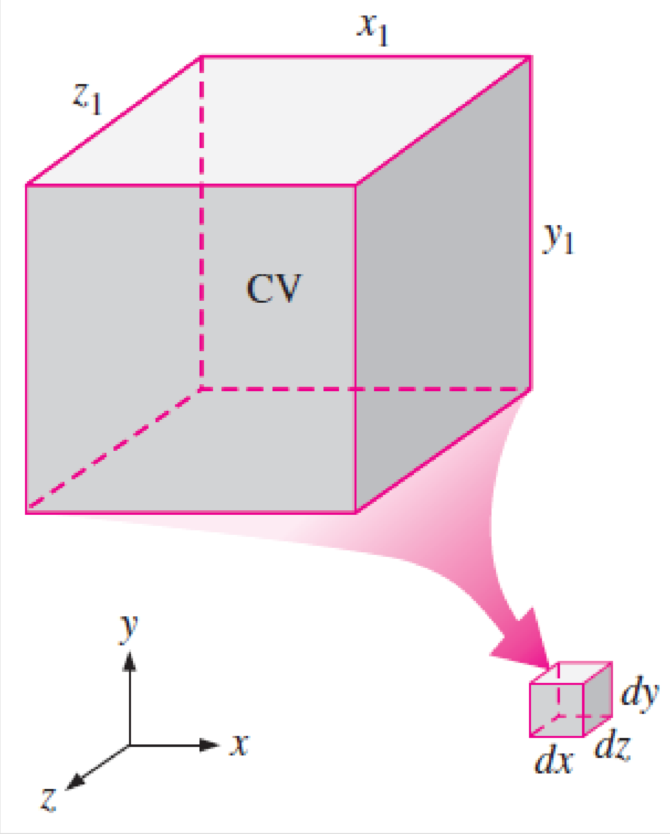
9–2 ■ KÜTLENİN KORUNUMU VE SÜREKLİLİK DENKLEMİ

Conservation of mass for a CV:

$$0 = \int_{CV} \frac{\partial \rho}{\partial t}\, dV + \int_{CS} \rho \vec{V} \cdot \vec{n}\, dA$$

$$\int_{CV} \frac{\partial \rho}{\partial t}\, dV = \sum_{in} \dot{m} - \sum_{out} \dot{m}$$

Kontrol hacmi içerisindeki kütlenin birim zamandaki net değişim hızı, kontrol hacmine giren ve kontrol hacminden çıkan kütlesel debilerin farkına eşittir.



Bir diferansiyel korunum denklemini türetmek kontrol hacmini sonsuz küçük boyuta küçülttüğümüzü düşünüyoruz.

Diverjans Teoremini Kullanarak Türetme

Kütlenin korunumunun diferansiyel formunu türetmenin en hızlı ve anlaşılması en kolay yolu **diverjans teoremini** uygulamaktır

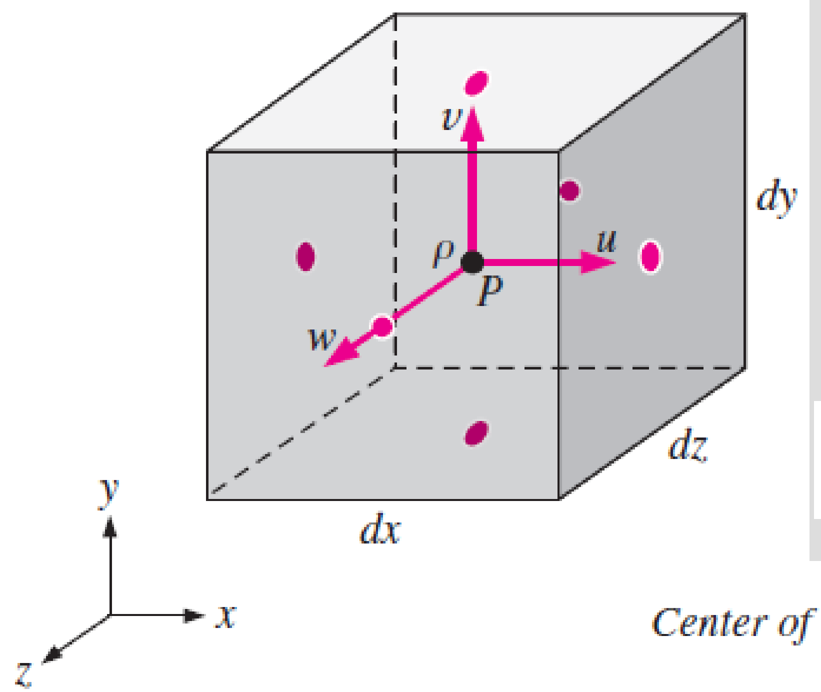
Divergence theorem: $$\int_V \vec{\nabla} \cdot \vec{G}\, dV = \oint_A \vec{G} \cdot \vec{n}\, dA$$

$$0 = \int_{CV} \frac{\partial \rho}{\partial t}\, dV + \int_{CV} \vec{\nabla} \cdot \left(\rho \vec{V}\right) dV$$

$$\int_{CV} \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V}\right)\right] dV = 0$$

Continuity equation: $$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0$$

Bu denklem, henüz sıkıştırılamaz akış kabulü yapmadığımız için süreklilik denkleminin sıkıştırılabilir akış için geçerli bir formudur ve akış alanındaki her noktada geçerlidir.



Sonsuz küçük kontrol hacmini kullanarak türetme

Kutu merkezinden uzak noktalar için kutu merkezi civarındaki **Taylor serisi açılımını** kullanalım.

$$(\rho u)_{\text{center of right face}} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} + \frac{1}{2!}\frac{\partial^2(\rho u)}{\partial x^2}\left(\frac{dx}{2}\right)^2 + \cdots$$

Kartezyen koordinatlarda kütlenin korunumununa ait diferansiyel denklemin türetilmesinde merkezi *P* noktası olan küçük bir kutu şeklindeki kontrol hacmi kullanılır: Mavi noktalar her bir yüzün merkezini gostörmektedir.

Center of right face: $$(\rho u)_{\text{center of right face}} \cong \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}$$

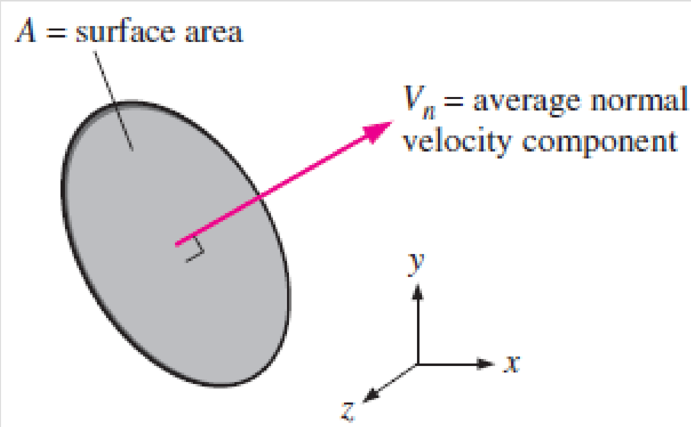
Center of left face: $$(\rho u)_{\text{center of left face}} \cong \rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}$$

Center of front face: $$(\rho w)_{\text{center of front face}} \cong \rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}$$

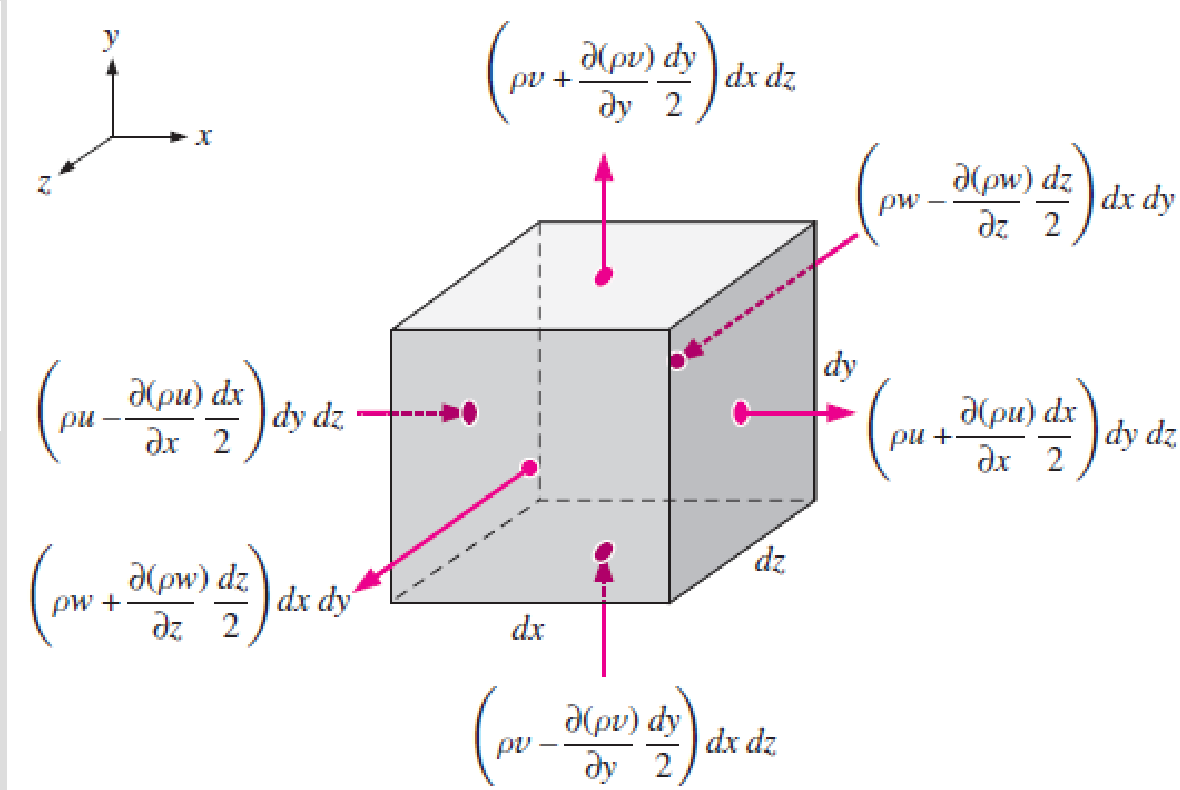
Center of rear face: $$(\rho w)_{\text{center of rear face}} \cong \rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}$$

Center of top face: $$(\rho v)_{\text{center of top face}} \cong \rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}$$

Center of bottom face: $$(\rho v)_{\text{center of bottom face}} \cong \rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}$$



Bir yüzeyden geçen kütlesel debi $\rho V_n A$ 'ya eşittir.



Rate of change of mass within CV:

$$\int_{CV} \frac{\partial \rho}{\partial t}\, dV \cong \frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

Diferansiyel kontrol hacminin her bir yüzünden kütle girişi ve çıkışı; mavi noktalar her bir yüzün merkezini göstermektedir.

Net mass flow rate into CV:

$$\sum_{\text{in}} \dot{m} \cong \underbrace{\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\, dz}_{\text{left face}} + \underbrace{\left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\, dz}_{\text{bottom face}} + \underbrace{\left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\, dy}_{\text{rear face}}$$

Net mass flow rate out of CV:

$$\sum_{\text{out}} \dot{m} \cong \underbrace{\left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\, dz}_{\text{right face}} + \underbrace{\left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\, dz}_{\text{top face}} + \underbrace{\left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\, dy}_{\text{front face}}$$

$$\frac{\partial \rho}{\partial t} dx\, dy\, dz = -\frac{\partial(\rho u)}{\partial x} dx\, dy\, dz - \frac{\partial(\rho v)}{\partial y} dx\, dy\, dz - \frac{\partial(\rho w)}{\partial z} dx\, dy\, dz$$

Continuity equation in Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

The Divergence Operation

Cartesian coordinates:

$$\vec{\nabla} \cdot (\rho \vec{V}) = \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)$$

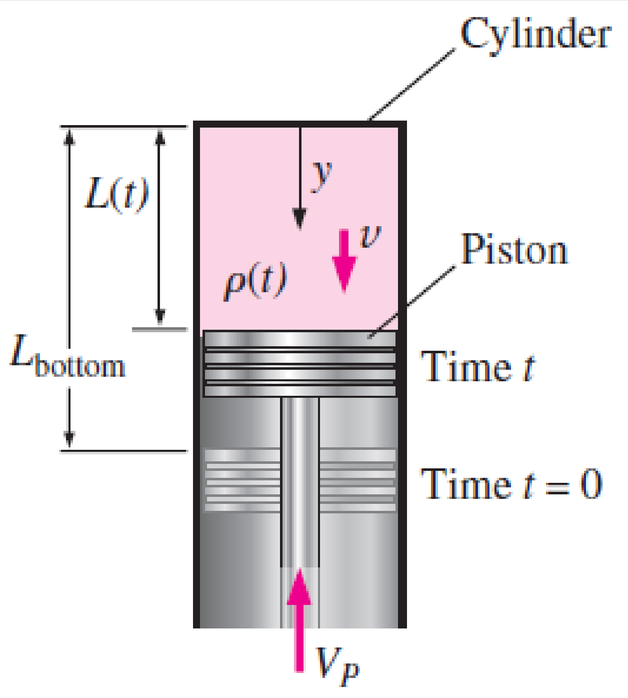
Cylindrical coordinates:

$$\vec{\nabla} \cdot (\rho \vec{V}) = \frac{1}{r}\frac{\partial(r \rho u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho u_\theta)}{\partial \theta} + \frac{\partial(\rho u_z)}{\partial z}$$

Kartezyen ve silindirik koordinatlarda diverjans işlemi.

Örnek 1: Hava-Yakıt Karışımının Sıkıtırılması

İçten yanmalı bir motorun silindirinde hava-yakıt karışımı bir piston ile sıkıştırılmaktadır. y koordinatının başlangıcı şekilde gösterildiği gibi silindirin tavanında ve yönü aşağı doğrudur. Pistonun yukarıya doğru sabit bir v_p hızıyla gittiği varsayılmaktadır. Silindirin tavanı ile piston kafası arasındaki L mesafesi $L=L_{bottom}-V_p t$ ilişkisi uyarınca zamanla azalmakta olup burada L_{bottom} şekilde gosterildigi gibi t=0 anında piston alt ölü noktadayken pistonun konumunu gostermektedir. t=0 anında hava-yakıt karışımının yoğunluğunun silindir içerisinde heryerde aynı ve ρ(0) olduğu bilindigine göre, pistonun yukarı çıkışı sırasında hava-yakıt karışımının yoğunluğunu zamanın ve verilen parametrelerin fonksiyonu olarak elde ediniz.



İçten yanmalı bir havanın pistonla motorun silindirinde yakıt ve havanın sıkıştırılması

Finally then, we have the desired expression for ρ as a function of time,

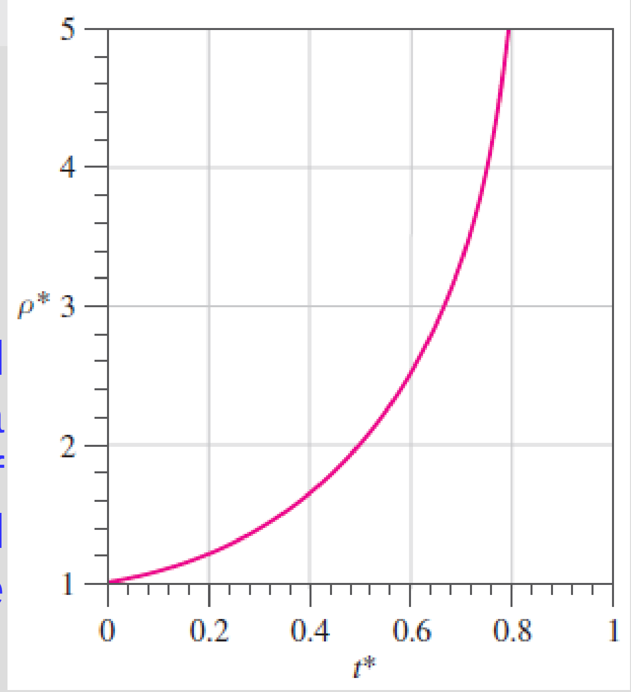
$$\rho = \rho(0)\frac{L_{\text{bottom}}}{L_{\text{bottom}} - V_p t} \tag{4}$$

In keeping with the convention of nondimensionalizing results, Eq. 4 is rewritten as

$$\frac{\rho}{\rho(0)} = \frac{1}{1 - V_p t/L_{\text{bottom}}} \quad \rightarrow \quad \rho^* = \frac{1}{1 - t^*} \tag{5}$$

where $\rho^* = \rho/\rho(0)$ and $t^* = V_p t/L_{\text{bottom}}$. Equation 5 is plotted in Fig. 9–8.

Discussion At $t^* = 1$, the piston hits the top of the cylinder and ρ goes to infinity. In an actual internal combustion engine, the piston stops before reaching the top of the cylinder, forming what is called the *clearance volume*, which typically constitutes 4 to 12 percent of the maximum cylinder volume. The assumption of uniform density within the cylinder is the weakest link in this simplified analysis. In reality, ρ may be a function of both space and time.



Nondimensional density as a function of nondimensional time for Example 9–1.

Analysis First we need to establish an expression for velocity component v as a function of y and t. Clearly $v = 0$ at $y = 0$ (the top of the cylinder), and $v = -V_P$ at $y = L$. For simplicity, we approximate that v varies linearly between these two boundary conditions,

$$\textit{Vertical velocity component:} \quad v = -V_P \frac{y}{L} \tag{1}$$

where L is a function of time, as given. The compressible continuity equation in Cartesian coordinates (Eq. 9–8) is appropriate for solution of this problem.

$$\frac{\partial \rho}{\partial t} + \underbrace{\frac{\partial(\rho u)}{\partial x}}_{0 \text{ since } u=0} + \frac{\partial(\rho v)}{\partial y} + \underbrace{\frac{\partial(\rho w)}{\partial z}}_{0 \text{ since } w=0} = 0 \quad \rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial y} = 0$$

By assumption 1, however, density is not a function of y and can therefore come out of the y-derivative. Substituting Eq. 1 for v and the given expression for L, differentiating, and simplifying, we obtain

$$\frac{\partial \rho}{\partial t} = -\rho\frac{\partial v}{\partial y} = -\rho\frac{\partial}{\partial y}\left(-V_P\frac{y}{L}\right) = \rho\frac{V_P}{L} = \rho\frac{V_P}{L_{\text{bottom}} - V_P t} \tag{2}$$

By assumption 1 again, we replace $\partial\rho/\partial t$ by $d\rho/dt$ in Eq. 2. After separating variables we obtain an expression that can be integrated analytically,

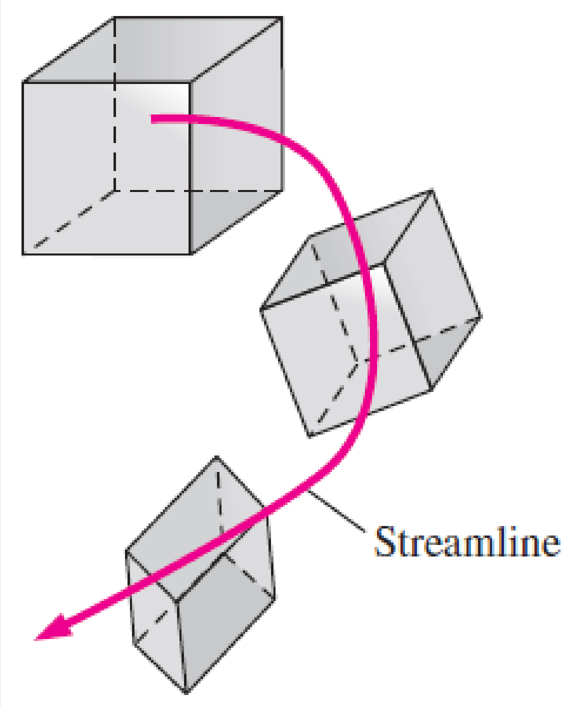
$$\int_{\rho=\rho(0)}^{\rho} \frac{d\rho}{\rho} = \int_{t=0}^{t} \frac{V_P}{L_{\text{bottom}} - V_P t}\,dt \quad \rightarrow \quad \ln\frac{\rho}{\rho(0)} = \ln\frac{L_{\text{bottom}}}{L_{\text{bottom}} - V_P t} \tag{3}$$

Süreklilik denkleminin alternatif formu

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = \underbrace{\frac{\partial \rho}{\partial t} + \vec{V} \cdot \vec{\nabla} \rho}_{\text{Material derivative of } \rho} + \rho \vec{\nabla} \cdot \vec{V} = 0$$

Alternative form of the continuity equation:

$$\frac{1}{\rho} \frac{D\rho}{Dt} + \vec{\nabla} \cdot \vec{V} = 0 \qquad \text{(9–10)}$$



Bir maddesel eleman akış alanında hareket ederken yoğunluğu Denklem 9-10 uyarınca değişir.

Equation 9–10 shows that as we follow a fluid element through the flow field (we call this a **material element**), its density changes as $\vec{\nabla} \cdot \vec{V}$ changes (Fig. 9–9). On the other hand, if changes in the density of the material element are negligibly small compared to the magnitudes of the velocity gradients in $\vec{\nabla} \cdot \vec{V}$ as the element moves around, $\rho^{-1} D\rho/Dt \cong 0$, and the flow is approximated as **incompressible**.

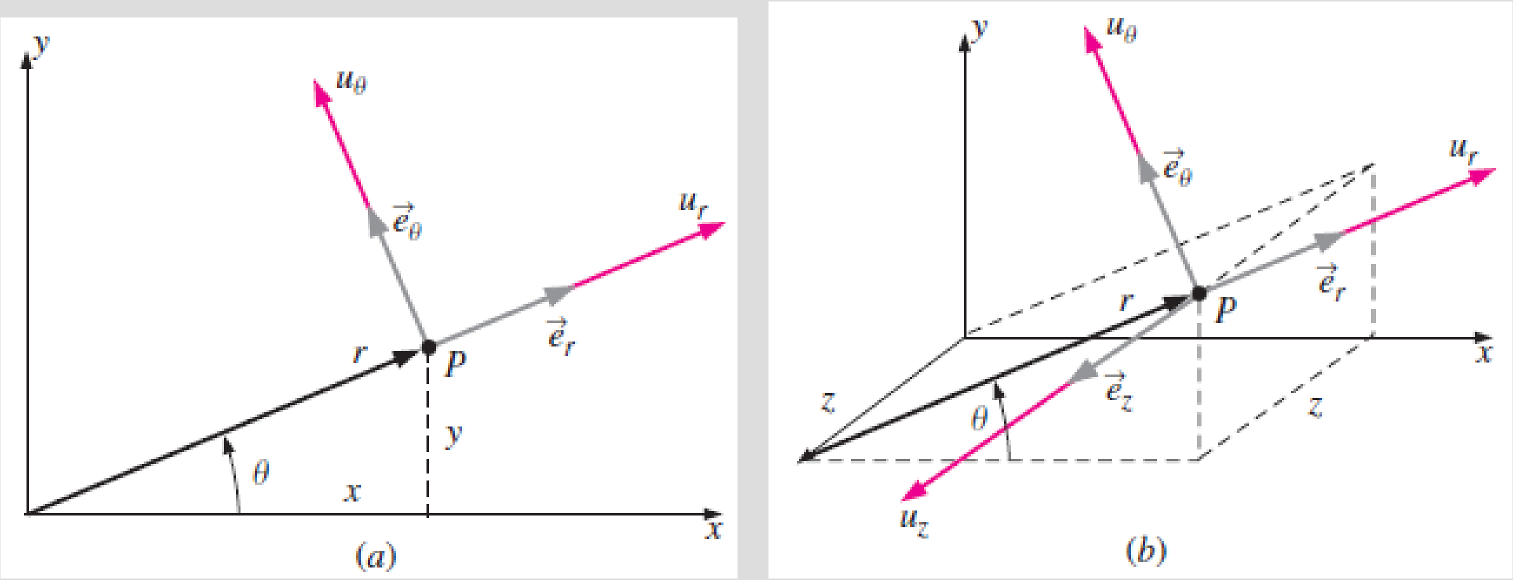
Silindirik Koordinatlarda Süreklilik Denklemi

Coordinate transformations:

$$r = \sqrt{x^2 + y^2} \qquad x = r\cos\theta \qquad y = r\sin\theta \qquad \theta = \tan^{-1}\frac{y}{x}$$

Continuity equation in cylindrical coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\rho u_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho u_\theta)}{\partial \theta} + \frac{\partial (\rho u_z)}{\partial z} = 0$$



Silindirik koordinartlarda hız bileşenleri ve birim vektörler: (a) xy- veya $r\theta$ düzleminde iki boyutlu akış, (b) üç boyutlu akış

Süreklilik Denkleminin Özel Durumları

Özel Durum 1: Daimi Sıkıştırılabilir Akış

Steady continuity equation: $\vec{\nabla} \cdot (\rho \vec{V}) = 0$ **(9–13)**

In Cartesian coordinates, Eq. 9–13 reduces to

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

In cylindrical coordinates, Eq. 9–13 reduces to

$$\frac{1}{r}\frac{\partial(r\rho u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho u_\theta)}{\partial \theta} + \frac{\partial(\rho u_z)}{\partial z} = 0$$

Özel Durum 2: Sıkıştırılamaz Akış

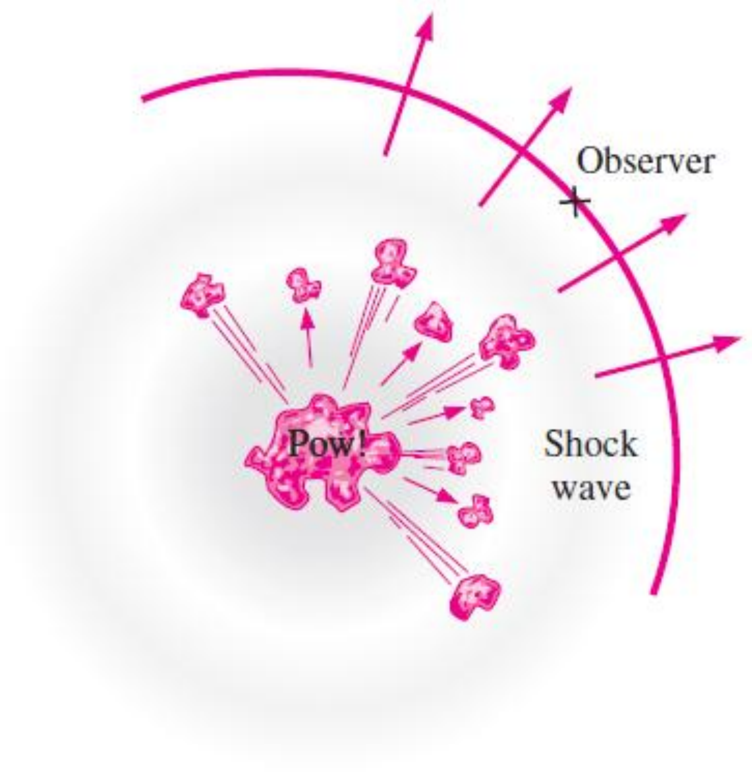
Incompressible continuity equation: $\vec{\nabla} \cdot \vec{V} = 0$

Incompressible continuity equation in Cartesian coordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Incompressible continuity equation in cylindrical coordinates:

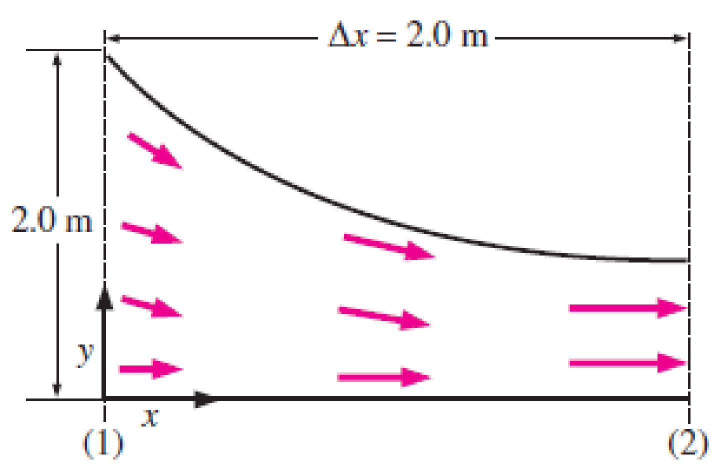
$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial (u_\theta)}{\partial \theta} + \frac{\partial (u_z)}{\partial z} = 0$$



Bir patlamadan sonra oluşan düzensizlik şok dalgası gözlemciye ulaşıncaya kadar hissedilmez.

EXAMPLE 9–2 Design of a Compressible Converging Duct

A two-dimensional converging duct is being designed for a high-speed wind tunnel. The bottom wall of the duct is to be flat and horizontal, and the top wall is to be curved in such a way that the axial wind speed u increases approximately linearly from $u_1 = 100$ m/s at section (1) to $u_2 = 300$ m/s at section (2) (Fig. 9–12). Meanwhile, the air density ρ is to decrease approximately linearly from $\rho_1 = 1.2\ \text{kg/m}^3$ at section (1) to $\rho_2 = 0.85\ \text{kg/m}^3$ at section (2). The converging duct is 2.0 m long and is 2.0 m high at section (1). (*a*) Predict the *y-component* of velocity, $v(x, y)$, in the duct. (*b*) Plot the approximate shape of the duct, ignoring friction on the walls. (*c*) How high should the duct be at section (2), the exit of the duct?



Yüksek-hızlı bir rüzgar tüneli için tasarlanmış yakınsak kanal (çizim ölçekli değildir).

Properties The fluid is air at room temperature (25°C). The speed of sound is about 346 m/s, so the flow is subsonic, but compressible.

Analysis (*a*) We write expressions for *u* and ρ, forcing them to be linear in *x*,

$$u = u_1 + C_u x \qquad \text{where} \qquad C_u = \frac{u_2 - u_1}{\Delta x} = \frac{(300 - 100)\ \text{m/s}}{2.0\ \text{m}} = 100\ \text{s}^{-1} \tag{1}$$

and

$$\rho = \rho_1 + C_\rho x \qquad \text{where} \qquad C_\rho = \frac{\rho_2 - \rho_1}{\Delta x} = \frac{(0.85 - 1.2)\ \text{kg/m}^3}{2.0\ \text{m}} = -0.175\ \text{kg/m}^4 \tag{2}$$

The steady continuity equation (Eq. 9–14) for this two-dimensional compressible flow simplifies to

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \underbrace{\frac{\partial(\rho w)}{\partial z}}_{0\ (2\text{-D})} = 0 \quad \rightarrow \quad \frac{\partial(\rho v)}{\partial y} = -\frac{\partial(\rho u)}{\partial x} \tag{3}$$

Substituting Eqs. 1 and 2 into Eq. 3 and noting that C_u and C_ρ are constants,

$$\frac{\partial(\rho v)}{\partial y} = -\frac{\partial[(\rho_1 + C_\rho x)(u_1 + C_u x)]}{\partial x} = -(\rho_1 C_u + u_1 C_\rho) - 2C_u C_\rho x$$

Integration with respect to *y* gives

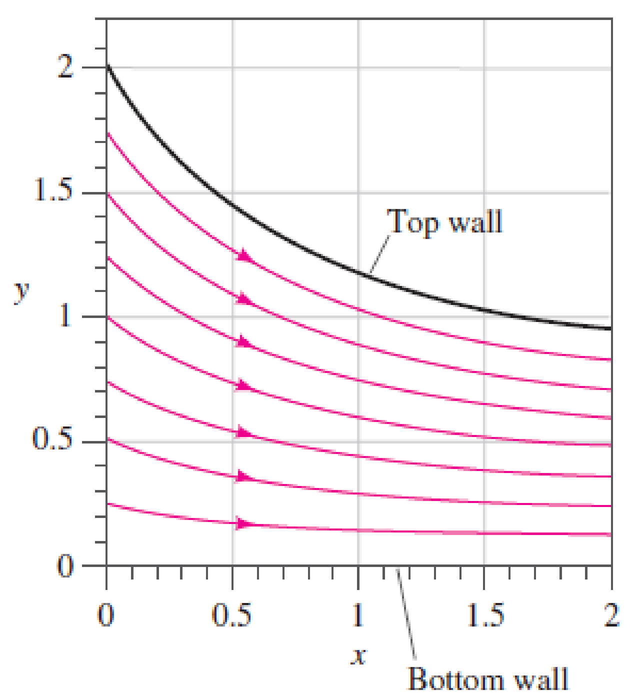
$$\rho v = -(\rho_1 C_u + u_1 C_\rho)y - 2C_u C_\rho xy + f(x) \tag{4}$$

Note that since the integration is a *partial* integration, we have added an arbitrary function of x instead of simply a constant of integration. Next, we apply boundary conditions. We argue that since the bottom wall is flat and horizontal, v must equal zero at $y = 0$ for any x. This is possible only if $f(x) = 0$. Solving Eq. 4 for v gives

$$v = \frac{-(\rho_1 C_u + u_1 C_\rho)y - 2C_u C_\rho xy}{\rho} \quad \rightarrow \quad v = \frac{-(\rho_1 C_u + u_1 C_\rho)y - 2C_u C_\rho xy}{\rho_1 + C_\rho x} \tag{5}$$

(*b*) Using Eqs. 1 and 5 and the technique described in Chap. 4, we plot several streamlines between $x = 0$ and $x = 2.0$ m in Fig. 9–13. The streamline starting at $x = 0$, $y = 2.0$ m approximates the top wall of the duct.

(*c*) At section (2), the top streamline crosses $y = 0.941$ m at $x = 2.0$ m. Thus, the predicted height of the duct at section (2) is **0.941 m**.



Örnek 9–2’deki yakınsak kanala ait akım çizgileri.

EXAMPLE 9–3 **Incompressibility of an Unsteady Two-Dimensional Flow**

Consider the velocity field of Example 4–5—an unsteady, two-dimensional velocity field given by $\vec{V} = (u, v) = (0.5 + 0.8x)\vec{i} + [1.5 + 2.5 \sin(\omega t) - 0.8y]\vec{j}$, where angular frequency ω is equal to 2π rad/s (a physical frequency of 1 Hz). Verify that this flow field can be approximated as incompressible.

SOLUTION We are to verify that a given velocity field is incompressible.
Assumptions **1** The flow is two-dimensional, implying no *z*-component of velocity and no variation of u or v with z.
Analysis The components of velocity in the *x*- and *y*-directions, respectively, are

$$u = 0.5 + 0.8x \qquad \text{and} \qquad v = 1.5 + 2.5 \sin(\omega t) - 0.8y$$

If the flow is incompressible, Eq. 9–16 must apply. More specifically, in Cartesian coordinates Eq. 9–17 must apply. Let's check:

$$\underbrace{\frac{\partial u}{\partial x}}_{0.8} + \underbrace{\frac{\partial v}{\partial y}}_{-0.8} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0 \text{ since 2-D}} = 0 \qquad \rightarrow \qquad 0.8 - 0.8 = 0$$

So we see that the incompressible continuity equation is indeed satisfied at any instant in time, and **this flow field may be approximated as incompressible.**
Discussion Although there is an unsteady term in v, it has no *y*-derivative and drops out of the continuity equation.

EXAMPLE 9–4 **Finding a Missing Velocity Component**

Two velocity components of a steady, incompressible, three-dimensional flow field are known, namely, $u = ax^2 + by^2 + cz^2$ and $w = axz + byz^2$, where a, b, and c are constants. The y velocity component is missing (Fig. 9–14). Generate an expression for v as a function of x, y, and z.

Analysis Since the flow is steady and incompressible, and since we are working in Cartesian coordinates, we apply Eq. 9–17 to the flow field,

Condition for incompressibility:

$$\frac{\partial v}{\partial y} = -\underbrace{\frac{\partial u}{\partial x}}_{2ax} - \underbrace{\frac{\partial w}{\partial z}}_{ax + 2byz} \quad \rightarrow \quad \frac{\partial v}{\partial y} = -3ax - 2byz$$

Next we integrate with respect to y. Since the integration is a *partial* integration, we add some arbitrary function of x and z instead of a simple constant of integration.

Solution: $$v = -3axy - by^2z + f(x,z)$$

Süreklilik denklemi eksik hız bileşeni bulmak için kullanılabilir

EXAMPLE 9–5 **Two-Dimensional, Incompressible, Vortical Flow**

Consider a two-dimensional, incompressible flow in cylindrical coordinates; the tangential velocity component is $u_\theta = K/r$, where K is a constant. This represents a class of vortical flows. Generate an expression for the other velocity component, u_r.

Analysis The incompressible continuity equation (Eq. 9–18) for this two-dimensional case simplifies to

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \underbrace{\frac{\partial u_z}{\partial z}}_{0\ (2\text{-D})} = 0 \quad \rightarrow \quad \frac{\partial(ru_r)}{\partial r} = -\frac{\partial u_\theta}{\partial \theta} \qquad (1)$$

The given expression for u_θ is not a function of θ, and therefore Eq. 1 reduces to

$$\frac{\partial(ru_r)}{\partial r} = 0 \quad \rightarrow \quad ru_r = f(\theta, t) \qquad (2)$$

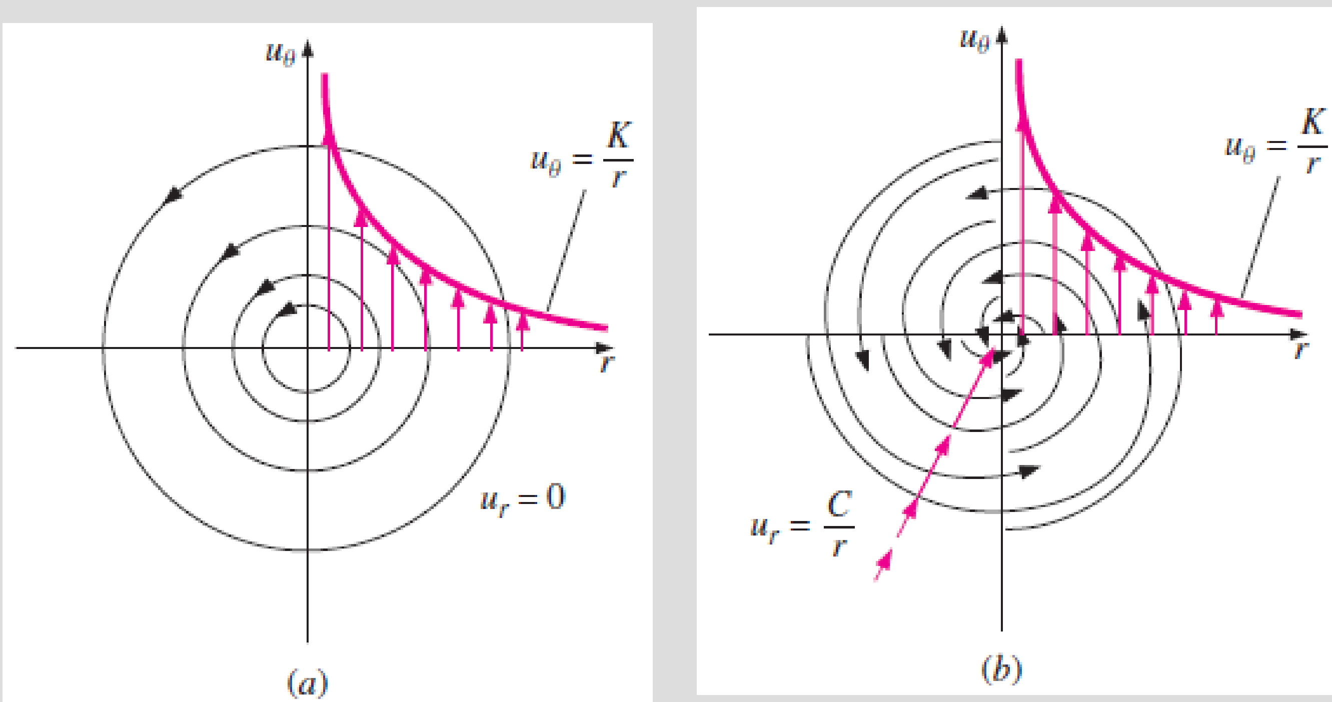
where we have introduced an arbitrary function of θ and t instead of a constant of integration, since we performed a *partial* integration with respect to r. Solving for u_r,

$$u_r = \frac{f(\theta, t)}{r} \qquad (3)$$

Thus, *any radial velocity component of the form given by Eq. 3 yields a two-dimensional, incompressible velocity field that satisfies the continuity equation.*

We discuss some specific cases. The simplest case is when $f(\theta, t) = 0$ ($u_r = 0$, $u_\theta = K/r$). This yields the **line vortex** discussed in Chap. 4, as sketched in Fig. 9–15*a*. Another simple case is when $f(\theta, t) = C$, where C is a constant. This yields a radial velocity whose magnitude decays as $1/r$. For negative C, imagine a spiraling line vortex/sink flow, in which fluid elements not only revolve around the origin, but get sucked into a sink at the origin (actually a line sink along the z-axis). This is illustrated in Fig. 9–15*b*.

Discussion Other more complicated flows can be obtained by setting $f(\theta, t)$ to some other function. For any function $f(\theta, t)$, the flow satisfies the two-dimensional, incompressible continuity equation at a given instant in time.



(*a*) Çizgisel çevri akışı ve (*b*) spiral şekilli çizgisel çevri/kuyu akışına ait akım çizgileri ve hız profilleri

EXAMPLE 9–6 **Comparison of Continuity and Volumetric Strain Rate**

Recall the *volumetric strain rate*, defined in Chap. 4. In Cartesian coordinates,

$$\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad (1)$$

Show that volumetric strain rate is zero for incompressible flow. Discuss the physical interpretation of volumetric strain rate for incompressible and compressible flows.

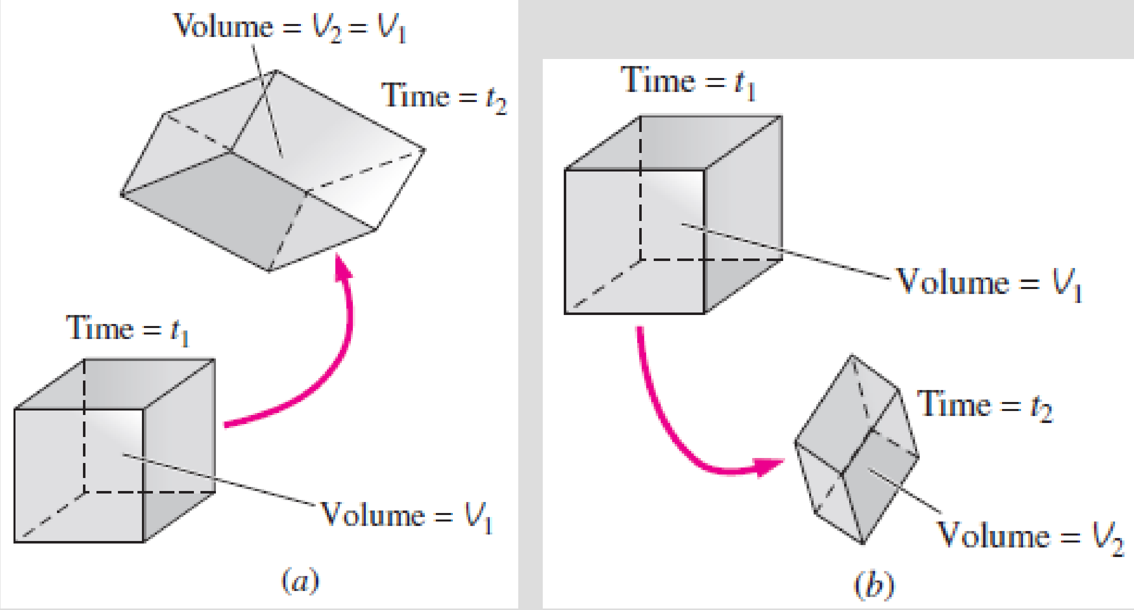
Analysis If the flow is incompressible, Eq. 9–16 applies. More specifically, Eq. 9–17, in Cartesian coordinates, applies. Comparing Eq. 9–17 to Eq. 1,

$$\frac{1}{V}\frac{DV}{Dt} = 0 \qquad \text{for incompressible flow}$$

Thus, *volumetric strain rate is zero in an incompressible flow field*. In fact, you can *define* incompressibility by $DV/Dt = 0$. Physically, as we follow a fluid element, parts of it may stretch while other parts shrink, and the element may translate, distort, and rotate, but its volume remains constant along its entire path through the flow field (Fig. 9–16*a*). This is true whether the flow is steady or unsteady, as long as it is incompressible. If the flow were compressible, the volumetric strain rate would not be zero, implying that fluid elements may expand in volume (dilate) or shrink in volume as they move around in the flow field (Fig. 9–16*b*). Specifically, consider Eq. 9–10, an alternative form of the continuity equation for compressible flow. By definition, $\rho = m/V$, where m is the mass of a fluid element. For a material element (following the fluid element as it moves through the flow field), m must be constant. Applying some algebra to Eq. 9–10 yields

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{V}{m}\frac{D(m/V)}{Dt} = -\frac{V}{m}\frac{m}{V^2}\frac{DV}{Dt} = -\frac{1}{V}\frac{DV}{Dt} = -\vec{\nabla}\cdot\vec{V} \quad \rightarrow \quad \frac{1}{V}\frac{DV}{Dt} = \vec{\nabla}\cdot\vec{V}$$

Discussion
The final result is general—not limited to Cartesian coordinates. It applies to unsteady as well as steady flows.



(*a*) Sıkıştırılamaz bir akış alanında akışkan elemanları ötelenebilir, dönebilir ve şekilleri değişebilir fakat hacimleri değişmez; (*b*) sıkıştırılabilir bir akış alanında ise akışkan elemanaları ötelenirken, dönerken ve şekilleri bozulurken hacimleri değişebilir.

EXAMPLE 9–7 Conditions for Incompressible Flow

Consider a steady velocity field given by $\vec{V} = (u, v, w) = a(x^2y + y^2)\vec{i} + bxy^2\vec{j} + cx\vec{k}$, where *a*, *b*, and *c* are constants. Under what conditions is this flow field incompressible?

SOLUTION We are to determine a relationship between constants *a*, *b*, and *c* that ensures incompressibility.

Assumptions **1** The flow is steady. **2** The flow is incompressible (under certain constraints to be determined).

Analysis We apply Eq. 9–17 to the given velocity field,

$$\underbrace{\frac{\partial u}{\partial x}}_{2axy} + \underbrace{\frac{\partial v}{\partial y}}_{2bxy} + \underbrace{\frac{\partial w}{\partial z}}_{0} = 0 \quad \rightarrow \quad 2axy + 2bxy = 0$$

Thus to guarantee incompressibility, constants *a* and *b* must be equal in magnitude but opposite in sign.

Condition for incompressibility: $$a = -b$$

Discussion If *a* were not equal to $-b$, this might still be a valid flow field, but density would have to vary with location in the flow field. In other words, the flow would be *compressible*, and Eq. 9–14 would need to be satisfied in place of Eq. 9–17.

9–3 ■ AKIM FONKSİYONU

Kartezyen Koordinatlarda Akım Fonksiyonu

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Kartezyen koordinatlarda, sıkıştırılamaz, iki boyutlu akım fonksiyonu:

$$u = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x}$$

Akım fonksiyonu ψ

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial y\, \partial x} = 0$$

Akım fonksiyonunun değişik koordinat sistemlerine göre tanımlamaları

Stream Function

- 2-D, incompressible, Cartesian coordinates:

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

- 2-D, incompressible, cylindrical coordinates:

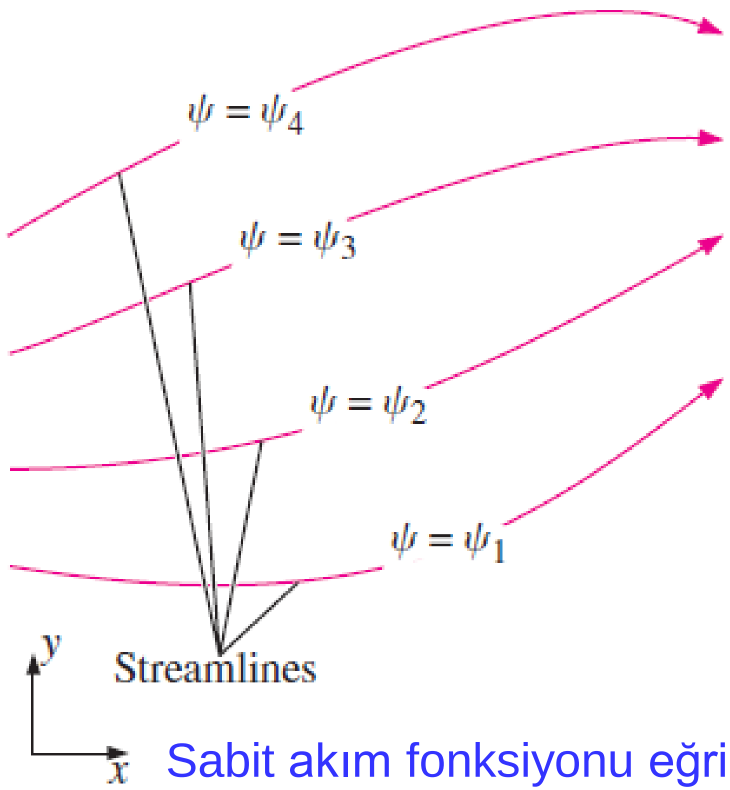
$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

- Axisymmetric, incompressible, cylindrical coordinates:

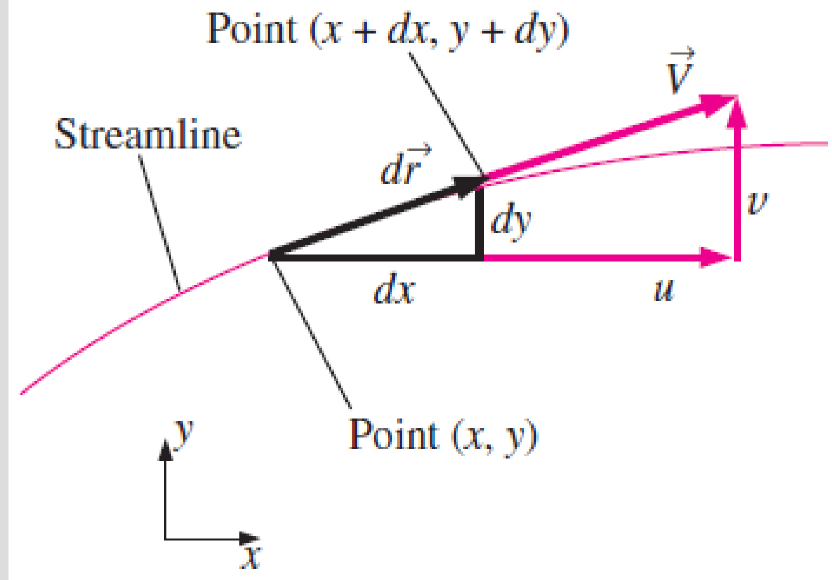
$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \quad \text{and} \quad u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$$

- 2-D, compressible, Cartesian coordinates:

$$\rho u = \frac{\partial \psi_\rho}{\partial y} \quad \text{and} \quad \rho v = -\frac{\partial \psi_\rho}{\partial x}$$



Sabit akım fonksiyonu eğrileri akışın akım çizgilerini temsil eder



Arc length $d\vec{r} = (dx, dy)$ and local velocity vector $\vec{V} = (u, v)$ along a two-dimensional streamline in the xy-plane.

Along a streamline: $$\frac{dy}{dx} = \frac{v}{u} \quad \rightarrow \quad \underbrace{-v}_{\partial\psi/\partial x}\, dx + \underbrace{u}_{\partial\psi/\partial y}\, dy = 0$$

Sabit ψ eğrileri akışın ***akım çizgileridir.***

Along a streamline: $$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

Total change of ψ*:* $$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

EXAMPLE 9–8 Calculation of the Velocity Field from the Stream Function

A steady, two-dimensional, incompressible flow field in the *xy*-plane has a stream function given by $\psi = ax^3 + by + cx$, where *a*, *b*, and *c* are constants: $a = 0.50\ (\text{m} \cdot \text{s})^{-1}$, $b = -2.0$ m/s, and $c = -1.5$ m/s. (*a*) Obtain expressions for velocity components *u* and *v*. (*b*) Verify that the flow field satisfies the incompressible continuity equation. (*c*) Plot several streamlines of the flow in the upper-right quadrant.

Analysis (*a*) We use Eq. 9–20 to obtain expressions for *u* and *v* by differentiating the stream function,

$$u = \frac{\partial \psi}{\partial y} = b \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x} = -3ax^2 - c$$

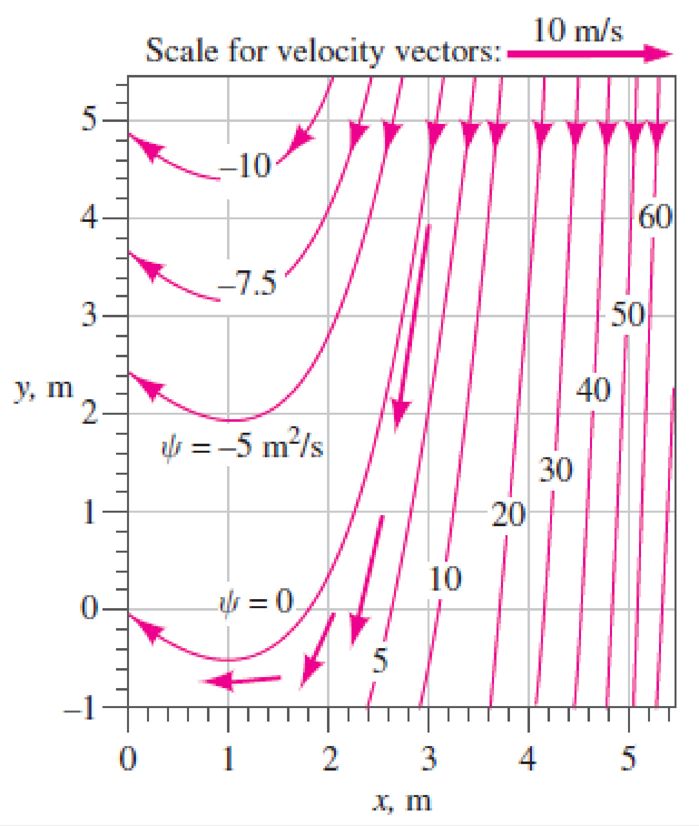
(*b*) Since *u* is not a function of *x*, and *v* is not a function of *y*, we see immediately that the two-dimensional, incompressible continuity equation (Eq. 9–19) is satisfied. In fact, since ψ is smooth in *x* and *y*, the two-dimensional, incompressible continuity equation in the *xy*-plane is automatically satisfied by the very definition of ψ. We conclude that **the flow is indeed incompressible.**

(*c*) To plot streamlines, we solve the given equation for either *y* as a function of *x* and ψ, or *x* as a function of *y* and ψ. In this case, the former is easier, and we have

Equation for a streamline: $$y = \frac{\psi - ax^3 - cx}{b}$$

This equation is plotted in Fig. 9–20 for several values of ψ, and for the provided values of a, b, and c. The flow is nearly straight down at large values of x, but veers upward for $x < 1$ m.

Discussion You can verify that $v = 0$ at $x = 1$ m. In fact, v is negative for $x > 1$ m and positive for $x < 1$ m. The direction of the flow can also be determined by picking an arbitrary point in the flow, say ($x = 3$ m, $y = 4$ m), and calculating the velocity there. We get $u = -2.0$ m/s and $v = -12.0$ m/s at this point, either of which shows that fluid flows to the lower left in this region of the flow field. For clarity, the velocity vector at this point is also plotted in Fig. 9–20; it is clearly parallel to the streamline near that point. Velocity vectors at three other locations are also plotted.



$$y = \frac{\psi - ax^3 - cx}{b}$$

Örnek 9-8'deki hız alanına ait akım çizgileri her bir akım çizgisi için sabit ψ değerleri ve dört farklı konumda hız vektorleri gösterilmiştir.

ÖRNEK 9–9 Bilinen Bir Hız Alanı İçin Akım Fonksiyonunun Hesaplanması

Daimi, iki-boyutlu, sıkıştırılamaz bir hız alanı $u = ax + b$ ve $v = -ay + cx$ ile verilmektedir. Burada *a*, *b* ve *c* sabittir ve $a = 0.50\ s^{-1}$, $b = 1.5$ m/s ve $c = 0.35\ s^{-1}$'dir. Akım fonksiyonu için bir ifade geliştiriniz ve üst sağ dördülde akışın bazı akım çizgilerini çiziniz.

ÇÖZÜM Verilen bir hız alanı için akım fonksiyonu ψ için bir ifade elde edeceğiz ve verilen *a*, *b* ve *c* sabitlerinin değerleri için bazı akım çizgilerini çizeceğiz.
Kabuller **1** Akış daimidir. **2** Akış sıkıştırılamazdır. **3** Akış *xy*-düzleminde iki-boyutludur yani $w = 0$, ne *u* ne de *v*, *z*'nin fonksiyonudur.
Analiz Akım fonksiyonunu tanımlayan Denklem 9–20'nin iki kısmından biriyle başlayalım (hangisiyle başlandığının önemi yoktur, çözüm aynı olacaktır):

$$\frac{\partial \psi}{\partial y} = u = ax + b$$

Ardından bunun bir *kısmi* integrasyon olduğuna dikkat ederek *y*.' ye göre integralini alalım. Bunun sonucu olarak, integrasyon sabiti yerine diğer değişken olan *x*'e bağlı bir fonksiyon eklenecektir.

$$\psi = axy + by + g(x) \qquad \textbf{(1)}$$

Şimdi de Denklem 9–20'nin diğer kısmını seçelim, Denklem 1'in türevini alalım ve aşağıdaki gibi düzenleyelim.

$$v = -\frac{\partial \psi}{\partial y} = -ay - g'(x) \qquad \textbf{(2)}$$

Burada *g' (x)*, *dg/dx*'i göstermektedir. Zira, *g* sadece *x*'in bir fonksiyonudur. Böylece *v* hız bileşeni için, problemde verilen ifade ve Denklem 2 olmak üzere iki ifade elde etmiş oluruz. *g (x)*'i bulmak için bu iki denklemi birbirine eşitleyip integre edelim:

$$v = -ay + cx = -ay - g'(x) \;\rightarrow\; g'(x) = -cx \;\rightarrow\; g(x) = -c\frac{x^2}{2} + C \quad \textbf{(3)}$$

g' nin yalnızca *x*'in bir fonksiyonu olmasından dolayı keyfi bir *C* bir integrası sabiti eklediğimize dikkat ediniz. Son olarak, Denklem 3'ü Denklem 1'de yerine koyarak ψ için son ifade

Çözüm:
$$\psi = axy + by - c\frac{x^2}{2} + C \quad \textbf{(4)}$$

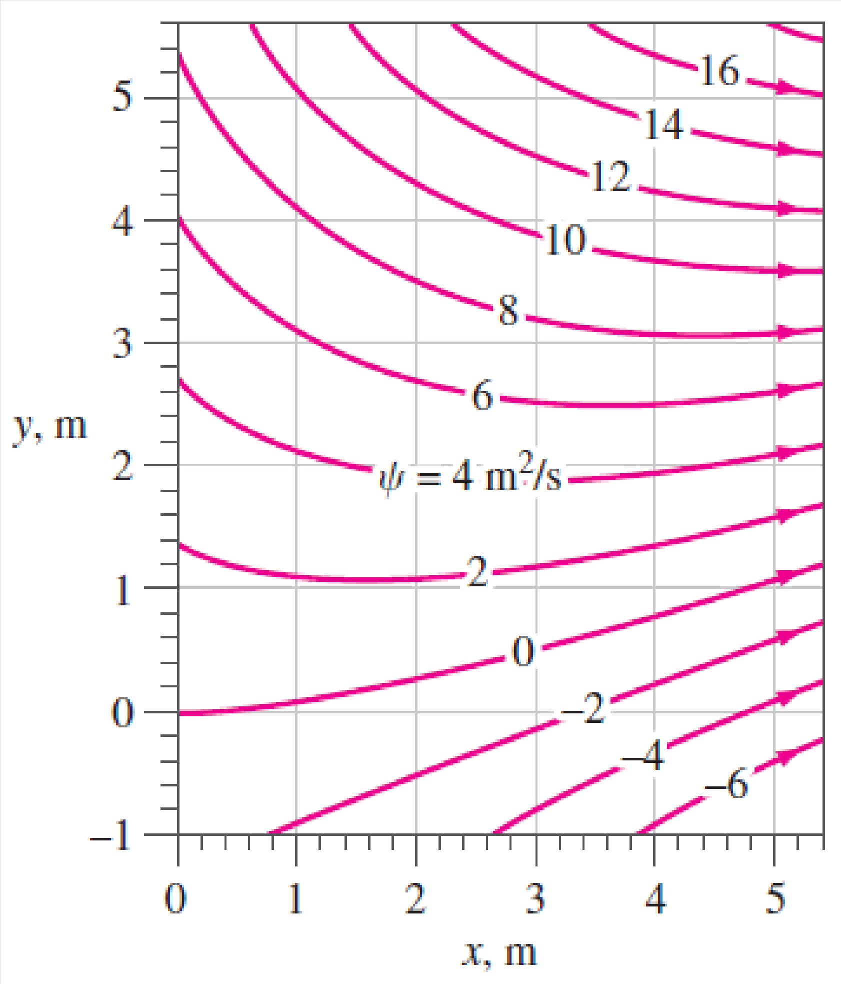
şeklinde elde edilir.

Akım çizgilerini çizmek için Denklem 4'ün bir eğri *ailesi*ni temsil ettiğini, yani her bir $(\psi - C)$ sabitinin değeri için tek bir eğri bulunduğunu belirtelim. *C* keyfi sabit olduğundan genellikle sıfır alınması tercih edilir, ancak başka değerler de alınabilir. Basitlik için $C = 0$ alıp Denklem 4'ten *x*'in bir fonksiyonu olarak *y*'yi çözersek,

Akım çizgileri denklemi:
$$y = \frac{\psi + cx^2/2}{ax + b} \quad \textbf{(5)}$$

elde ederiz. Verilen *a, b* ve *c* sabitlerinin değerleri ve çeşitli ψ değerleri için Denklem 5 Şekil 9–20'de çizilmiştir. Bu sabit ψ eğrileri akışın akım çizgileridir. Şekil 9–20'den bunun üst sağ dördülde düzgün şekilde daralan bir akış olduğunu görülmektedir.

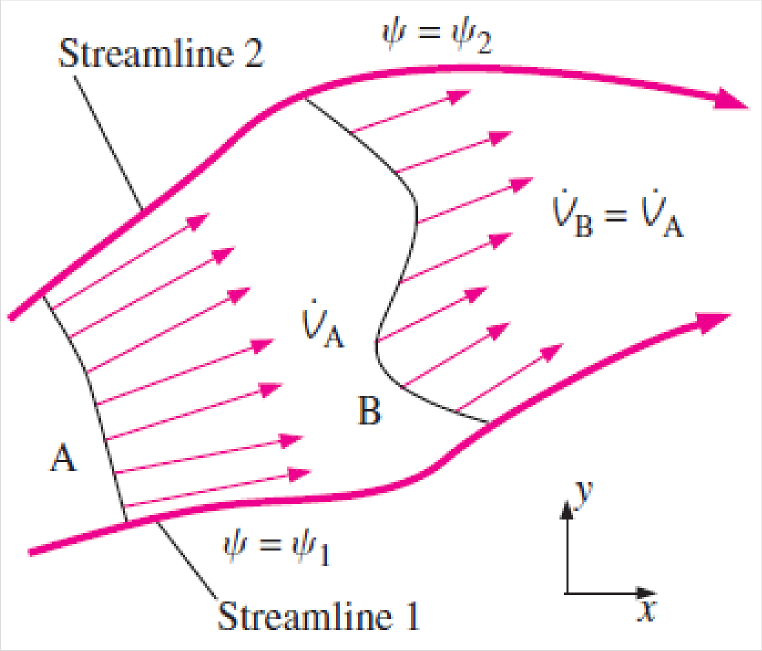
İrdeleme Yaptığımız cebirsel işlemlerin doğruluğunu kontrol etmek her zaman faydalıdır. Bu örnekte doğru hız bileşenlerinin elde edildiğini göstermek için Denklem 4'ü Denklem 9–20'de yerine koymalısınız.



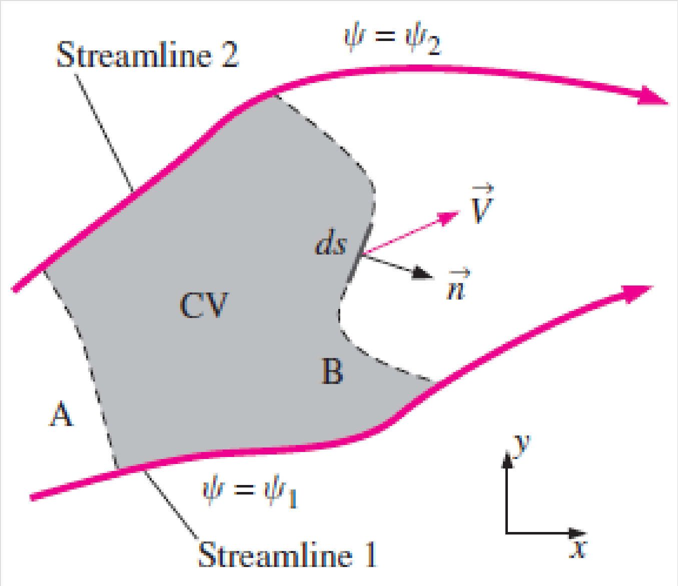
$$y = \frac{\psi + cx^2/2}{ax + b}$$

Örnek 9-9 'daki hız alanına ait akım çizgileri her bir akım çizgisi için sabit ψ değerleri gösterilmiştir

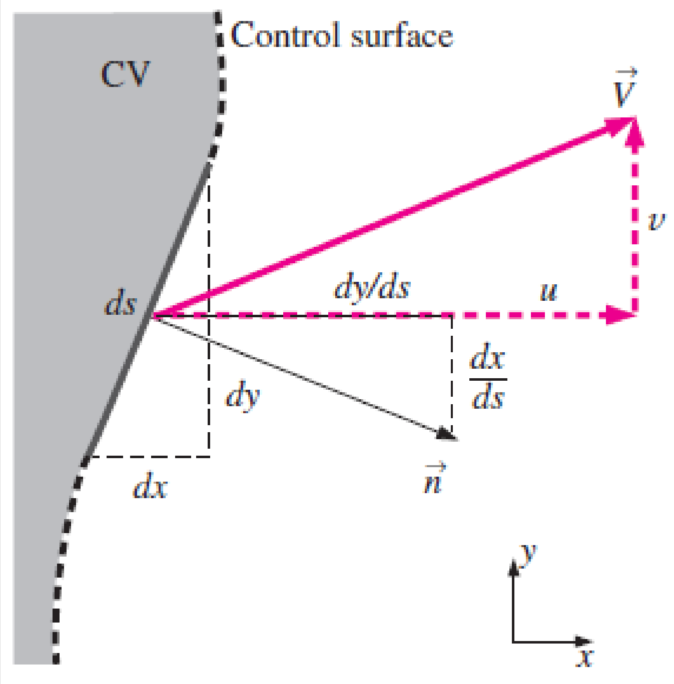
Bir akım çizgisinden diğerine ψ değerleri arasındaki fark, ***birim genişlik başına*** bu iki akım çizgisi arasından geçen ***hacimsel debiye*** eşittir.



xy-düzleminde iki-boyutlu akım çizgileri için iki akım çizgisi arasından birim genişlik başına geçen hacimsel debi $\dot{V}$ her en-kesitte aynıdır.



(a)

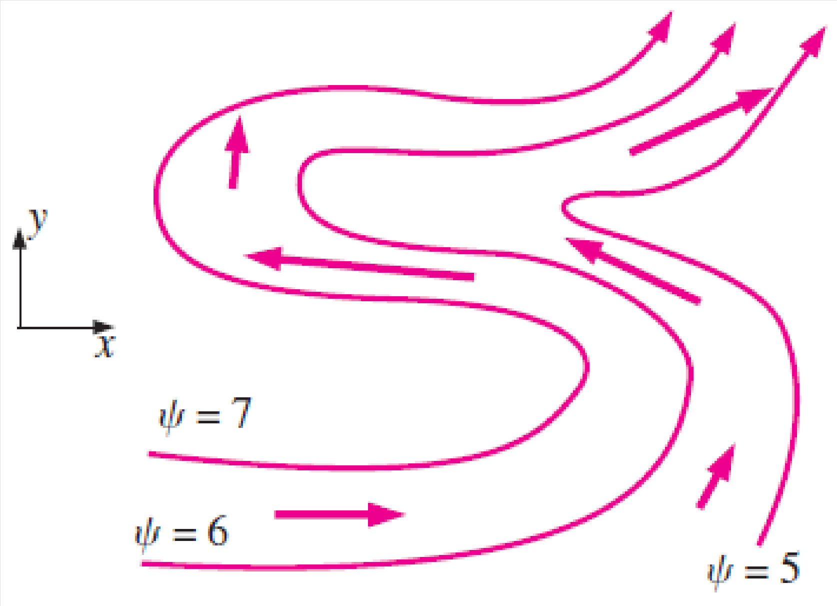


(b)

(a) xy düzleminde akım çizgileri ψ_1 ve ψ_2 ile sınırlandırılmış kontrol hacmi ile A ve B dilimleri;

(b) Sonsuz küçük ds uzunluğu etrafındaki bölgenin büyütülmüş görüntüsü.

Akım fonksiyonu *ψ'nın değeri* xy-düzleminde akış yönünün soluna doğru artar



Sol-taraf kuralı çizim üzerinde gösterilmiştir. xy düzleminde akım fonksiyonunun değeri akış yönünün soluna doğru artar.

Şekilden görülebileceği gibi, akım fonksiyonu, akışın ne kadar kıvrıldığına ve döndüğüne bakılmaksızın akış yönünün soluna doğru artmaktadır.

Akım çizgilerinin birbirinden uzaklaştığı (sağ alt kısım) kısımlardaki hızın büyüklüğü (akışkan hızı), akım çizgilerinin birbirlerine daha yalan olduğu kısımlardakine (orta kısım) oranla daha küçük kalır.

Akım çizgileri daraldığında aralarındaki en-kesit alanı azalır ve akım çizgileri arasındaki debiyi korumak için hız artmalıdır.

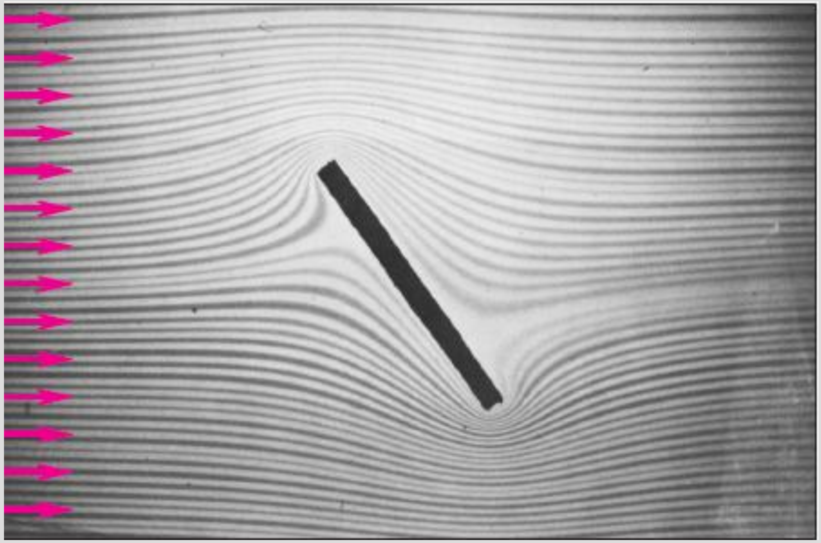
Örnek 9. Akım Çizgilerinden Bulunan Bağıl Hız

Hele-Shaw akışı, bir sıvıyı paralel plakalar arasındaki İnce bir aralıktan geçmeye zorlamak suretiyle elde edilir. Şekilde eğik plaka üzerindeki akış için örnek bir Hele-Shaw akışı verilmiştir. Çıkış çizgileri, yukarı akım bölgesinde eşit aralıklı noktalardan salınan mürekkep ile oluşturulmuştur. Akış daimi olduğundan akım çizgileri ile çıkış çizgileri çakışmaktadır. Akışkan su olup cam plakalar 1.0 'er mm aralıkla yerleştirilmiştir. Akım çizgisi deseninden yola çıkarak akış hızının, akış alanının belirli bir bölgesinde yüksek veya düşük (göreceli olarak) olduğunu nasıl söyleyebileceğimizi irdeleyiniz.

Kabuller: 1 Akış daimidir. 2 Akış sıkıştırılamazdır. 3 Akış xy-düzleminde iki-boyutlu potansiyel akışı modellemektedir.

Analiz: Bir akım fonksiyonunun eşit aralıklı akım çizgilerinin birbirlerinden uzaklaşması, o bölgede akış hızının düştüğünü gösterin Benzer şekilde eğer akım çizgileri birbirlerine yaklaşıyorsa bu bölgede akış hızı artıyordur. Şekilden, akım çizgileri eşit aralıklı olduğundan plakanın yukarıakımının uzağında akışın düz ve üniform olduğu anlaşılmaktadır: Akışkan plakanın.alt tarafına doğru yaklaştıkça, özellikle durma noktası civarında akım çizgileri arasındaki geniş aralıklarından anlaşıldığı gibi akışın hızı yavaşlamaktadır. Öte yandan akış, plakanın keskin köşeleri civarında: birbirine.iyice yaklaşmış akım çizgilerinin gösterdiği gibi yüksek hızlara doğru ivmelenmektedir: :

İrdeleme: Hele-Shaw akışının çıkış çizgileri, Bölüm 10'da tartışılacağı gibi potansiyel akışa benzer bir yapı sergiler.

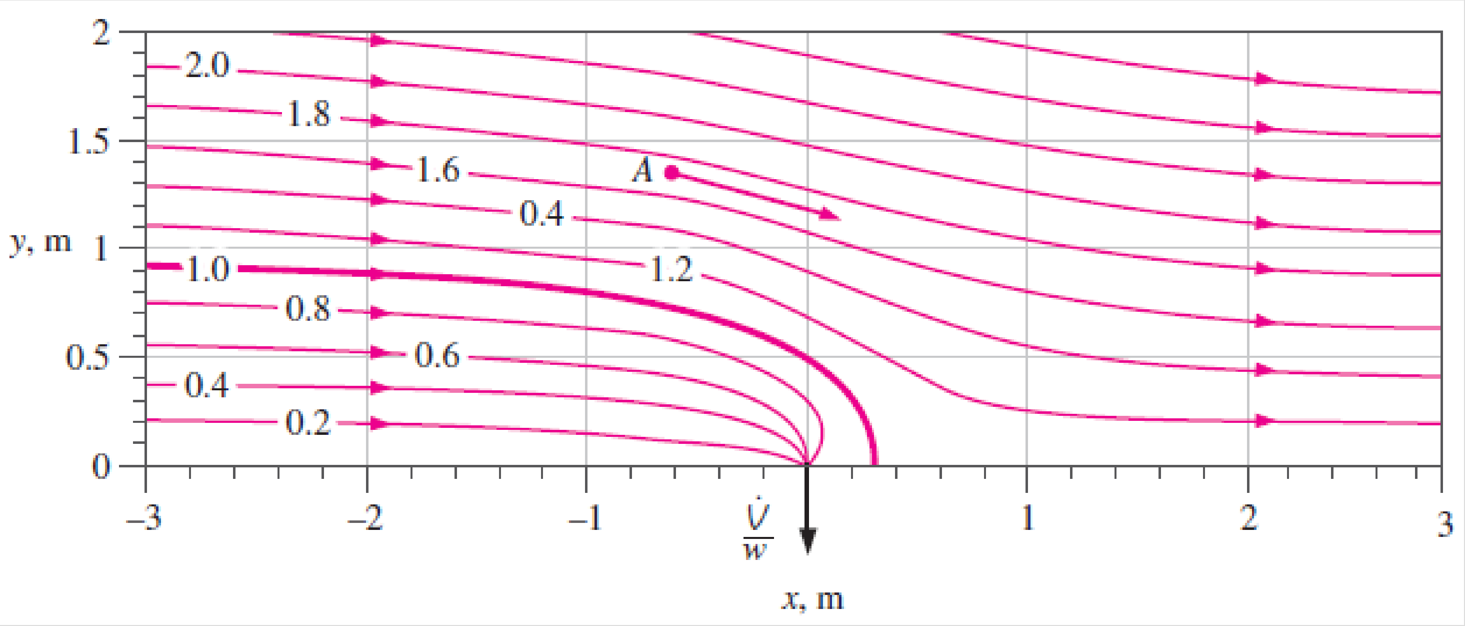


Eğik bir plaka üzerinden Hele-Shaw akışı ile oluşturulan çıkış çizgileri. Çıkış çizgileri, aynı en-kesit şekline sahip iki-boyutlu eğik bir plaka üzerindeki potansiyel akışa ait (Bölüm 10) akım çizgilerini modellemektedir.

Örnek 11: Hacimsel Debinin Akım Çizgilerinden Bulunması

Bir su kanalının taban yüzeyindeki dar bir yarıktan su emişi yapılmaktadır. Kanaldaki su V= 1.0 m/s üniform hızıyla soldan sağa doğru akmaktadır. Yarık xy-düzlemine dik ve tüm kanal kesitinde z-ekseni boyunca uzamakta olup genişliği w= 2.0 m'dir. Dolayısıyla akışın xy-düzleminde iki-boyutlu olduğu düşünülmektedir.

Şekilde bazı akım çizgileri çizilmiş ve işaretlenmiştir. Kalın akım çizgisi akışı ikiye ayırdığından **bölen akım çizgisi** olarak adlandırılır. Bu bölen akım çizgisinin altındaki tüm su yarık tarafından emilirken, bölen akım çizgisinin üzerinde kalan su aşağı akım yönünde yoluna devam etmektedir. Yarıktan emilen suyun hacimsel debisi nedir? A noktasındaki hızın büyüklüğünü belirleyiniz.



Üzerinde dar bir emiş yarığı bulunan bir çeper boyunca olan serbest akım akışına ait akım çizgileri; akım çizgilerinin değerleri m^2/s birimindedir; kalın olan akım çizgisi bölen akım çizgisidir. A noktasındaki biz vektörünün yönü sol-taraf kuralına göre belirlenmiştir.

Analysis By Eq. 9–25, the volume flow rate per unit width between the bottom wall ($\psi_{wall} = 0$) and the dividing streamline ($\psi_{dividing} = 1.0\ m^2/s$) is

$$\frac{\dot{V}}{w} = \psi_{dividing} - \psi_{wall} = (1.0 - 0)\ m^2/s = 1.0\ m^2/s$$

All of this flow must go through the slot. Since the channel is 2.0 m wide, the total volume flow rate through the slot is

$$\dot{V} = \frac{\dot{V}}{w} w = (1.0\ m^2/s)(2.0\ m) = 2.0\ m^3/s$$

To estimate the speed at point *A*, we measure the distance δ between the two streamlines that enclose point *A*. We find that streamline 1.8 is about 0.21 m away from streamline 1.6 in the vicinity of point *A*. The volume flow rate per unit width (into the page) between these two streamlines is equal to the difference in value of the stream function. We thus estimate the speed at point *A*,

$$V_A \cong \frac{\dot{V}}{w\delta} = \frac{1}{\delta}\frac{\dot{V}}{w} = \frac{1}{\delta}(\psi_{1.8} - \psi_{1.6}) = \frac{1}{0.21\ m}(1.8 - 1.6)\ m^2/s = 0.95\ m/s$$

Our estimate is close to the known free-stream speed (1.0 m/s), indicating that the fluid in the vicinity of point *A* flows at nearly the same speed as the free-stream flow, but points slightly downward.

Discussion The streamlines of Fig. 9–26 were generated by superposition of a uniform stream and a line sink, assuming irrotational (potential) flow. We discuss such superposition in Chap. 10.

Silindirik Koordinatlarda Akım Fonksiyonu

$$\frac{\partial(ru_r)}{\partial r} + \frac{\partial(u_\theta)}{\partial \theta} = 0$$

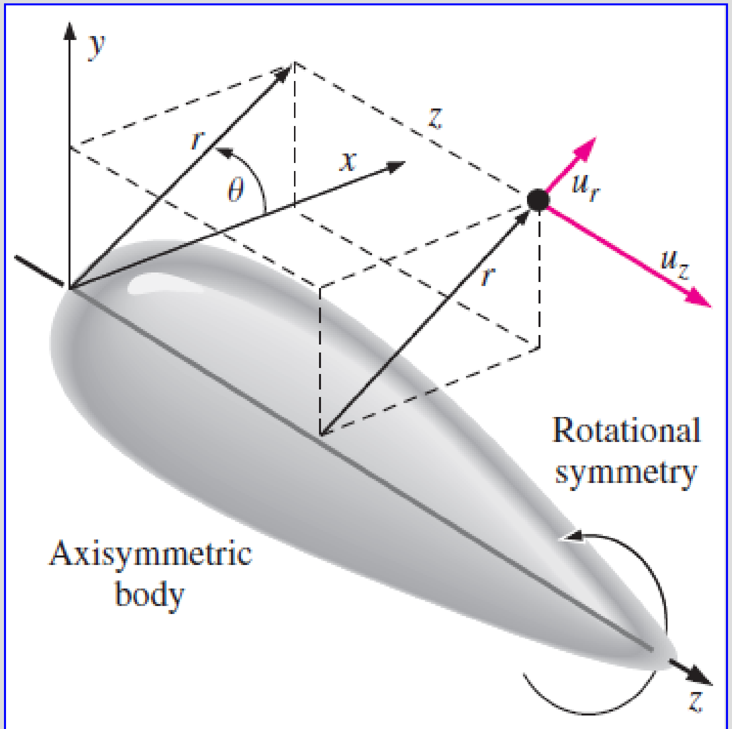
Incompressible, planar stream function in cylindrical coordinates:

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad u_\theta = -\frac{\partial\psi}{\partial r}$$

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial(u_z)}{\partial z} = 0$$

Incompressible, axisymmetric stream function in cylindrical coordinates:

$$u_r = -\frac{1}{r}\frac{\partial\psi}{\partial z} \quad \text{and} \quad u_z = \frac{1}{r}\frac{\partial\psi}{\partial r}$$



Silindirik koordinatlarda z-eksenine göre dönel simteriye sahip bir eksenel simetrik cisim üzerindeki akış; ne geometri ne de hız alanı θ'ya bağlıdır ve u_θ =0 'dır.

Örnek 12: Silindirik Koordinatlarda Akım Fonksiyonu

Hız bileşenlerinin u_r=0 ve u_θ = K/r (K-sabit) olarak verildiği, daimi, düzlemsel ve sıkıştırılamaz bir çizgisel çevriyi göz önüne alınız. Bu akış aşağıdaki şekilde gösterilmiştir. Akım fonksiyonu ψ(x,y) için bir ifade geliştiriniz ve akım çizgilerinin dairesel olduğunu gösteriniz.

Analysis We use the definition of stream function given by Eq. 9–27. We can choose either component to start with; we choose the tangential component,

$$\frac{\partial \psi}{\partial r} = -u_\theta = -\frac{K}{r} \qquad \rightarrow \qquad \psi = -K \ln r + f(\theta) \qquad \textbf{(1)}$$

Now we use the other component of Eq. 9–27,

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r} f'(\theta) \qquad \textbf{(2)}$$

where the prime denotes a derivative with respect to θ. By equating u_r from the given information to Eq. 2, we see that

$$f'(\theta) = 0 \qquad \rightarrow \qquad f(\theta) = C$$

where C is an arbitrary constant of integration. Equation 1 is thus

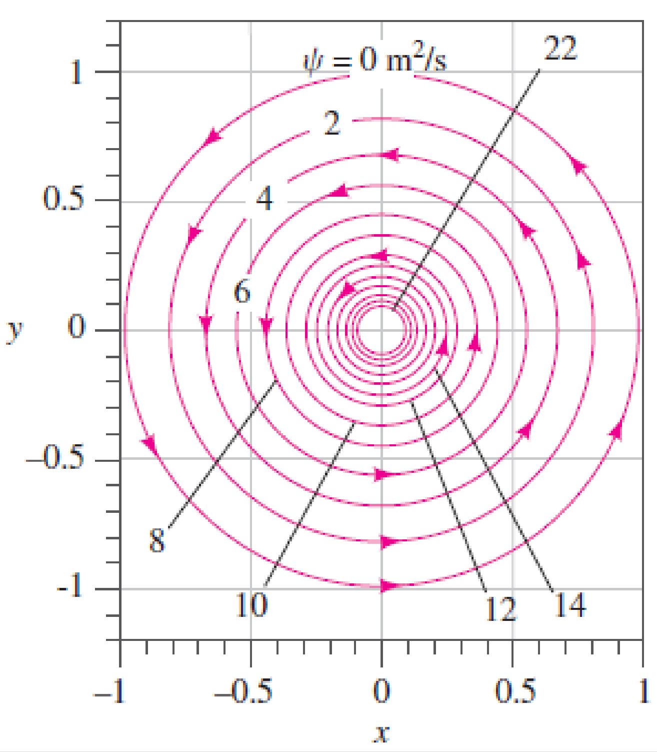
Solution: $$\psi = -K \ln r + C \qquad \textbf{(3)}$$

Finally, we see from Eq. 3 that curves of constant ψ are produced by setting r to a constant value. Since curves of constant r are circles by definition, streamlines (curves of constant ψ) must therefore be circles about the origin, as in Fig. 9–15*a*.

For given values of C and ψ, we solve Eq. 3 for r to plot the streamlines,

Equation for streamlines: $$r = e^{-(\psi - C)/K} \qquad \textbf{(4)}$$

For K = 10 m²/s and C = 0, streamlines from ψ = 0 to 22 are plotted in Fig. 9–28.



Hız alanına ait akım çizgileri K = 10 m²/s and C = 0 alınarak çizilmiştir. Bazı akım çizgileri için sabit ψ değerleri gösterilmiştir.;

Sıkıştırılabilir Akım Fonksiyonu

We extend the stream function concept to steady, *compressible*, two-dimensional flow in the *xy*-plane. The compressible continuity equation (Eq. 9–14) in Cartesian coordinates reduces to the following for steady two-dimensional flow:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad \text{(9–30)}$$

We introduce a **compressible stream function,** which we denote as ψ_ρ,

Steady, compressible, two-dimensional stream function in Cartesian coordinates:

$$\rho u = \frac{\partial \psi_\rho}{\partial y} \quad \text{and} \quad \rho v = -\frac{\partial \psi_\rho}{\partial x} \qquad \text{(9–31)}$$

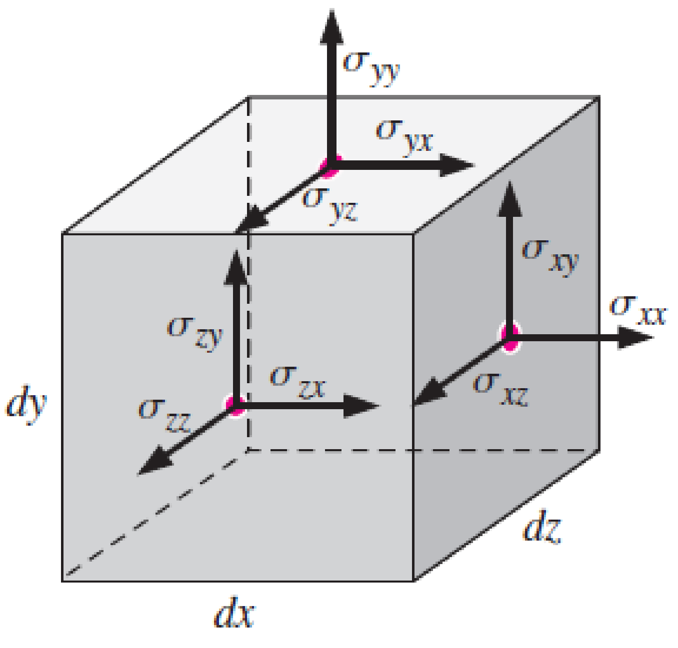
By definition, ψ_ρ of Eq. 9–31 satisfies Eq. 9–30 exactly, provided that ψ_ρ is a smooth function of x and y. Many of the features of the compressible stream function are the same as those of the incompressible ψ as discussed previously. For example, curves of constant ψ_ρ are still streamlines. However, the difference in ψ_ρ from one streamline to another is *mass* flow rate per unit width rather than volume flow rate per unit width. Although not as popular as its incompressible counterpart, the compressible stream function finds use in some commercial CFD codes.

9–4 ■ DOĞRUSAL MOMENTUM KORUNUMU–CAUCHY DENKLEMİ

RTT uygulanmasıyla bir kontrol hacmi için doğrusal momentum denkleminin genel ifadesi.

$$\sum \vec{F} = \int_{CV} \rho\vec{g}\, dV + \int_{CS} \sigma_{ij} \cdot \vec{n}\, dA = \int_{CV} \frac{\partial}{\partial t}(\rho\vec{V})\, dV + \int_{CS} (\rho\vec{V})\vec{V} \cdot \vec{n}\, dA$$

$$\sum \vec{F} = \sum \vec{F}_{\text{body}} + \sum \vec{F}_{\text{surface}} = \int_{CV} \frac{\partial}{\partial t}(\rho\vec{V})\, dV + \sum_{\text{out}} \beta\dot{m}\vec{V} - \sum_{\text{in}} \beta\dot{m}\vec{V}$$



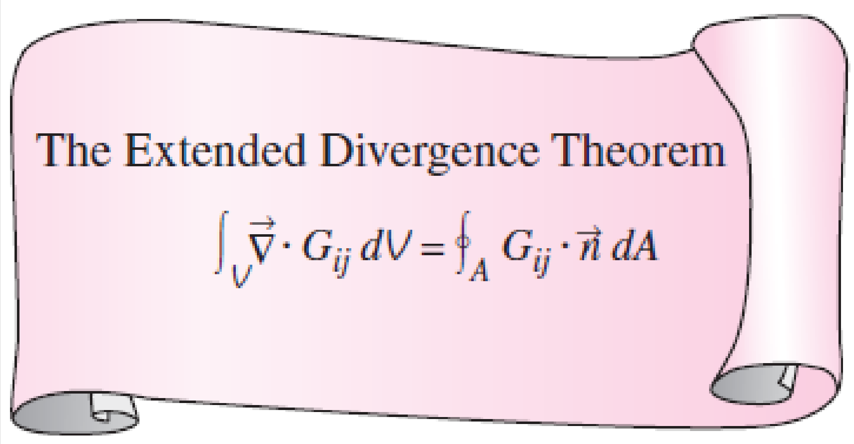
Kartezyen koordinatlarda düzgün altıyüzlü şeklindeki sonsuz küçük bir kontrol hacminin pozitif yüzlerindeki (sağ, üst ve ön) pozitif gerilme tansörü bileşenleri. Mavi noktalar her bir yüzün merkezini göstermektedir. Negatif yüzlerdeki (sol, alt ve arka) pozitif bileşenler şekilde gösterilenlerin tersi yönlerdedir.

Diverjans Teoremi Kullanarak Türetilmesi

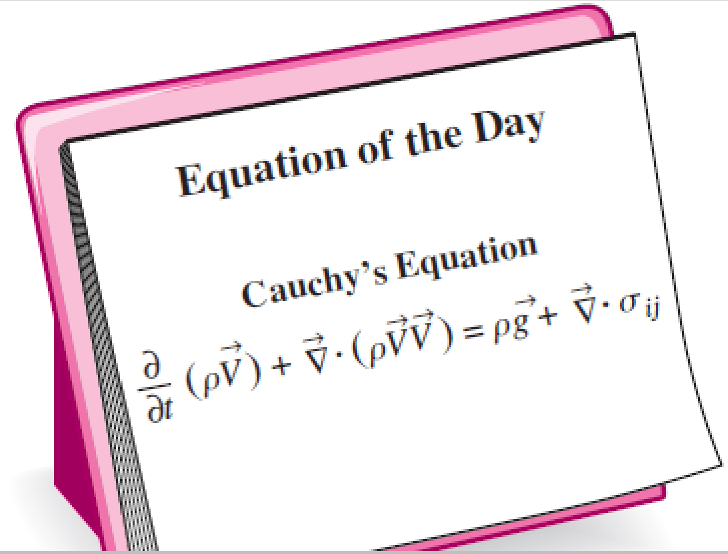
$$\int_{CS} (\rho\vec{V})\vec{V}\cdot\vec{n}\, dA = \int_{CV} \vec{\nabla}\cdot(\rho\vec{V}\vec{V})\, dV \qquad \int_{CS} \sigma_{ij}\cdot\vec{n}\, dA = \int_{CV} \vec{\nabla}\cdot\sigma_{ij}\, dV$$

$$\int_{CV} \left[\frac{\partial}{\partial t}(\rho\vec{V}) + \vec{\nabla}\cdot(\rho\vec{V}\vec{V}) - \rho\vec{g} - \vec{\nabla}\cdot\sigma_{ij}\right] dV = 0$$

Cauchy's equation: $$\frac{\partial}{\partial t}(\rho\vec{V}) + \vec{\nabla}\cdot(\rho\vec{V}\vec{V}) = \rho\vec{g} + \vec{\nabla}\cdot\sigma_{ij}$$



Diverjans teoreminin genişletilmiş formu sadece vektörler için değil aynı zamanda tensörler için de kullanışlıdır. Denklemde G_{ij} ikinci mertebeden bir tensör, *W hacim ve A ise hacmi çevreleyen ve tanımlayan yüzeyin alanıdır.*



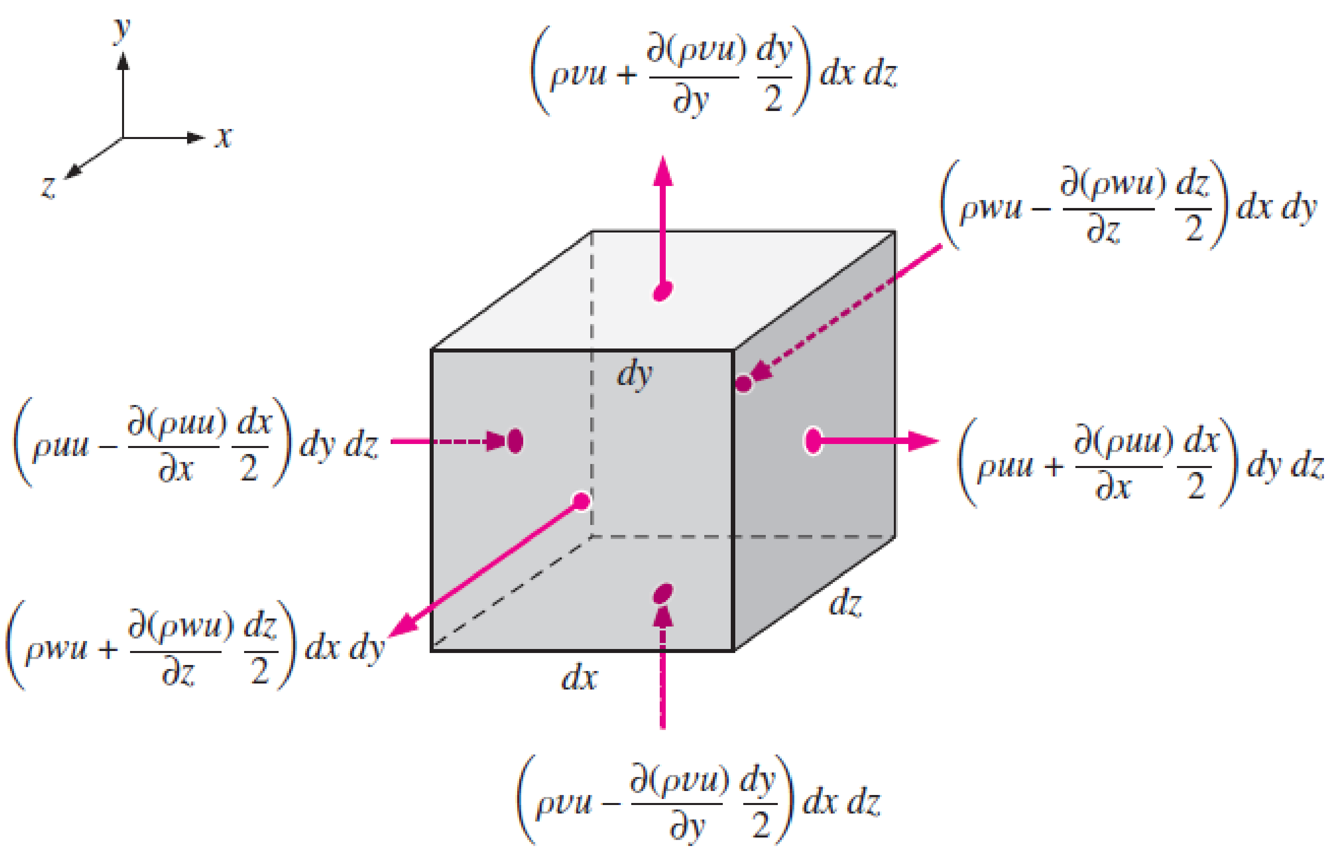
Cauchy denklemi doğrusal momentumun korunumu yasasının diferansiyel formudur. Her akışkan tipine uygulanır.

Sonsuz Küçük Kontrol Hacmi Kullanarak Türetilmesi

$$\sum F_x = \sum F_{x,\text{ body}} + \sum F_{x,\text{ surface}} = \int_{CV} \frac{\partial}{\partial t}(\rho u)\, dV + \sum_{\text{out}} \beta \dot{m} u - \sum_{\text{in}} \beta \dot{m} u$$

Rate of change of x-momentum within the control volume:

$$\int_{CV} \frac{\partial}{\partial t}(\rho u)\, dV \cong \frac{\partial}{\partial t}(\rho u)\, dx\, dy\, dz$$



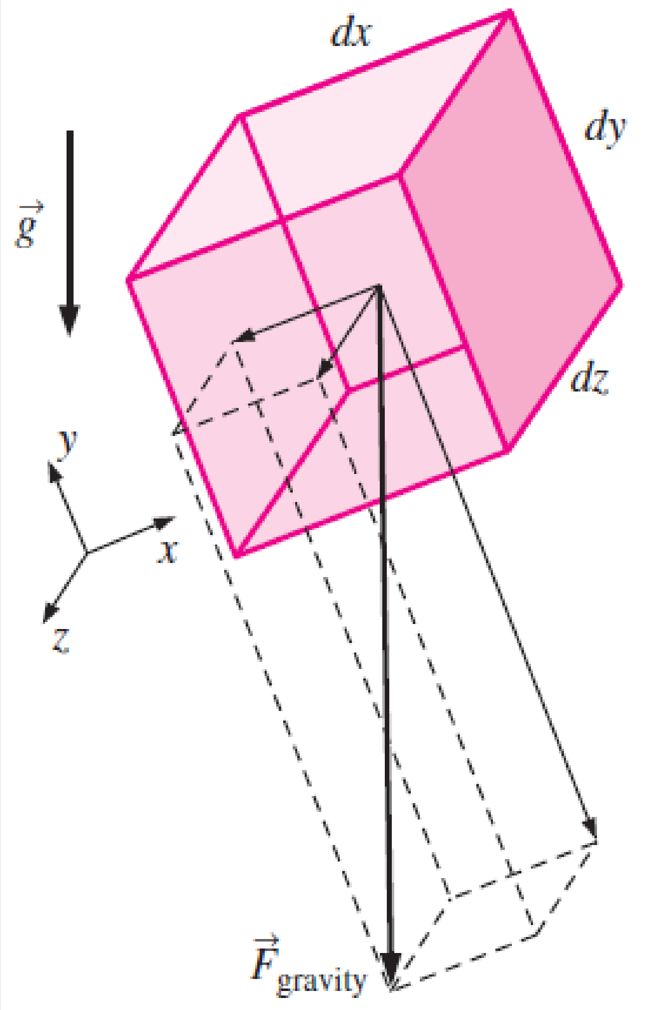
Sonsuz küçük bir kontrol hacminin her bir yüzünden geçen doğrusal momentumun x-bileşeninin giriş ve çıkışları; kırmızı noktalar her bir yüzün merkezini göstermektedir.

Net outflow of x-momentum through the control surface:

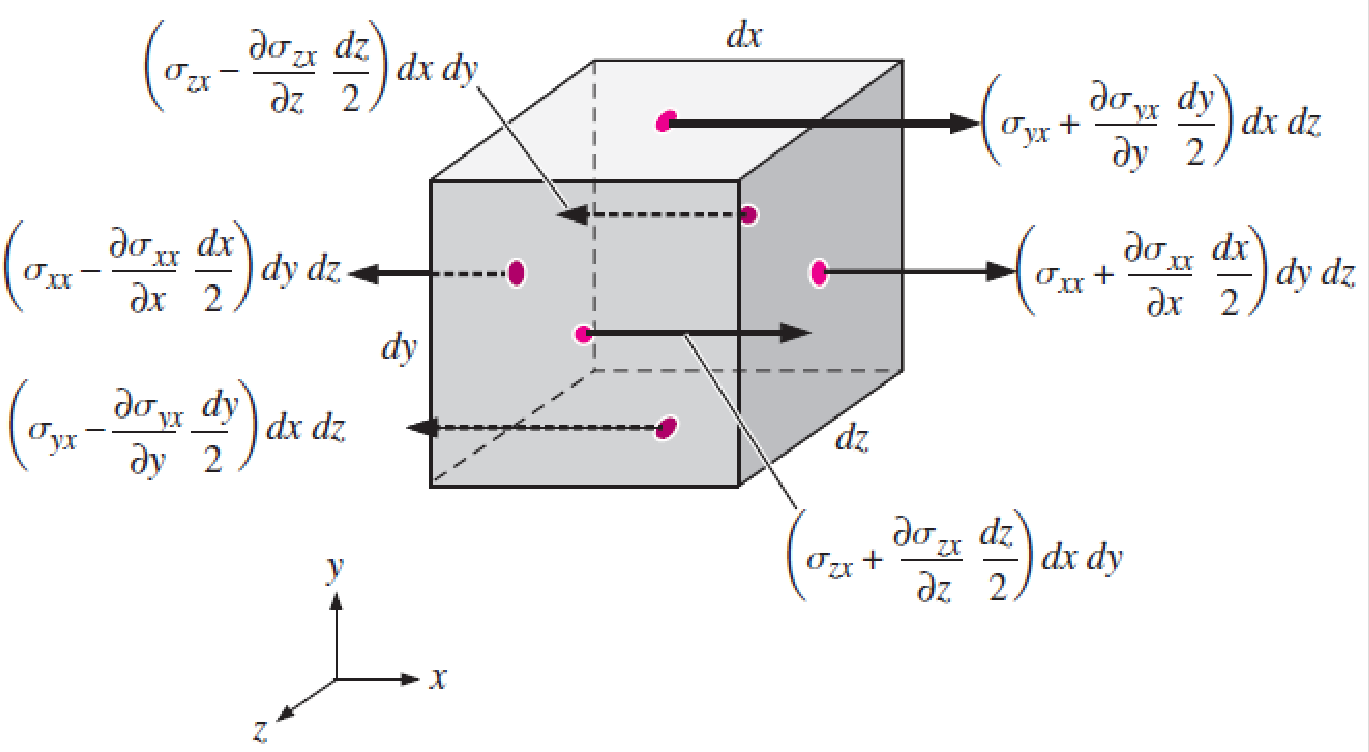
$$\sum_{\text{out}} \beta \dot{m} u - \sum_{\text{in}} \beta \dot{m} u \cong \left(\frac{\partial}{\partial x}(\rho u u) + \frac{\partial}{\partial y}(\rho v u) + \frac{\partial}{\partial z}(\rho w u) \right) dx\, dy\, dz$$

$$\vec{g} = g_x \vec{i} + g_y \vec{j} + g_z \vec{k}$$

$$\sum F_{x,\text{ body}} = \sum F_{x,\text{ gravity}} \cong \rho g_x\, dx\, dy\, dz$$



Yerçekimi vektörünün mutlaka belirli bir eksen ile hizalanması gerekmez. En genel halde sonsuz küçük bir akışkan elemanına etkiyen ağırlık kuvvetinin üç bileşeni vardır.



Diferansiyel bir kontol hacminin her bir yüzü üzerindeki ilgili gerilme tansörü bileşeninden kaynaklanan ve x-yönünde etkiyen kuvvetleri gösteren şematik çizim; kırmızı noktalar her bir yüzün merkezini göstermektedir.

$$\sum F_{x,\text{ surface}} \cong \left(\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{yx} + \frac{\partial}{\partial z}\sigma_{zx}\right) dx\, dy\, dz$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} + \frac{\partial(\rho v u)}{\partial y} + \frac{\partial(\rho w u)}{\partial z} = \rho g_x + \frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{yx} + \frac{\partial}{\partial z}\sigma_{zx}$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho u v)}{\partial x} + \frac{\partial(\rho v v)}{\partial y} + \frac{\partial(\rho w v)}{\partial z} = \rho g_y + \frac{\partial}{\partial x}\sigma_{xy} + \frac{\partial}{\partial y}\sigma_{yy} + \frac{\partial}{\partial z}\sigma_{zy}$$

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho u w)}{\partial x} + \frac{\partial(\rho v w)}{\partial y} + \frac{\partial(\rho w w)}{\partial z} = \rho g_z + \frac{\partial}{\partial x}\sigma_{xz} + \frac{\partial}{\partial y}\sigma_{yz} + \frac{\partial}{\partial z}\sigma_{zz}$$

Cauchy's equation: $$\frac{\partial}{\partial t}(\rho\vec{V}) + \vec{\nabla}\cdot(\rho\vec{V}\vec{V}) = \rho\vec{g} + \vec{\nabla}\cdot\sigma_{ij}$$

$$\vec{V}\vec{V} = \begin{bmatrix} uu & uv & uw \\ vu & vv & vw \\ wu & wv & ww \end{bmatrix}$$

$\vec{V} = (u, v, w)$ vektörünün kendisi ile dış çarpımı ikinci mertebeden bir tensördür. Gösterilen çarpım Kartezyen koordinatlardadır ve dokuz bileşenli bir matris olarak gösterilmiştir.

Cauchy Denkleminin Alternatif Formu

$$\frac{\partial}{\partial t}(\rho\vec{V}) = \rho\,\frac{\partial\vec{V}}{\partial t} + \vec{V}\,\frac{\partial\rho}{\partial t}$$

$$\vec{\nabla}\cdot(\rho\vec{V}\vec{V}) = \vec{V}\,\vec{\nabla}\cdot(\rho\vec{V}) + \rho(\vec{V}\cdot\vec{\nabla})\vec{V}$$

$$\rho\frac{\partial\vec{V}}{\partial t} + \vec{V}\left[\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V})\right] + \rho(\vec{V}\cdot\vec{\nabla})\vec{V} = \rho\vec{g} + \vec{\nabla}\cdot\sigma_{ij}$$

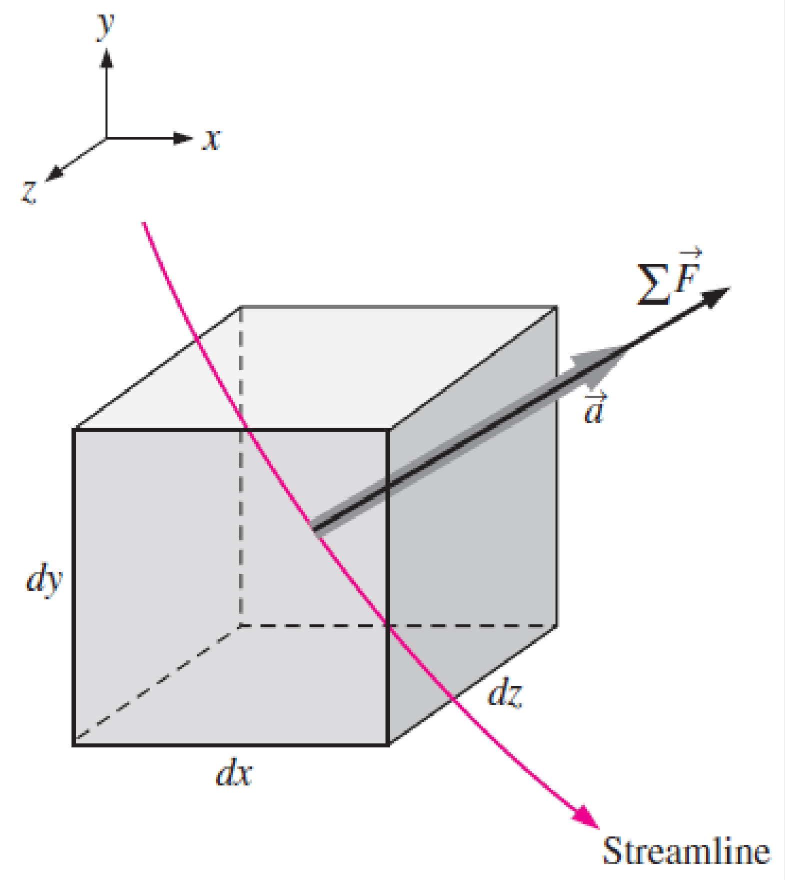
Alternative form of Cauchy's equation:

$$\rho\left[\frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} + \vec{\nabla}\cdot\sigma_{ij}$$

Newton’un İkinci Yasasını Kullanarak Türetme

$$\sum \vec{F} = m\vec{a} = m\frac{D\vec{V}}{Dt} = \rho\, dx\, dy\, dz \frac{D\vec{V}}{Dt}$$

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} + \vec{\nabla} \cdot \sigma_{ij}$$



Diferansiyel akış elemanı eğer maddesel bir eleman ise, akış ile birlikte hareket eder ve Newton'un ikinci yasası doğrudan uygulanabilir.

x-component:
$$\rho \frac{Du}{Dt} = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}$$

y-component:
$$\rho \frac{Dv}{Dt} = \rho g_y + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}$$

z-component:
$$\rho \frac{Dw}{Dt} = \rho g_z + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$

9–5 ■ NAVIER-STOKES DENKLEMİ

Giriş

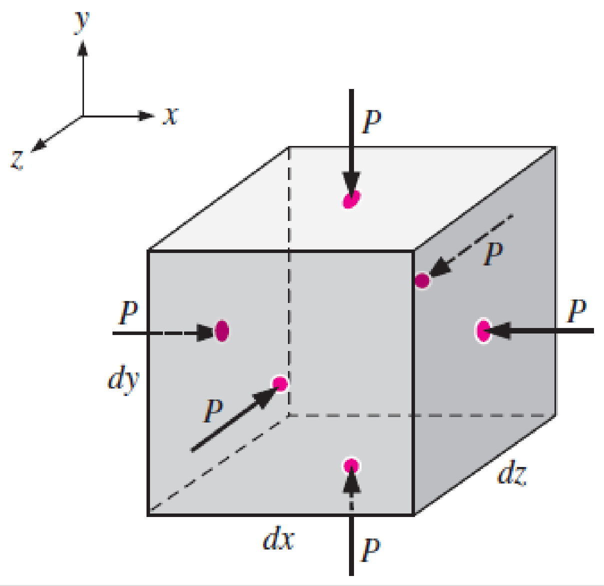
Fluid at rest: $$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

τ_{ij}, **viskoz gerilim tansörü**

Moving fluids:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

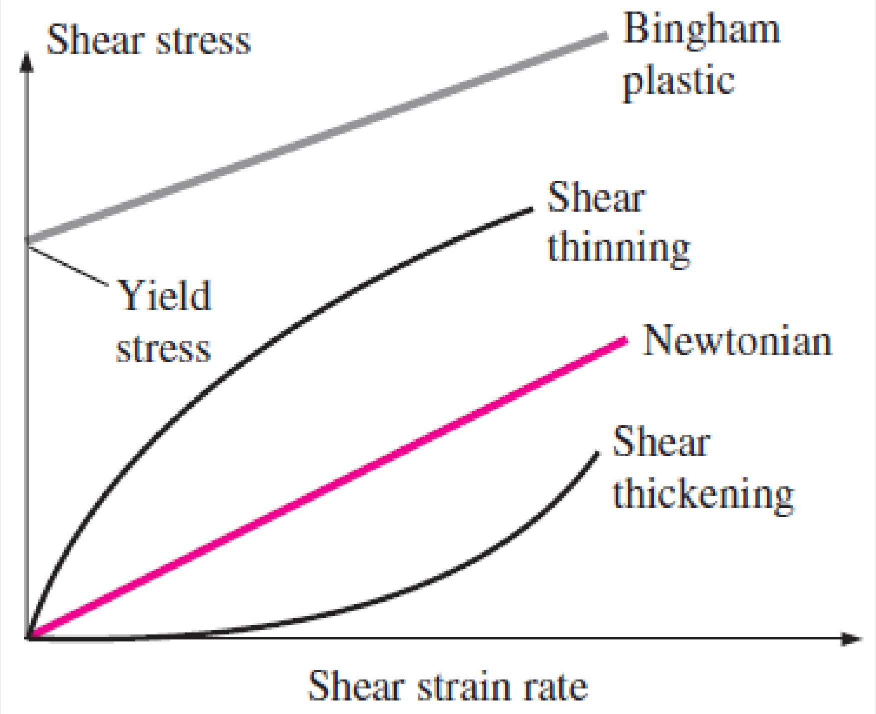
Mechanical pressure: $$P_m = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$



Mekanik basınç, bir akışkan elemanı üzerinde içe doğru etki eden ortalama normal gerilmedir.

Durgun haldeki akışkanlarda akışkan elemanı üzerindeki tek gerilme, daima her bir yüzeyin normali doğrultusunda ve içeri doğru etkiyen hidrostatik basınçtır.

Newton tipi ve Newton Tipi Olmayan Akışkanlar



Akışkanların reolojik davranışları; şekil değiştirme hızının fonksiyonu olarak kayma gerilmesi.

Bazı akışkanlar harekete geçirebilmek için **akma gerilmesi** denilen sonlu bir gerilmenin uygulanmasına ihtiyaç vardır. Bu tür akışkanlar **Bingham plastik akışkanları** olarak bilinir.

Reoloji: Akmakta olan akışkanların deformasyonunun incelenmesine denir.

Newton tipi akışkanlar: Kayma gerilmesinin şekil değiştirme hızıyla doğrusal değiştiği akışkanlardır.

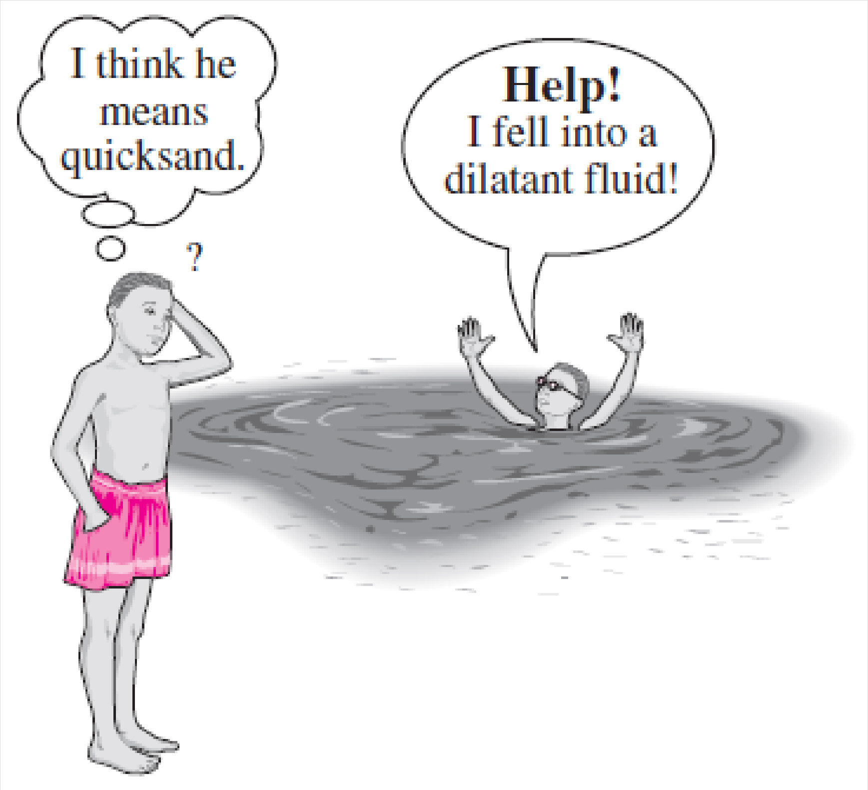
Newton tipi olmayan akışkanlar: Kayma gerilmesi ile şekil değiştirme hızının doğrusal olmadığı akışkanlardır.

Viskoelastik: Uygulanan gerilme kaldırıldığında baştaki asıl şekline (tamamen ya da kısmen) dönen akışkana denir.

Bazı Newton tipi olmayan akışkanlar, ne kadar hızlı şekil değişimine uğrarlarsa o denli daha az viskoz duruma geldiklerinden **incelen akışkanlar** veya **sanki- plastik akışkanlar** olarak adlandırılır.

Plastik akışkanlar incelme etkisinin en fazla görüldüğü akışkanlardır.

Kabalaşan akışkanlar veya kabaran (dilatant) akışkanlarda gerilme veya şekil değişimi hızı arttıkça akışkan *daha viskoz hale gelir.*



Bir mühendisin bataklık kumuna (bir dilatant akışkan) düştüğündeki hali. Daha çok hareket etmeye çalıştıkça akışkan daha viskoz hale gelmektedir.

Sıkıştırılamaz İzotermal Akış için Navier-Stokes Denkleminin Türetilmesi

For a fluid flow that is both incompressible and isothermal:

- $\rho = \text{constant}$
- $\mu = \text{constant}$

And therefore:

- $\nu = \text{constant}$

Sıkıştırılamaz akış yaklaştırmı yoğunluğun; izotermal yaklaştıranı ise viskozitenin sabit olduğu anlamına gelir.

Viscous stress tensor for an incompressible Newtonian fluid with constant properties:

$$\tau_{ij} = 2\mu\varepsilon_{ij} \tag{9–55}$$

$$\tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} 2\mu\dfrac{\partial u}{\partial x} & \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & 2\mu\dfrac{\partial v}{\partial y} & \mu\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \mu\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & 2\mu\dfrac{\partial w}{\partial z} \end{pmatrix}$$

$$\sigma_{ij} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} 2\mu \frac{\partial u}{\partial x} & \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & 2\mu \frac{\partial v}{\partial y} & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & 2\mu \frac{\partial w}{\partial z} \end{pmatrix}$$

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho g_x + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \mu \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\mu \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right) = \mu \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right)$$

$$\begin{aligned} \rho \frac{Du}{Dt} &= -\frac{\partial P}{\partial x} + \rho g_x + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x}\frac{\partial w}{\partial z} + \frac{\partial^2 u}{\partial z^2}\right] \\ &= -\frac{\partial P}{\partial x} + \rho g_x + \mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \end{aligned}$$

The Laplacian Operator

Cartesian coordinates:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Cylindrical coordinates:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Sıkışırılamaz Navier-Stokes denkleminin vizkoz terimlerinde görülen Laplace operatörünün Kartezyen ve silindirik koordinatlardaki açılımı.

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho g_x + \mu \nabla^2 u$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \rho g_y + \mu \nabla^2 v$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \rho g_z + \mu \nabla^2 w$$

Incompressible Navier–Stokes equation:

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} P + \rho \vec{g} + \mu \nabla^2 \vec{V} \quad (9\text{–}60)$$

Navier-Stokes denklemi akışkanlar mekaniğinin köşe taşıdır

Navier-Stokes denklemi; daimi olmayan, doğrusal olmayan, ikinci mertebeden bir kısmi diferansiyel denklemdir.

Denklem 9-60 dört bilinmeyene (**üç hız bileşeni ve basınç**) sahip olmakla birlikte sadece 3 denklemi (vektörel denklem olduğundan üç bileşeni vardır) temsil eder.

Açıkça anlaşılıyor ki bu denklemi çözülebilir hale getirmek için bir denkleme daha ihtiyaç vardır. Dördüncü denklem **sıkıştırılamaz süreklilik denklemidir**

Kartezyen Koordinatlarda Süreklilik ve Navier–Stokes Denklemleri

The continuity equation (Eq. 9–16) and the Navier–Stokes equation (Eq. 9–60) are expanded in Cartesian coordinates (x, y, z) and (u, v, w):

Incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(9–61a)}$$

x-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad \text{(9–61b)}$$

y-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad \text{(9–61c)}$$

z-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \qquad \text{(9–61d)}$$

Silindirik Koordinatlarda Süreklilik ve Navier–Stokes Denklemleri

Incompressible continuity equation:

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta}+\frac{\partial(u_z)}{\partial z}=0 \tag{9–62a}$$

r-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)$$

$$=-\frac{\partial P}{\partial r}+\rho g_r+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right] \tag{9–62b}$$

θ-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)$$

$$=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\rho g_\theta+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right] \tag{9–62c}$$

z-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)$$

$$=-\frac{\partial P}{\partial z}+\rho g_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right] \tag{9–62d}$$

Alternative Form of the Viscous Terms

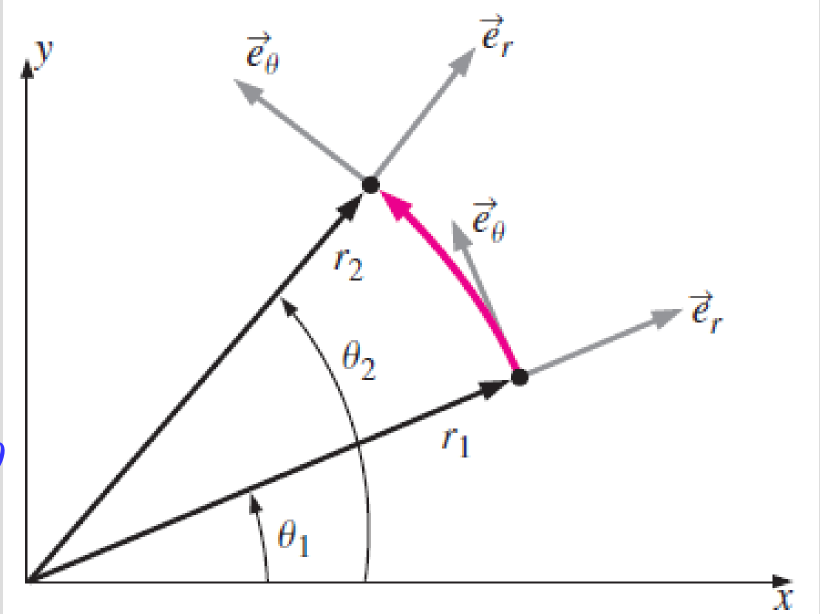
It can be shown that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)-\frac{u_r}{r^2}$$
$$=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}$$
$$=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)$$

Navier–Stokes denklemindeki *r*- ve θ bileşenlerinde ilk iki viskoz terim için alternatif bir form



Silindirik koordinatlarda $\vec{e}_r$ ve $\vec{e}_\theta$ birim vektörleri bağlıdır: θ-yönünde hareket edildiğinde $\vec{e}_r$'nin yönünde de değişim meydana gelir ve Navier Stokes Denkleminin *r*- ve θ- bileşenlerinde ek terimler ortaya çıkar.

$$\tau_{ij}=\begin{pmatrix}\tau_{rr} & \tau_{r\theta} & \tau_{rz}\\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z}\\ \tau_{zr} & \tau_{z\theta} & \tau_{zz}\end{pmatrix}$$

$$=\begin{pmatrix} 2\mu\frac{\partial u_r}{\partial r} & \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right] & \mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)\\ \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right] & 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right) & \mu\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)\\ \mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right) & \mu\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right) & 2\mu\frac{\partial u_z}{\partial z}\end{pmatrix}$$

9–6 ■ AKIŞ PROBLEMLERİNİN DİFERANSİYEL ANALİZİ

Diferansiyel hareket denklemlerinin (süreklilik ve N&S) kullanışlı olduğu iki tür problem vardır.

- Bilinen bir hız alanı için basınç alanı hesabı
- Bilinen geometri ve sınır şartları için hem hız hem de basınç alanlarının hesaplanması

Three-Dimensional Incompressible Flow

Four variables or unknowns:

- Pressure P
- Three components of velocity $\vec{V}$

Four equations of motion:

- Continuity,

$$\vec{\nabla} \cdot \vec{V} = 0$$

- Three components of Navier–Stokes,

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} P + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

Sabit özelliklere sahip genel üç-boyutlu fakat sıkıştırılamaz akış alanında dört bilinmeyeni bulmak için dört denkleme ihtiyaç vardır.

Bilinen Bir Hız Alanı için Basınç Alanının Hesaplanması

Birinci örnek gurubu, bilinen bir hız alanı için basınç alanının hesaplanmasını içermektedir.

Süreklilik denkleminde basınç bulunmadığından, hız alanını teorik olarak sadece kütlenin korunumuna dayanarak oluşturabiliriz.

Bununla birlikte hız, hem süreklilik hem de Navier-Stokes denkleminde bulunduğundan bu iki denklem *bağlıdır*.

Buna ilave olarak basınç, Navier-Stokes denkleminin her üç bileşeninde de yer alır ve böylece hız ve basınç alanları da bağlı haldedir.

Hız ve basınç alanları arasındaki bu bağlılık, bilinen bir hız alanı için basınç alanını hesaplamamıza olanak verir.

EXAMPLE 9–13 Calculating the Pressure Field in Cartesian Coordinates

Consider the steady, two-dimensional, incompressible velocity field of Example 9–9, namely, $\vec{V} = (u, v) = (ax + b)\vec{i} + (-ay + cx)\vec{j}$. Calculate the pressure as a function of x and y.

SOLUTION For a given velocity field, we are to calculate the pressure field.

Assumptions **1** The flow is steady and incompressible. **2** The fluid has constant properties. **3** The flow is two-dimensional in the xy-plane. **4** Gravity does not act in either the x- or y-direction.

Analysis First we check whether the given velocity field satisfies the two-dimensional, incompressible continuity equation:

$$\underbrace{\frac{\partial u}{\partial x}}_{a} + \underbrace{\frac{\partial v}{\partial y}}_{-a} + \underbrace{\frac{\partial w}{\partial z}}_{0\text{ (2-D)}} = a - a = 0 \qquad (1)$$

Thus, continuity is indeed satisfied by the given velocity field. If continuity were *not* satisfied, we would stop our analysis—the given velocity field would not be physically possible, and we could not calculate a pressure field.

Next, we consider the y-component of the Navier–Stokes equation:

$$\rho\left(\underbrace{\frac{\partial v}{\partial t}}_{0\text{ (steady)}} + \underbrace{u\frac{\partial v}{\partial x}}_{(ax+b)c} + \underbrace{v\frac{\partial v}{\partial y}}_{(-ay+cx)(-a)} + \underbrace{w\frac{\partial v}{\partial z}}_{0\text{ (2-D)}}\right) = -\frac{\partial P}{\partial y} + \underbrace{\rho g_y}_{0} + \mu\left(\underbrace{\frac{\partial^2 v}{\partial x^2}}_{0} + \underbrace{\frac{\partial^2 v}{\partial y^2}}_{0} + \underbrace{\frac{\partial^2 v}{\partial z^2}}_{0\text{ (2-D)}}\right)$$

The y-momentum equation reduces to

$$\frac{\partial P}{\partial y} = \rho(-acx - bc - a^2y + acx) = \rho(-bc - a^2y) \qquad \textbf{(2)}$$

The y-momentum equation is satisfied if we can generate a pressure field that satisfies Eq. 2. In similar fashion, the x-momentum equation reduces to

$$\frac{\partial P}{\partial x} = \rho(-a^2x - ab) \qquad \textbf{(3)}$$

The x-momentum equation is satisfied if we can generate a pressure field that satisfies Eq. 3.

In order for a steady flow solution to exist, P cannot be a function of time. Furthermore, a physically realistic steady, incompressible flow field requires a pressure field $P(x, y)$ that is a smooth function of x and y (there can be no sudden discontinuities in either P or a derivative of P). Mathematically, this requires that the order of differentiation (x then y versus y then x) should not matter (Fig. 9–46). We check whether this is so by cross-differentiating Eqs. 2 and 3, respectively,

$$\frac{\partial^2 P}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right) = 0 \quad \text{and} \quad \frac{\partial^2 P}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right) = 0 \qquad \textbf{(4)}$$

Equation 4 shows that P is indeed a smooth function of x and y. Thus, *the given velocity field satisfies the steady, two-dimensional, incompressible Navier– Stokes equation.*

If at this point in the analysis, the cross-differentiation of pressure were to yield two incompatible relationships (in other words if the equation in Fig. 9–46 were not satisfied) we would conclude that the given velocity field could not satisfy the steady, two-dimensional, incompressible Navier–Stokes equation, and we would abandon our attempt to calculate a steady pressure field.

To calculate $P(x, y)$, we partially integrate Eq. 2 (with respect to y) to obtain an expression for $P(x, y)$,

Pressure field from y-momentum:

$$P(x, y) = \rho\left(-bcy - \frac{a^2y^2}{2}\right) + g(x) \quad \textbf{(5)}$$

Note that we add an arbitrary function of the other variable x rather than a constant of integration since this is a partial integration. We then take the partial derivative of Eq. 5 with respect to x to obtain

$$\frac{\partial P}{\partial x} = g'(x) = \rho(-a^2x - ab) \quad \textbf{(6)}$$

Cross-Differentiation, *xy*-Plane

$P(x, y)$ is a smooth function of x and y only if the order of differentiation does not matter:

$$\frac{\partial^2 P}{\partial x\, \partial y} = \frac{\partial^2 P}{\partial y\, \partial x}$$

For a two-dimensional flow field in the *xy*-plane, cross-differentiation reveals whether pressure *P* is a smooth function.

where we have equated our result to Eq. 3 for consistency. We now integrate Eq. 6 to obtain the function $g(x)$:

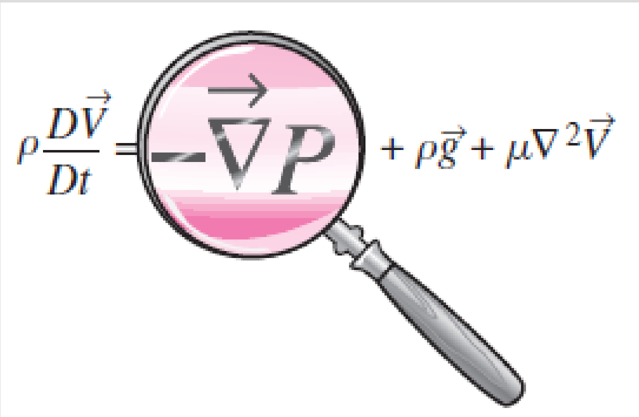
$$g(x) = \rho\left(-\frac{a^2x^2}{2} - abx\right) + C_1 \qquad \textbf{(7)}$$

where C_1 is an arbitrary constant of integration. Finally, we substitute Eq. 7 into Eq. 5 to obtain our final expression for $P(x, y)$. The result is

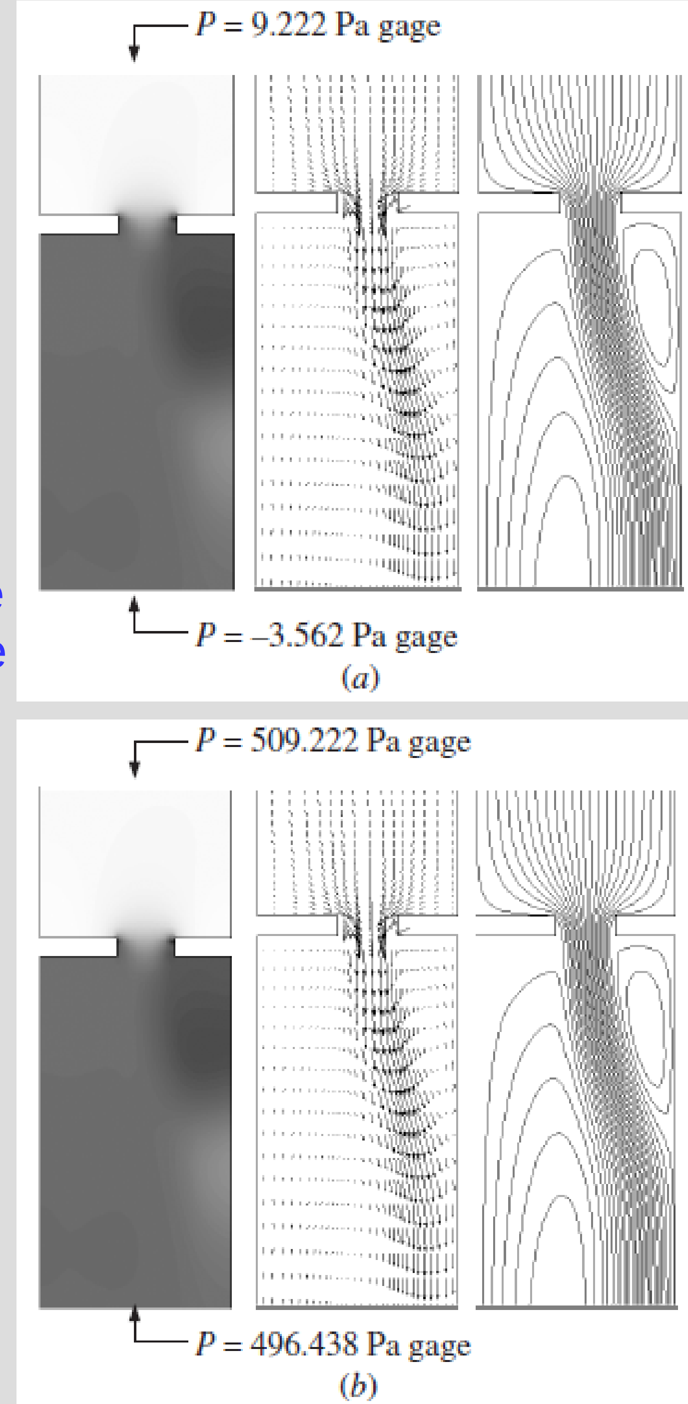
$$P(x, y) = \rho\left(-\frac{a^2x^2}{2} - \frac{a^2y^2}{2} - abx - bcy\right) + C_1 \qquad \textbf{(8)}$$

Discussion For practice, and as a check of our algebra, you should differentiate Eq. 8 with respect to both y and x, and compare to Eqs. 2 and 3. In addition, try to obtain Eq. 8 by starting with Eq. 3 rather than Eq. 2; you should get the same answer.

The velocity field in an incompressible flow is not affected by the absolute magnitude of pressure, but only by pressure differences.



Since pressure appears only as a gradient in the incompressible Navier–Stokes equation, the absolute magnitude of pressure is not relevant—only pressure *differences* matter.



Filled pressure contour plot, velocity vector plot, and streamlines for downward flow of air through a channel with blockage: (*a*) case 1; (*b*) case 2—identical to case 1, except *P* is everywhere increased by 500 Pa. On the gray-scale contour plots, dark is low pressure and light is high pressure.

Finally, we note that most CFD codes do *not* calculate pressure by integration of the Navier–Stokes equation as we have done in Example 9–13. Instead, some kind of **pressure correction algorithm** is used. Most of the commonly used algorithms work by combining the continuity and Navier–Stokes equations in such a way that pressure appears in the continuity equation. The most popular pressure correction algorithms result in a form of **Poisson's equation** for the change in pressure ΔP from one iteration (n) to the next $(n + 1)$,

Poisson's equation for ΔP: $$\nabla^2(\Delta P) = \text{RHS}_{(n)} \tag{9–64}$$

Then, as the computer iterates toward a solution, the modified continuity equation is used to "correct" the pressure field at iteration $(n + 1)$ from its values at iteration (n),

Correction for P: $$P_{(n+1)} = P_{(n)} + \Delta P$$

Details associated with the development of pressure correction algorithms is beyond the scope of the present text. An example for two-dimensional flows is developed in Gerhart, Gross, and Hochstein (1992).

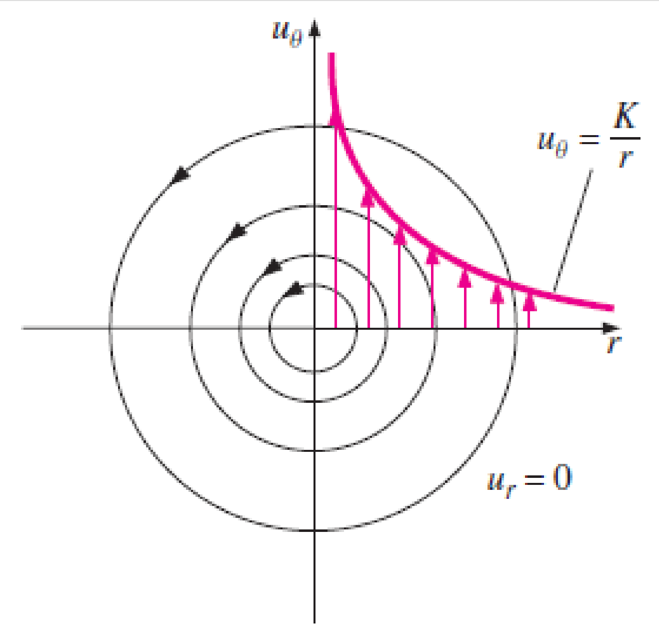
EXAMPLE 9–14 Calculating the Pressure Field in Cylindrical Coordinates

Consider the steady, two-dimensional, incompressible velocity field of Example 9–5 with function $f(\theta, t)$ equal to 0. This represents a line vortex whose axis lies along the *z*-coordinate (Fig. 9–49). The velocity components are $u_r = 0$ and $u_\theta = K/r$, where K is a constant. Calculate the pressure as a function of r and θ.

Analysis The flow field must satisfy both the continuity and the momentum equations, Eqs. 9–62. For steady, two-dimensional, incompressible flow,

Incompressible continuity:

$$\underbrace{\frac{1}{r}\frac{\partial(ru_r)}{\partial r}}_{0} + \underbrace{\frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta}}_{0} + \underbrace{\frac{\partial(u_z)}{\partial z}}_{0} = 0$$



Streamlines and velocity profiles for a line vortex.

Thus, the incompressible continuity equation is satisfied. Now we look at the θ component of the Navier–Stokes equation (Eq. 9–62c):

$$\rho\left(\underbrace{\frac{\partial u_\theta}{\partial t}}_{0\text{ (steady)}} + \underbrace{u_r\frac{\partial u_\theta}{\partial r}}_{(0)\left(-\frac{K}{r^2}\right)} + \underbrace{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}}_{\left(\frac{K}{r^2}\right)(0)} + \underbrace{\frac{u_r u_\theta}{r}}_{0} + \underbrace{u_z\frac{\partial u_\theta}{\partial z}}_{0\text{ (2-D)}}\right)$$

$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \underbrace{\rho g_\theta}_{0} + \mu\left(\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)}_{\frac{K}{r^3}} - \underbrace{\frac{u_\theta}{r^2}}_{\frac{K}{r^3}} + \underbrace{\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}}_{0} + \underbrace{\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}}_{0} + \underbrace{\frac{\partial^2 u_\theta}{\partial z^2}}_{0\text{ (2-D)}}\right)$$

The θ-momentum equation therefore reduces to

θ-momentum: $$\frac{\partial P}{\partial \theta} = 0 \qquad \textbf{(1)}$$

Thus, the θ-momentum equation is satisfied if we can generate an appropriate pressure field that satisfies Eq. 1. In similar fashion, the r-momentum equation (Eq. 9–62b) reduces to

r-momentum: $$\frac{\partial P}{\partial r} = \rho\frac{K^2}{r^3} \qquad \textbf{(2)}$$

Thus, the r-momentum equation is satisfied if we can generate a pressure field that satisfies Eq. 2.

In order for a steady flow solution to exist, P cannot be a function of time. Furthermore, a physically realistic steady, incompressible flow field requires a pressure field $P(r, \theta)$ that is a smooth function of r and θ. Mathematically, this requires that the order of differentiation (r then θ versus θ then r) should not matter (Fig. 9–50). We check whether this is so by cross-differentiating the pressure:

$$\frac{\partial^2 P}{\partial r\, \partial \theta} = \frac{\partial}{\partial r}\left(\frac{\partial P}{\partial \theta}\right) = 0 \qquad \text{and} \qquad \frac{\partial^2 P}{\partial \theta\, \partial r} = \frac{\partial}{\partial \theta}\left(\frac{\partial P}{\partial r}\right) = 0 \tag{3}$$

Equation 3 shows that P is indeed a smooth function of r and θ. Thus, *the given velocity field satisfies the steady, two-dimensional, incompressible Navier–Stokes equation.*

We integrate Eq. 1 with respect to θ to obtain an expression for $P(r, \theta)$,

Pressure field from θ-momentum: $$P(r, \theta) = 0 + g(r) \tag{4}$$

Note that we added an arbitrary function of the other variable r, rather than a constant of integration, since this is a partial integration. We take the partial derivative of Eq. 4 with respect to r to obtain

$$\frac{\partial P}{\partial r} = g'(r) = \rho \frac{K^2}{r^3} \tag{5}$$

Cross-Differentiation, $r\theta$-Plane

$P(r, \theta)$ is a smooth function of r and θ only if the order of differentiation does not matter:

$$\frac{\partial^2 P}{\partial r\, \partial \theta} = \frac{\partial^2 P}{\partial \theta\, \partial r}$$

For a two-dimensional flow field in the $r\theta$-plane, cross-differentiation reveals whether pressure P is a smooth function.

where we have equated our result to Eq. 2 for consistency. We integrate Eq. 5 to obtain the function $g(r)$:

$$g(r) = -\frac{1}{2}\rho\frac{K^2}{r^2} + C \qquad (6)$$

where C is an arbitrary constant of integration. Finally, we substitute Eq. 6 into Eq. 4 to obtain our final expression for $P(r, \theta)$. The result is

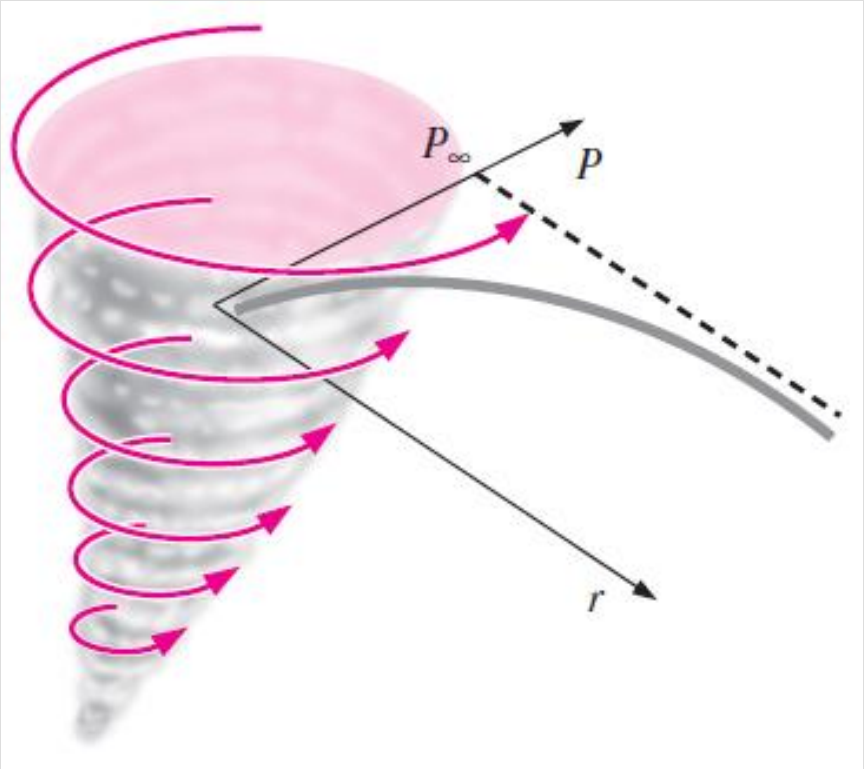
$$P(r, \theta) = -\frac{1}{2}\rho\frac{K^2}{r^2} + C \qquad (7)$$

Thus the pressure field for a line vortex decreases like $1/r^2$ as we approach the origin. (The origin itself is a singular point.) This flow field is a simplistic model of a tornado or hurricane, and the low pressure at the center is the "eye of the storm" (Fig. 9–51). We note that this flow field is irrotational, and thus Bernoulli's equation can be used instead to calculate the pressure. If we call the pressure P_∞ far away from the origin ($r \to \infty$), where the local velocity approaches zero, Bernoulli's equation shows that at any distance r from the origin,

Bernoulli equation: $$P + \frac{1}{2}\rho V^2 = P_\infty \quad \rightarrow \quad P = P_\infty - \frac{1}{2}\rho\frac{K^2}{r^2} \qquad (8)$$

Equation 8 agrees with our solution (Eq. 7) from the Navier–Stokes equation if we set constant C equal to P_∞. A region of rotational flow near the origin would avoid the singularity there and would yield a more physically realistic model of a tornado.

Discussion For practice, try to obtain Eq. 7 by starting with Eq. 2 rather than Eq. 1; you should get the same answer.



The two-dimensional line vortex is a simple approximation of a tornado; the lowest pressure is at the center of the vortex.

Süreklilik ve Navier–Stokes Denklemlerinin Kesin Çözümleri

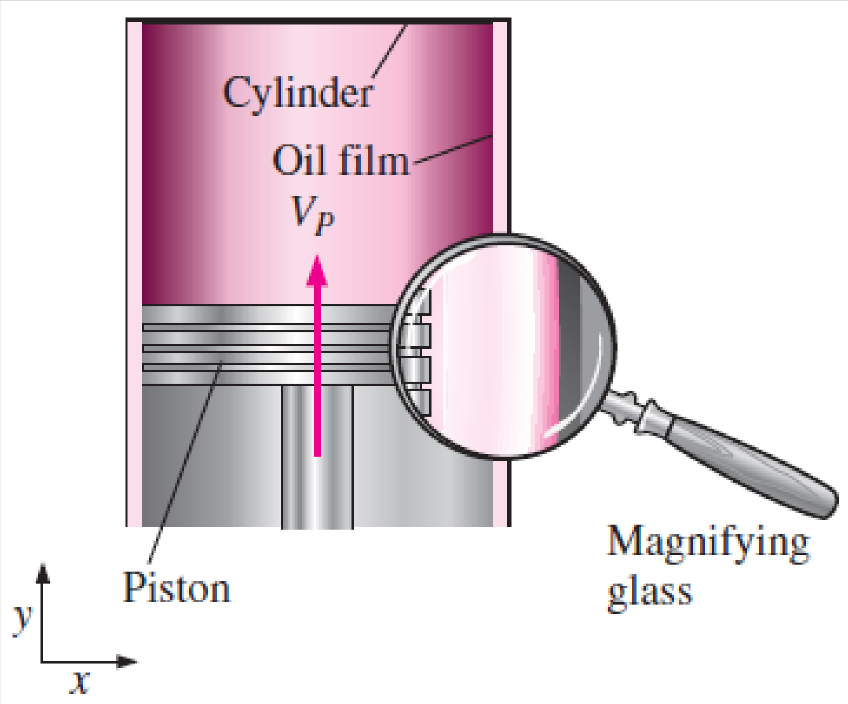
Adım 1: İlgili tüm boyut ve parametreleri tespit ederek problem ve geometri kurulur.
Adım 2: Uygun olan tüm kabuller, yaklaştırımlar, basitleştirmeler ve sınır şartları sıralanır.
Adım 3: Diferansiyel hareket denklemleri (süreklilik ve N&S) mümkün olduğunca basitleştirilir.
Adım 4: Denklemler integre edilerek bir veya daha fazla integral sabitine bağlı ifadeler elde edilir.
Adım 5: İntegral sabitlerini bulmak için sınır şartları uygulanır.
Adım 6: Sonuçlar doğrulanır.

Sıkıştırılamaz süreklilik ve Navier–Stokes denklemlerinin çözümünde izlenecek yol.

Sınır Şartları

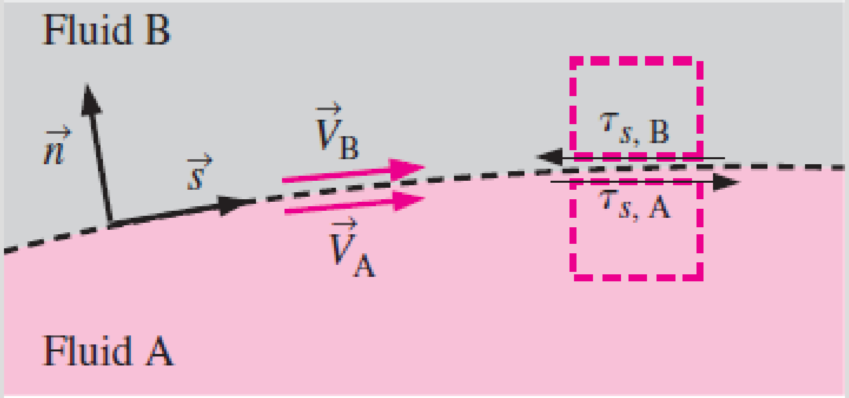
No-slip boundary condition:

$$\vec{V}_{\text{fluid}} = \vec{V}_{\text{wall}}$$

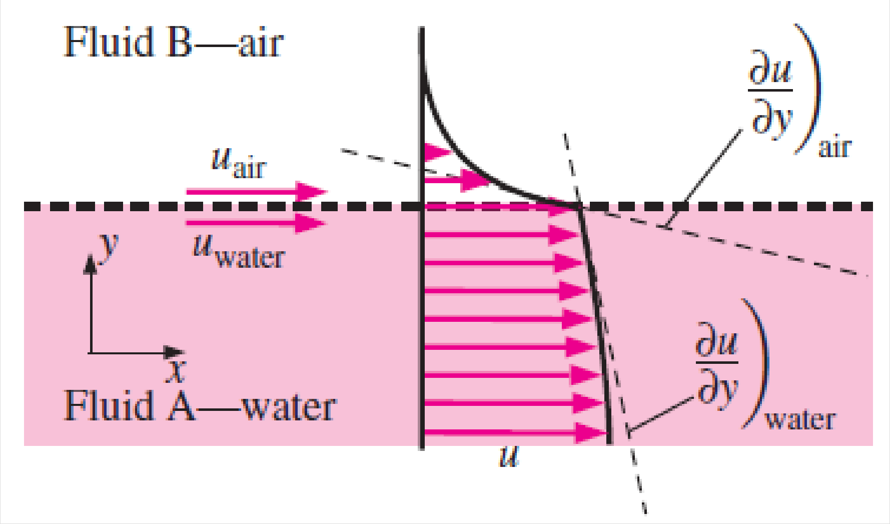


Bir silindir içinde V_p hızıyla hareket eden bir piston. Piston ile silindir arasında ince bir yağ filmi vardır. Şekilde filmin büyütülmüş görünüşü verilmiştir. Kaymama sınır koşulu çepere bitişik akışkanın hızının çeperin hızına eşit olmasını gerektirir.

Interface boundary conditions: $\vec{V}_A = \vec{V}_B$ and $\tau_{s,A} = \tau_{s,B}$



At an interface between two fluids, the velocity of the two fluids must be equal. In addition, the shear stress parallel to the interface must be the same in both fluids.

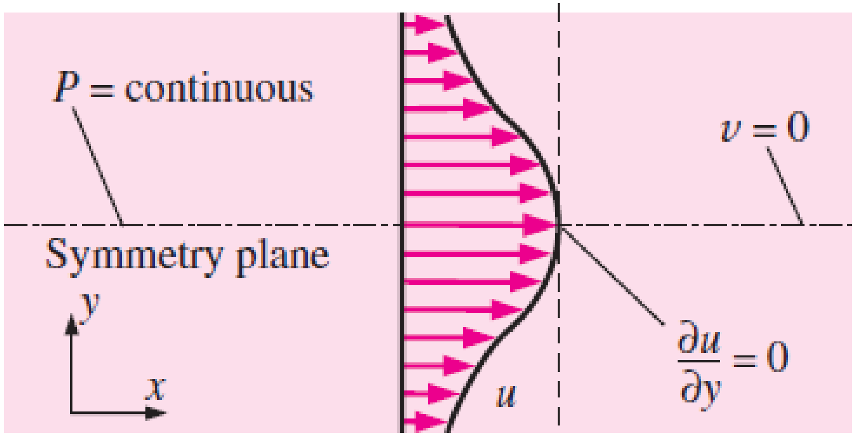


Along a horizontal *free surface* of water and air, the water and air velocities must be equal and the shear stresses must match. However, since $\mu_{air} << \mu_{water}$, a good approximation is that the shear stress at the water surface is negligibly small.

Boundary conditions at water–air interface:

$$u_{\text{water}} = u_{\text{air}} \quad \text{and} \quad \tau_{s,\text{ water}} = \mu_{\text{water}} \left.\frac{\partial u}{\partial y}\right)_{\text{water}} = \tau_{s,\text{ air}} = \mu_{\text{air}} \left.\frac{\partial u}{\partial y}\right)_{\text{air}}$$

Free-surface boundary conditions: $P_{\text{liquid}} = P_{\text{gas}}$ and $\tau_{s,\text{ liquid}} \cong 0$



Boundary conditions along a plane of symmetry are defined so as to ensure that the flow field on one side of the symmetry plane is a *mirror image* of that on the other side, as shown here for a horizontal symmetry plane.

Other boundary conditions arise depending on the problem setup.

For example, we often need to define **inlet boundary conditions** at a boundary of a flow domain where fluid enters the domain.

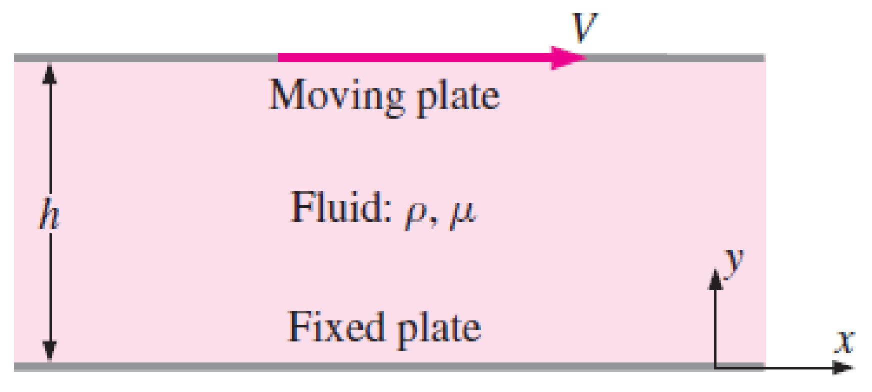
Likewise, we define **outlet boundary conditions** at an outflow.

Symmetry boundary conditions are useful along an axis or plane of symmetry.

For unsteady flow problems we also need to define **initial conditions** (at the starting time, usually $t = 0$).

EXAMPLE 9–15 Fully Developed Couette Flow

Consider steady, incompressible, laminar flow of a Newtonian fluid in the narrow gap between two infinite parallel plates (Fig. 9–57). The top plate is moving at speed V, and the bottom plate is stationary. The distance between these two plates is h, and gravity acts in the negative z-direction (into the page in Fig. 9–57). There is no applied pressure other than hydrostatic pressure due to gravity. This flow is called **Couette flow**. Calculate the velocity and pressure fields, and estimate the shear force per unit area acting on the bottom plate.



Geometry of Example 9–15: viscous flow between two infinite plates; upper plate moving and lower plate stationary.

Analysis To obtain the velocity and pressure fields, we follow the step-by-step procedure outlined in Fig. 9–52.

Step 1 *Set up the problem and the geometry*. See Fig. 9–57.

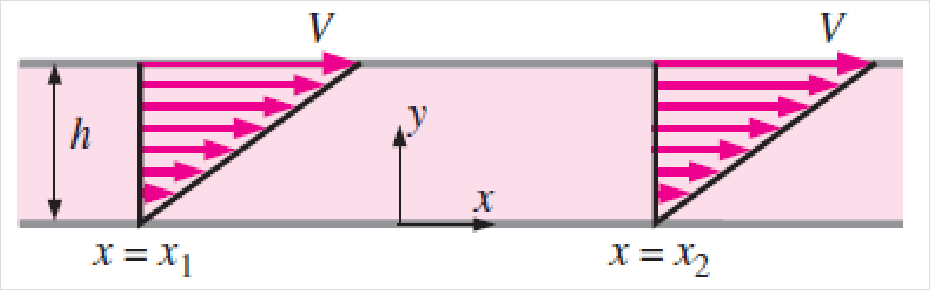
Step 2 *List assumptions and boundary conditions*. We have numbered and listed seven assumptions (above). The boundary conditions come from imposing the no-slip condition: (1) At the bottom plate ($y = 0$), $u = v = w = 0$. (2) At the top plate ($y = h$), $u = V$, $v = 0$, and $w = 0$.

Step 3 *Simplify the differential equations.* We start with the incompressible continuity equation in Cartesian coordinates, Eq. 9–61a,

$$\frac{\partial u}{\partial x} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{\text{assumption 3}} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{\text{assumption 6}} = 0 \quad \rightarrow \quad \frac{\partial u}{\partial x} = 0 \qquad (1)$$

Equation 1 tells us that u is not a function of x. In other words, it doesn't matter where we place our origin—the flow is the same at any x-location. The phrase **fully developed** is often used to describe this situation (Fig. 9–58). This can also be obtained directly from assumption 1, which tells us that there is nothing special about any x-location since the plates are infinite in length. Furthermore, since u is not a function of time (assumption 2) or z (assumption 6), we conclude that u is at most a function of y,

Result of continuity: $$u = u(y) \text{ only} \qquad (2)$$



A *fully developed* region of a flow field is a region where the velocity profile does not change with downstream distance. Fully developed flows are encountered in long, straight channels and pipes. Fully developed Couette flow is shown here—the velocity profile at x_2 is identical to that at x_1.

We now simplify the x-momentum equation (Eq. 9–61b) as far as possible. It is good practice to list the reason for crossing out a term, as we do here:

$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{assumption 2}} + \underbrace{u\cancel{\frac{\partial u}{\partial x}}}_{\text{continuity}} + \underbrace{\cancel{v}\frac{\partial u}{\partial y}}_{\text{assumption 3}} + \underbrace{w\cancel{\frac{\partial u}{\partial z}}}_{\text{assumption 6}}\right) = \underbrace{-\cancel{\frac{\partial P}{\partial x}}}_{\text{assumption 5}} + \underbrace{\cancel{\rho g_x}}_{\text{assumption 7}}$$

$$+ \mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{continuity}} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{\text{assumption 6}}\right) \quad \rightarrow \quad \frac{d^2 u}{dy^2} = 0 \quad \textbf{(3)}$$

Notice that the material acceleration (left-hand side of Eq. 3) is zero, implying that fluid particles are not accelerating in this flow field, neither by local (unsteady) acceleration, nor by advective acceleration. Since the advective acceleration terms make the Navier–Stokes equation nonlinear, this greatly simplifies the problem. In fact, all other terms in Eq. 3 have disappeared except for a lone viscous term, which must then itself equal zero. Also notice that we have changed from a partial derivative ($\partial/\partial y$) to a total derivative (d/dy) in Eq. 3 as a direct result of Eq. 2. We do not show the details here, but you can show in similar fashion that every term except the pressure term in the y-momentum equation (Eq. 9–61c) goes to zero, forcing that lone term to also be zero,

$$\frac{\partial P}{\partial y} = 0 \qquad \textbf{(4)}$$

In other words, P is not a function of y. Since P is also not a function of time (assumption 2) or x (assumption 5), P is at most a function of z,

Result of y-momentum: $\quad P = P(z)$ only $\qquad$ **(5)**

Finally, by assumption 6 the z-component of the Navier–Stokes equation (Eq. 9–61d) simplifies to

$$\frac{\partial P}{\partial z} = -\rho g \qquad \rightarrow \qquad \frac{dP}{dz} = -\rho g \tag{6}$$

where we used Eq. 5 to convert from a partial derivative to a total derivative.

Step 4 *Solve the differential equations.* Continuity and y-momentum have already been "solved," resulting in Eqs. 2 and 5, respectively. Equation 3 (x-momentum) is integrated twice to get

$$u = C_1 y + C_2 \tag{7}$$

where C_1 and C_2 are constants of integration. Equation 6 (z-momentum) is integrated once, resulting in

$$P = -\rho g z + C_3 \tag{8}$$

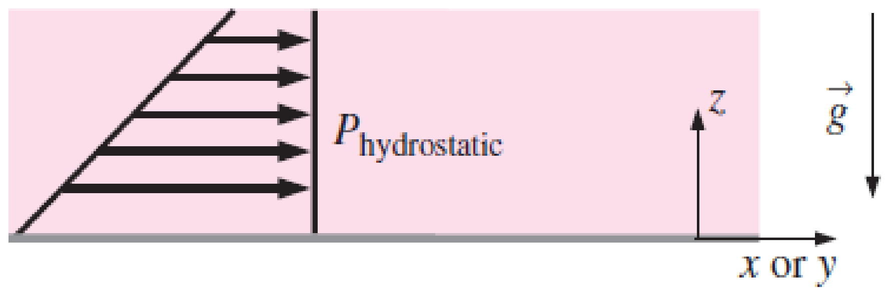
Step 5 *Apply boundary conditions.* We begin with Eq. 8. Since we have not specified boundary conditions for pressure, C_3 remains an arbitrary constant. (Recall that for incompressible flow, the absolute pressure can be specified only if P is known somewhere in the flow.) For example, if we let $P = P_0$ at $z = 0$, then $C_3 = P_0$ and Eq. 8 becomes

Final solution for pressure field: $$P = P_0 - \rho g z \tag{9}$$

Alert readers will notice that Eq. 9 represents a simple **hydrostatic pressure distribution** (pressure decreasing linearly as z increases). We conclude that, at least for this problem, *hydrostatic pressure acts independently of the flow.* More generally, we make the following statement (see also Fig. 9–59):

For incompressible flow fields without free surfaces, hydrostatic pressure does not contribute to the dynamics of the flow field.

In fact, in Chap. 10 we show how hydrostatic pressure can actually be *removed* from the equations of motion through use of a modified pressure.



For incompressible flow fields *without free surfaces*, hydrostatic pressure does not contribute to the dynamics of the flow field.

We next apply boundary conditions (1) and (2) from step 2 to obtain constants C_1 and C_2.

Boundary condition (1): $u = C_1 \times 0 + C_2 = 0 \quad \rightarrow \quad C_2 = 0$

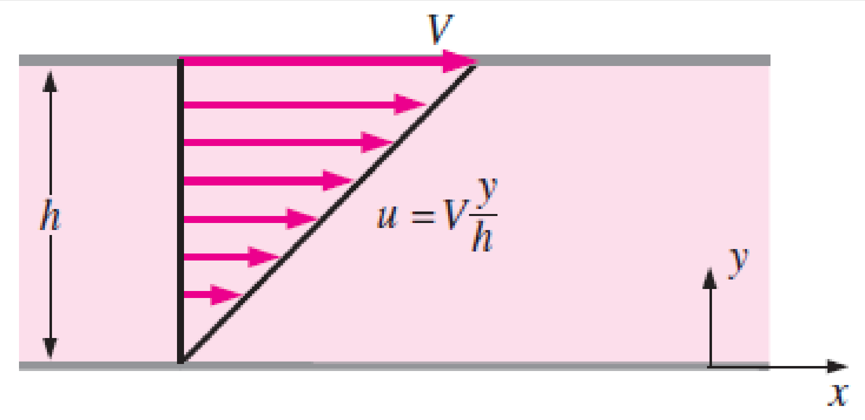
and

Boundary condition (2): $u = C_1 \times h + 0 = V \quad \rightarrow \quad C_1 = V/h$

Finally, Eq. 7 becomes

Final result for velocity field: $$u = V\frac{y}{h} \qquad \textbf{(10)}$$

The velocity field reveals a simple linear velocity profile from $u = 0$ at the bottom plate to $u = V$ at the top plate, as sketched in Fig. 9–60.

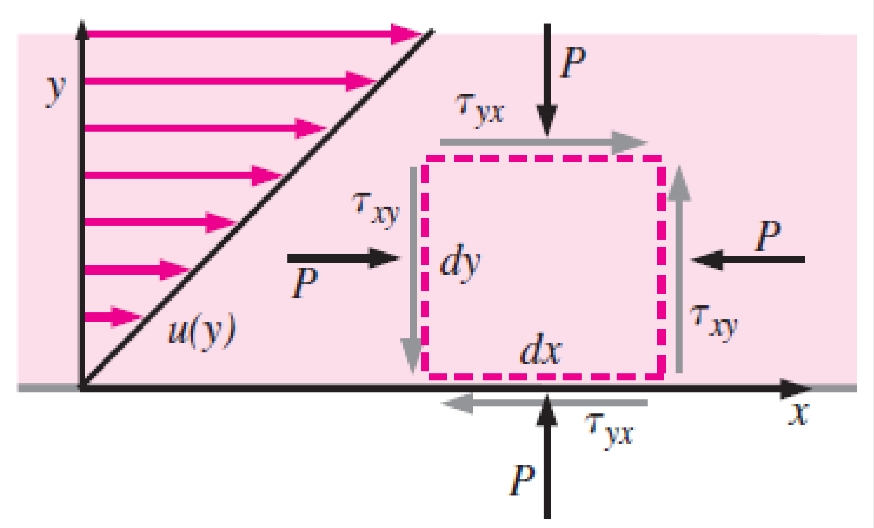


The linear velocity profile of Example 9–15: Couette flow between parallel plates.

Step 6 *Verify the results.* Using Eqs. 9 and 10, you can verify that all the differential equations and boundary conditions are satisfied.

To calculate the shear force per unit area acting on the bottom plate, we consider a rectangular fluid element whose bottom face is in contact with the bottom plate (Fig 9–61). Mathematically positive viscous stresses are shown. In this case, these stresses are in the proper direction since fluid above the differential element pulls it to the right while the wall below the element pulls it to the left. From Eq. 9–56, we write out the components of the viscous stress tensor,

$$\tau_{ij} = \begin{pmatrix} 2\mu \dfrac{\partial u}{\partial x} & \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & 2\mu \dfrac{\partial v}{\partial y} & \mu\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \mu\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & 2\mu \dfrac{\partial w}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & \mu\dfrac{V}{h} & 0 \\ \mu\dfrac{V}{h} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (11)$$



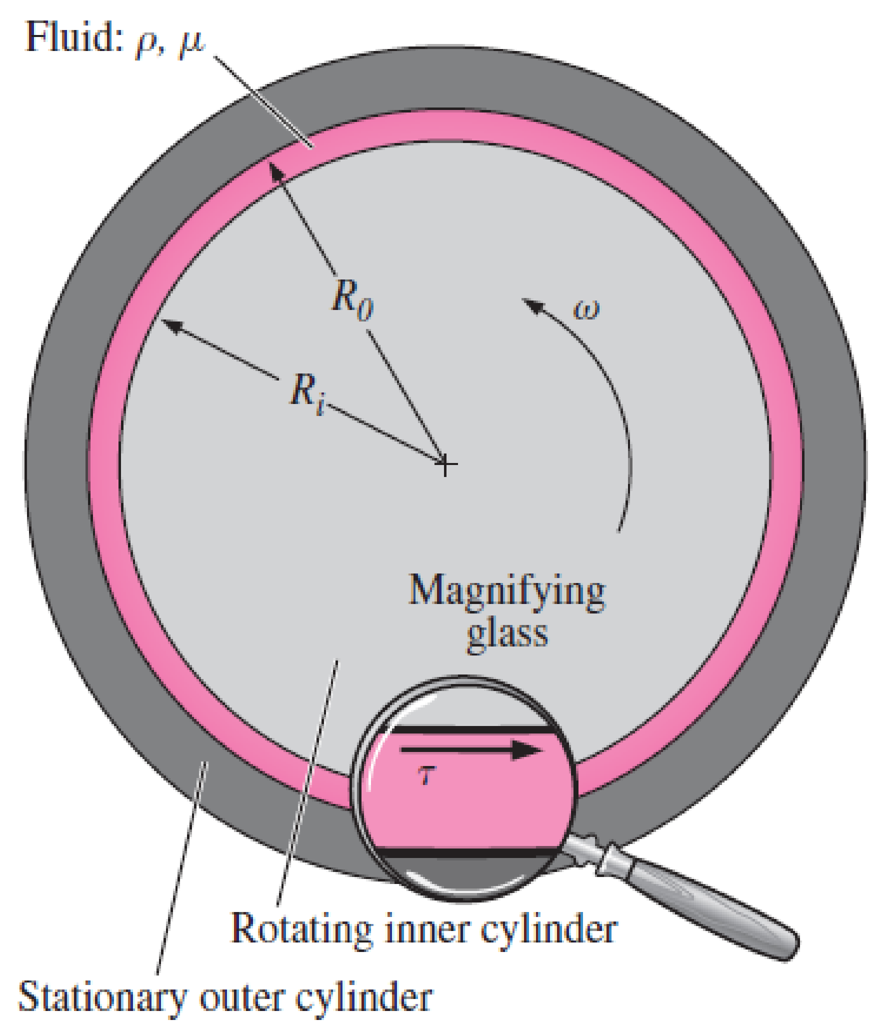
Stresses acting on a differential two-dimensional rectangular fluid element whose bottom face is in contact with the bottom plate of Example 9–15.

Since the dimensions of stress are force per unit area by definition, the force per unit area acting on the bottom face of the fluid element is equal to $\tau_{yx} = \mu V/h$ and acts in the negative x-direction, as sketched. The shear force per unit area on the *wall* is equal and opposite to this (Newton's third law); hence,

Shear force per unit area acting on the wall:
$$\frac{\vec{F}}{A} = \mu \frac{V}{h}\vec{i} \qquad \textbf{(12)}$$

The direction of this force agrees with our intuition; namely, the fluid tries to pull the bottom wall to the right, due to viscous effects (friction).

Discussion The z-component of the linear momentum equation is *uncoupled* from the rest of the equations; this explains why we get a hydrostatic pressure distribution in the z-direction, even though the fluid is not static, but moving. Equation 11 reveals that the viscous stress tensor is constant *everywhere* in the flow field, not just at the bottom wall (notice that none of the components of τ_{ij} is a function of location).



A rotational viscometer; the inner cylinder rotates at angular velocity ω, and a torque $T_{applied}$ is applied, from which the viscosity of the fluid is calculated.

$$\tau = \tau_{yx} \cong \mu \frac{V}{R_o - R_i} = \mu \frac{\omega R_i}{R_o - R_i}$$

$$T_{viscous} = \tau A R_i \cong \mu \frac{\omega R_i}{R_o - R_i}\left(2\pi R_i L\right) R_i$$

Viscosity of the fluid: $$\mu = T_{applied} \frac{(R_o - R_i)}{2\pi \omega R_i^3 L}$$

EXAMPLE 9–16 Couette Flow with an Applied Pressure Gradient

Consider the same geometry as in Example 9–15, but instead of pressure being constant with respect to *x*, let there be an applied pressure gradient in the *x*-direction (Fig. 9–63). Specifically, let the pressure gradient in the *x*-direction, $\partial P/\partial x$, be some constant value given by

Applied pressure gradient: $$\frac{\partial P}{\partial x} = \frac{P_2 - P_1}{x_2 - x_1} = \text{constant} \quad (1)$$

where x_1 and x_2 are two arbitrary locations along the *x*-axis, and P_1 and P_2 are the pressures at those two locations. Everything else is the same as for Example 9–15. (*a*) Calculate the velocity and pressure field. (*b*) Plot a family of velocity profiles in dimensionless form.

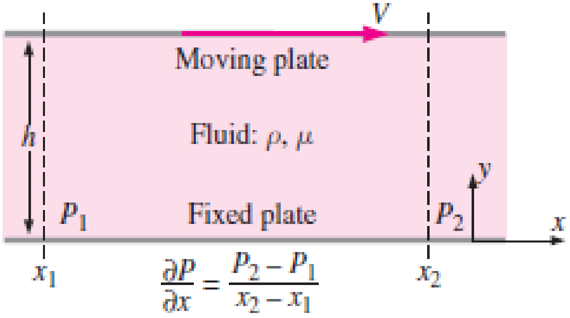


FIGURE 9–63
Geometry of Example 9–16: viscous flow between two infinite plates with a constant applied pressure gradient $\partial P/\partial x$; the upper plate is moving and the lower plate is stationary.

SOLUTION We are to calculate the velocity and pressure field for the flow sketched in Fig. 9–63 and plot a family of velocity profiles in dimensionless form.

Assumptions The assumptions are identical to those of Example 9–15, except assumption 5 is replaced by the following: A constant pressure gradient is applied in the *x*-direction such that pressure changes linearly with respect to *x* according to Eq. 1.

Analysis (*a*) We follow the same procedure as in Example 9–15. Much of the algebra is identical, so to save space we discuss only the differences.

Step 1 See Fig. 9–63.

Step 2 Same as Example 9–15 except for assumption 5.

Step 3 The continuity equation is simplified in the same way as in Example 9–15,

Result of continuity: $$u = u(y) \text{ only} \quad (2)$$

The x-momentum equation is simplified in the same manner as in Example 9–15 except that the pressure gradient term remains. The result is

Result of x-momentum:
$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{\partial P}{\partial x} \quad (3)$$

Likewise, the y-momentum and z-momentum equations simplify to

Result of y-momentum:
$$\frac{\partial P}{\partial y} = 0 \quad (4)$$

and

Result of z-momentum:
$$\frac{\partial P}{\partial z} = -\rho g \quad (5)$$

We cannot convert from a partial derivative to a total derivative in Eq. 5, because P is a function of both x and z in this problem, unlike in Example 9–15 where P was a function of z only.

Step 4 We integrate Eq. 3 (x-momentum) twice, noting that $\partial P/\partial x$ is a constant,

Integration of x-momentum:
$$u = \frac{1}{2\mu}\frac{\partial P}{\partial x}y^2 + C_1 y + C_2 \quad (6)$$

where C_1 and C_2 are constants of integration. Equation 5 (z-momentum) is integrated once, resulting in

Integration of z-momentum:
$$P = -\rho g z + f(x) \quad (7)$$

Note that since P is now a function of both x and z, we add a function of x instead of a constant of integration in Eq. 7. This is a *partial* integration with respect to z, and we must be careful when performing partial integrations (Fig. 9–64).

Step 5 From Eq. 7, we see that the pressure varies hydrostatically in the z-direction, and we have specified a linear change in pressure in the x-direction. Thus the function $f(x)$ must equal a constant plus $\partial P/\partial x$ times x. If we set $P = P_0$ along the line $x = 0$, $z = 0$ (the y-axis), Eq. 7 becomes

Final result for pressure field:
$$P = P_0 + \frac{\partial P}{\partial x}x - \rho g z \quad (8)$$

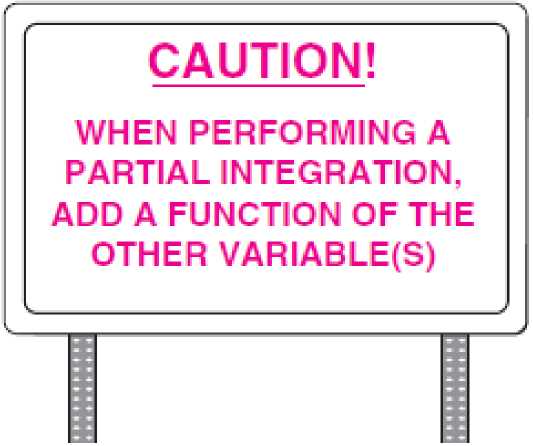


FIGURE 9–64
A caution about partial integration.

We next apply the velocity boundary conditions (1) and (2) from step 2 of Example 9–15 to obtain constants C_1 and C_2.

Boundary condition (1):

$$u = \frac{1}{2\mu}\frac{\partial P}{\partial x} \times 0 + C_1 \times 0 + C_2 = 0 \quad \rightarrow \quad C_2 = 0$$

and

Boundary condition (2):

$$u = \frac{1}{2\mu}\frac{\partial P}{\partial x} h^2 + C_1 \times h + 0 = V \quad \rightarrow \quad C_1 = \frac{V}{h} - \frac{1}{2\mu}\frac{\partial P}{\partial x} h$$

Finally, Eq. 6 becomes

$$u = \frac{Vy}{h} + \frac{1}{2\mu}\frac{\partial P}{\partial x}(y^2 - hy) \qquad \textbf{(9)}$$

Equation 9 indicates that the velocity field consists of the superposition of two parts: a linear velocity profile from $u = 0$ at the bottom plate to $u = V$ at the top plate, and a parabolic distribution that depends on the magnitude of the applied pressure gradient. If the pressure gradient is zero, the parabolic portion of Eq. 9 disappears and the profile is linear, just as in Example 9–15; this is sketched as the dashed line in Fig. 9–65. If the pressure gradient is negative (pressure decreasing in the x-direction, causing flow to be pushed from left to right), $\partial P/\partial x < 0$ and the velocity profile looks like the one sketched in Fig. 9–65. A special case is when $V = 0$ (top plate stationary); the linear portion of Eq. 9 vanishes, and the velocity profile is parabolic and symmetric about the center of the channel ($y = h/2$); this is sketched as the dotted line in Fig. 9–65.

Step 6 You can use Eqs. 8 and 9 to verify that all the differential equations and boundary conditions are satisfied.

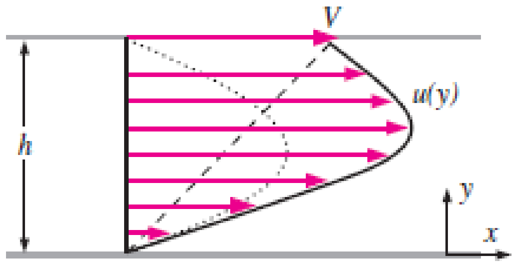


FIGURE 9–65
The velocity profile of Example 9–16: Couette flow between parallel plates with an applied negative pressure gradient; the dashed line indicates the profile for a zero pressure gradient, and the dotted line indicates the profile for a negative pressure gradient with the upper plate stationary ($V = 0$).

(*b*) We use dimensional analysis to generate the dimensionless groups (Π groups). We set up the problem in terms of velocity component u as a function of y, h, V, μ, and $\partial P/\partial x$. There are six variables (including the dependent variable u), and since there are three primary dimensions represented in the problem (mass, length, and time), we expect $6 - 3 = 3$ dimensionless groups. When we pick h, V, and μ as our repeating variables, we get the following result using the method of repeating variables (details are left for you to do on your own—this is a good review of Chap. 7 material):

Result of dimensional analysis:
$$\frac{u}{V} = f\left(\frac{y}{h}, \frac{h^2}{\mu V}\frac{\partial P}{\partial x}\right) \tag{10}$$

Using these three dimensionless groups, we rewrite Eq. 9 as

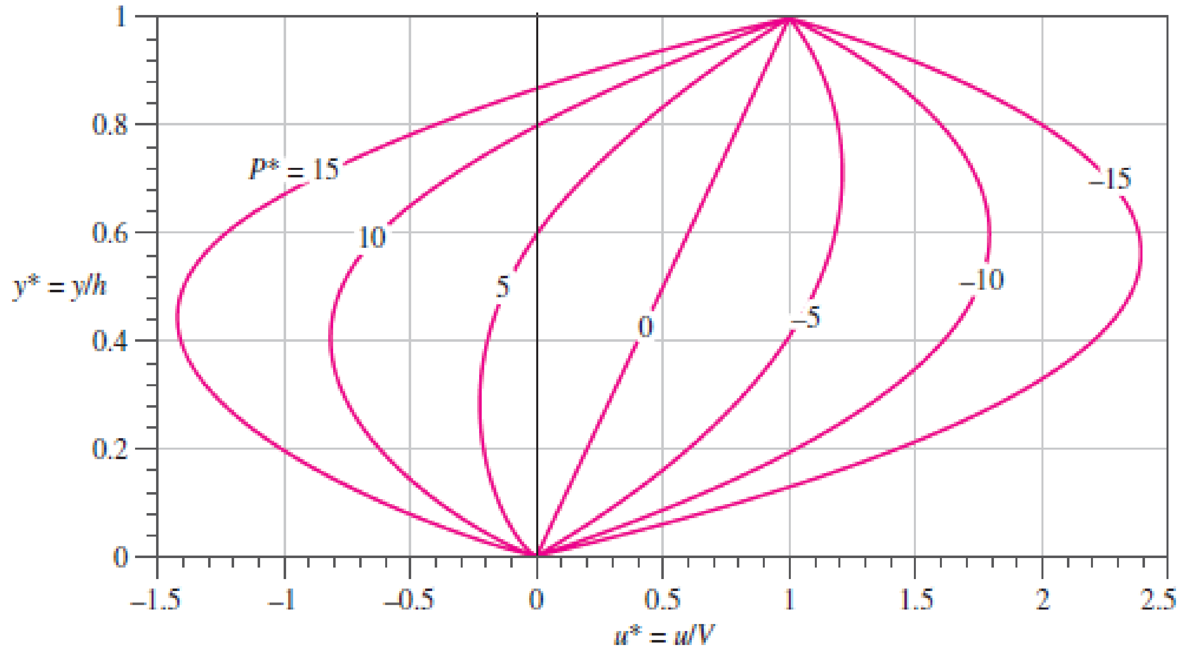
Dimensionless form of velocity field:
$$u^* = y^* + \frac{1}{2}P^*y^*(y^* - 1) \tag{11}$$

where the dimensionless parameters are

$$u^* = \frac{u}{V} \qquad y^* = \frac{y}{h} \qquad P^* = \frac{h^2}{\mu V}\frac{\partial P}{\partial x}$$

In Fig. 9–66, u^* is plotted as a function of y^* for several values of P^*, using Eq. 11.

FIGURE 9–66
Nondimensional velocity profiles for Couette flow with an applied pressure gradient; profiles are shown for several values of nondimensional pressure gradient.



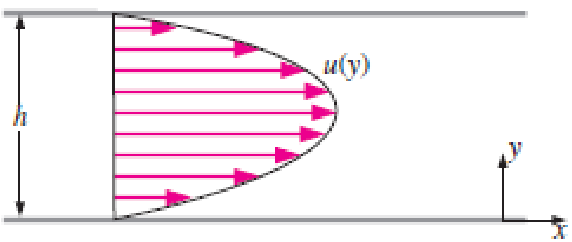


FIGURE 9–67
The velocity profile for fully developed two-dimensional channel flow (planar Poiseuille flow).

Discussion When the result is nondimensionalized, we see that Eq. 11 represents a *family* of velocity profiles. We also see that when the pressure gradient is *positive* (flow being pushed from right to left) and of sufficient magnitude, we can have *reverse flow* in the bottom portion of the channel. For all cases, the boundary conditions reduce to $u^* = 0$ at $y^* = 0$ and $u^* = 1$ at $y^* = 1$. If there is a pressure gradient but both walls are stationary, the flow is called two-dimensional channel flow, or **planar Poiseuille flow** (Fig. 9–67). We note, however, that most authors reserve the name *Poiseuille flow* for fully developed *pipe* flow—the axisymmetric analog of two-dimensional channel flow (see Example 9–18).

EXAMPLE 9–17 Oil Film Flowing Down a Vertical Wall by Gravity

Consider steady, incompressible, parallel, laminar flow of a film of oil falling slowly down an infinite vertical wall (Fig. 9–68). The oil film thickness is h, and gravity acts in the negative z-direction (downward in Fig. 9–68). There is no applied (forced) pressure driving the flow—the oil falls by gravity alone. Calculate the velocity and pressure fields in the oil film and sketch the normalized velocity profile. You may neglect changes in the hydrostatic pressure of the surrounding air.

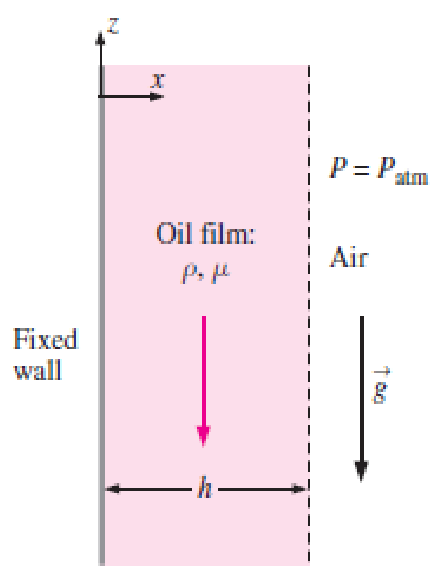


FIGURE 9–68
Geometry of Example 9–17: a viscous film of oil falling by gravity along a vertical wall.

SOLUTION For a given geometry and set of boundary conditions, we are to calculate the velocity and pressure fields and plot the velocity profile.

Assumptions **1** The wall is infinite in the yz-plane (y is into the page for a right-handed coordinate system). **2** The flow is steady (all partial derivatives with respect to time are zero). **3** The flow is parallel (the x-component of velocity, u, is zero everywhere). **4** The fluid is incompressible and Newtonian with constant properties, and the flow is laminar. **5** Pressure $P = P_{atm}$ = constant at the free surface. In other words, there is no applied pressure gradient pushing the flow; the flow establishes itself due to a balance between gravitational forces and viscous forces. In addition, since there is no gravity force in the horizontal direction, $P = P_{atm}$ *everywhere*. **6** The velocity field is purely two-dimensional, which implies that velocity component $v = 0$ and all partial derivatives with respect to y are zero. **7** Gravity acts in the negative z-direction. We express this mathematically as $\vec{g} = -g\vec{k}$, or $g_x = g_y = 0$ and $g_z = -g$.

Analysis We obtain the velocity and pressure fields by following the step-by-step procedure for differential fluid flow solutions. (Fig. 9–52).

Step 1 *Set up the problem and the geometry.* See Fig. 9–68.

Step 2 *List assumptions and boundary conditions.* We have listed seven assumptions. The boundary conditions are: (1) There is no slip at the wall; at $x = 0$, $u = v = w = 0$. (2) At the free surface ($x = h$), there is negligible shear (Eq. 9–68), which for a vertical free surface in this coordinate system means $\partial w/\partial x = 0$ at $x = h$.

Step 3 *Write out and simplify the differential equations.* We start with the incompressible continuity equation in Cartesian coordinates,

$$\underbrace{\frac{\partial u}{\partial x}}_{\text{assumption 3}} + \underbrace{\frac{\partial v}{\partial y}}_{\text{assumption 6}} + \frac{\partial w}{\partial z} = 0 \quad \rightarrow \quad \frac{\partial w}{\partial z} = 0 \qquad \textbf{(1)}$$

Equation 1 tells us that w is not a function of z; i.e., it doesn't matter where we place our origin—the flow is the same at *any* z-location. In other words, the flow is *fully developed*. Since w is not a function of time (assumption 2), z (Eq. 1), or y (assumption 6), we conclude that w is at most a function of x,

Result of continuity: $\qquad w = w(x)$ only $\qquad$ **(2)**

We now simplify each component of the Navier–Stokes equation as far as possible. Since $u = v = 0$ everywhere, and gravity does not act in the x- or y-directions, the x- and y-momentum equations are satisfied exactly (in fact all terms are zero in both equations). The z-momentum equation reduces to

$$\rho\Bigg(\underbrace{\frac{\partial w}{\partial t}}_{\text{assumption 2}} + \underbrace{u\frac{\partial w}{\partial x}}_{\text{assumption 3}} + \underbrace{v\frac{\partial w}{\partial y}}_{\text{assumption 6}} + \underbrace{w\frac{\partial w}{\partial z}}_{\text{continuity}}\Bigg) = \underbrace{-\frac{\partial P}{\partial z}}_{\text{assumption 5}} + \underbrace{\rho g_z}_{-\rho g}$$

$$+ \mu\Bigg(\frac{\partial^2 w}{\partial x^2} + \underbrace{\frac{\partial^2 w}{\partial y^2}}_{\text{assumption 6}} + \underbrace{\frac{\partial^2 w}{\partial z^2}}_{\text{continuity}}\Bigg) \quad \rightarrow \quad \frac{d^2 w}{dx^2} = \frac{\rho g}{\mu} \qquad \textbf{(3)}$$

The material acceleration (left side of Eq. 3) is zero, implying that fluid particles are not accelerating in this flow field, neither by local nor advective acceleration. Since the advective acceleration terms make the Navier–Stokes equation nonlinear, this greatly simplifies the problem. We have changed from a partial derivative ($\partial/\partial x$) to a total derivative (d/dx) in Eq. 3 as a direct result of Eq. 2, reducing the partial differential equation (PDE) to an ordinary differential equation (ODE). ODEs are of course much easier than PDEs to solve (Fig. 9–69).

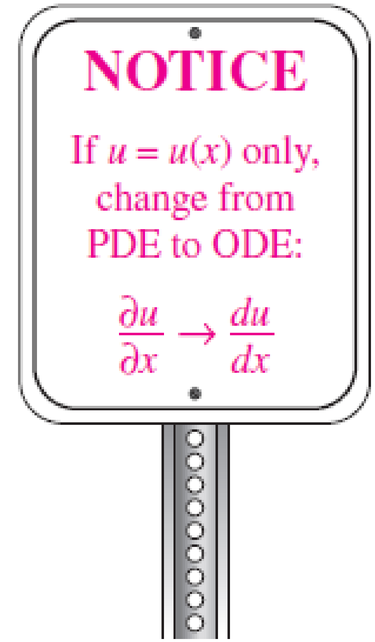


FIGURE 9–69
In Examples 9–15 through 9–18, the equations of motion are reduced from *partial differential equations* to *ordinary differential equations*, making them much easier to solve.

Step 4 *Solve the differential equations.* The continuity and *x*- and *y*-momentum equations have already been "solved." Equation 3 (*z*-momentum) is integrated twice to get

$$w = \frac{\rho g}{2\mu} x^2 + C_1 x + C_2 \tag{4}$$

Step 5 *Apply boundary conditions.* We apply boundary conditions (1) and (2) from step 2 to obtain constants C_1 and C_2,

Boundary condition (1): $w = 0 + 0 + C_2 = 0 \qquad C_2 = 0$

and

Boundary condition (2): $\left.\frac{dw}{dx}\right)_{x=h} = \frac{\rho g}{\mu} h + C_1 = 0 \quad \rightarrow \quad C_1 = -\frac{\rho g h}{\mu}$

Finally, Eq. 4 becomes

Velocity field: $$w = \frac{\rho g}{2\mu} x^2 - \frac{\rho g}{\mu} hx = \frac{\rho g x}{2\mu}(x - 2h) \tag{5}$$

Since $x < h$ in the film, w is negative everywhere, as expected (flow is downward). The pressure field is trivial; namely, $P = P_{atm}$ everywhere.

Step 6 *Verify the results.* You can verify that all the differential equations and boundary conditions are satisfied.

We normalize Eq. 5 by inspection: we let $x^* = x/h$ and $w^* = w\mu/(\rho g h^2)$. Equation 5 becomes

Normalized velocity profile: $$w^* = \frac{x^*}{2}(x^* - 2) \tag{6}$$

We plot the normalized velocity field in Fig. 9–70.

Discussion The velocity profile has a large slope near the wall due to the no-slip condition there ($w = 0$ at $x = 0$), but zero slope at the free surface, where the boundary condition is zero shear stress ($\partial w/\partial x = 0$ at $x = h$). We could have introduced a factor of -2 in the definition of w^* so that w^* would equal 1 instead of $-\frac{1}{2}$ at the free surface.

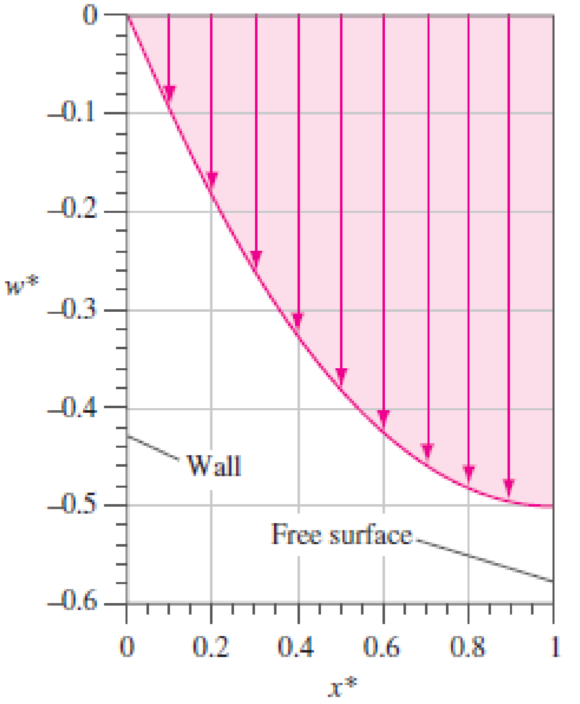


FIGURE 9–70
The normalized velocity profile of Example 9–17: an oil film falling down a vertical wall.

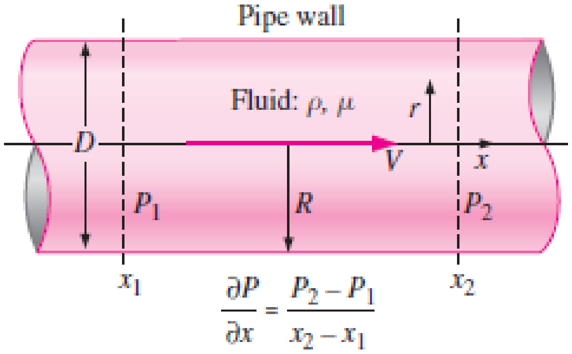


FIGURE 9–71
Geometry of Example 9–18: steady laminar flow in a long round pipe with an applied pressure gradient $\partial P/\partial x$ pushing fluid through the pipe. The pressure gradient is usually produced by a pump and/or gravity.

EXAMPLE 9–18 Fully Developed Flow in a Round Pipe—Poiseuille Flow

Consider steady, incompressible, laminar flow of a Newtonian fluid in an infinitely long round pipe of diameter D or radius $R = D/2$ (Fig. 9–71). We ignore the effects of gravity. A constant pressure gradient $\partial P/\partial x$ is applied in the x-direction,

Applied pressure gradient:
$$\frac{\partial P}{\partial x} = \frac{P_2 - P_1}{x_2 - x_1} = \text{constant} \qquad (1)$$

where x_1 and x_2 are two arbitrary locations along the x-axis, and P_1 and P_2 are the pressures at those two locations. Note that we adopt a modified cylindrical coordinate system here with x instead of z for the axial component, namely, (r, θ, x) and (u_r, u_θ, u). Derive an expression for the velocity field inside the pipe and estimate the viscous shear force per unit surface area acting on the pipe wall.

SOLUTION For flow inside a round pipe we are to calculate the velocity field, and then estimate the viscous shear stress acting on the pipe wall.

Assumptions **1** The pipe is infinitely long in the x-direction. **2** The flow is steady (all partial time derivatives are zero). **3** This is a parallel flow (the r-component of velocity, u_r, is zero). **4** The fluid is incompressible and Newtonian with constant properties, and the flow is laminar (Fig. 9–72). **5** A constant pressure gradient is applied in the x-direction such that pressure changes linearly with respect to x according to Eq. 1. **6** The velocity field is axisymmetric with no swirl, implying that $u_\theta = 0$ and all partial derivatives with respect to θ are zero. **7** We ignore the effects of gravity.

Analysis To obtain the velocity field, we follow the step-by-step procedure outlined in Fig. 9–52.

Step 1 *Lay out the problem and the geometry.* See Fig. 9–71.

Step 2 *List assumptions and boundary conditions.* We have listed seven assumptions. The first boundary condition comes from imposing the no-slip condition at the pipe wall: (1) at $r = R$, $\vec{V} = 0$. The second boundary condition comes from the fact that the centerline of the pipe is an axis of symmetry: (2) at $r = 0$, $\partial u/\partial r = 0$.

Step 3 *Write out and simplify the differential equations.* We start with the incompressible continuity equation in cylindrical coordinates, a modified version of Eq. 9–62a,

$$\underbrace{\cancel{\frac{1}{r}\frac{\partial(ru_r)}{\partial r}}}_{\text{assumption 3}} + \underbrace{\cancel{\frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta}}}_{\text{assumption 6}} + \frac{\partial u}{\partial x} = 0 \quad \rightarrow \quad \frac{\partial u}{\partial x} = 0 \qquad \textbf{(2)}$$

Equation 2 tells us that u is not a function of x. In other words, it doesn't matter where we place our origin—the flow is the same at any x-location. This can also be inferred directly from assumption 1, which tells us that there is nothing special about any x-location since the pipe is infinite in length—the flow is fully developed. Furthermore, since u is not a function of time (assumption 2) or θ (assumption 6), we conclude that u is at most a function of r,

Result of continuity: $$u = u(r) \text{ only} \qquad \textbf{(3)}$$

We now simplify the axial momentum equation (a modified version of Eq. 9–62d) as far as possible:

$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{assumption 2}} + \underbrace{\cancel{u_r\frac{\partial u}{\partial r}}}_{\text{assumption 3}} + \underbrace{\cancel{\frac{u_\theta}{r}\frac{\partial u}{\partial \theta}}}_{\text{assumption 6}} + \underbrace{\cancel{u\frac{\partial u}{\partial x}}}_{\text{continuity}}\right)$$

$$= -\frac{\partial P}{\partial x} + \underbrace{\cancel{\rho g_x}}_{\text{assumption 7}} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}}_{\text{assumption 6}} + \underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{continuity}}\right)$$

or

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\frac{\partial P}{\partial x} \qquad \textbf{(4)}$$

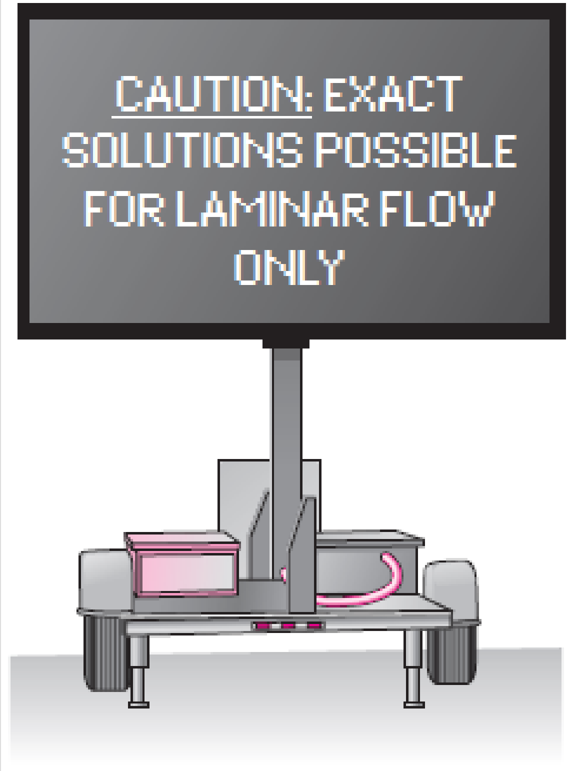


FIGURE 9–72
Exact analytical solutions of the Navier-Stokes equations, as in the examples provided here, are not possible if the flow is turbulent.

As in Examples 9–15 through 9–17, the material acceleration (entire left side of the x-momentum equation) is zero, implying that fluid particles are not accelerating at all in this flow field, and linearizing the Navier–Stokes equation (Fig. 9–73). We have replaced the partial derivative operators for the u-derivatives with total derivative operators because of Eq. 3.

In similar fashion, every term in the r-momentum equation (Eq. 9–62b) except the pressure gradient term is zero, forcing that lone term to also be zero,

r-momentum:
$$\frac{\partial P}{\partial r} = 0 \tag{5}$$

In other words, P is not a function of r. Since P is also not a function of time (assumption 2) or θ (assumption 6), P can be at most a function of x,

Result of r-momentum:
$$P = P(x) \text{ only} \tag{6}$$

Therefore, we replace the partial derivative operator for the pressure gradient in Eq. 4 by the total derivative operator since P varies only with x. Finally, all terms of the θ-component of the Navier–Stokes equation (Eq. 9–62c) go to zero.

Step 4 *Solve the differential equations.* Continuity and r-momentum have already been "solved," resulting in Eqs. 3 and 6, respectively. The θ-momentum equation has vanished, and thus we are left with Eq. 4 (x-momentum). After multiplying both sides by r, we integrate once to obtain

$$r\frac{du}{dr} = \frac{r^2}{2\mu}\frac{dP}{dx} + C_1 \tag{7}$$

where C_1 is a constant of integration. Note that the pressure gradient dP/dx is a constant here. Dividing both sides of Eq. 7 by r, we integrate a second time to get

$$u = \frac{r^2}{4\mu}\frac{dP}{dx} + C_1 \ln r + C_2 \tag{8}$$

where C_2 is a second constant of integration.

The Navier–Stokes Equation

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \boxed{(\vec{V}\cdot\vec{\nabla})\vec{V}}\right) = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V}$$

Nonlinear term

FIGURE 9–73
For incompressible flow solutions in which the advective terms in the Navier–Stokes equation are zero, the equation becomes *linear* since the advective term is the only nonlinear term in the equation.

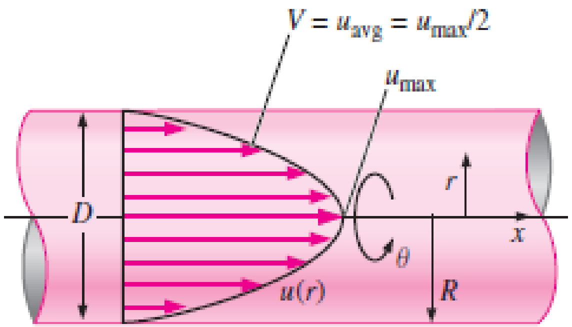


FIGURE 9–74
Axial velocity profile of Example 9–18: steady laminar flow in a long round pipe with an applied constant-pressure gradient dP/dx pushing fluid through the pipe.

Step 5 *Apply boundary conditions*. First, we apply boundary condition (2) to Eq. 7,

$$\textit{Boundary condition (2):} \qquad 0 = 0 + C_1 \qquad \rightarrow \qquad C_1 = 0$$

An alternative way to interpret this boundary condition is that u must remain finite at the centerline of the pipe. This is possible only if constant C_1 is equal to 0, since ln(0) is undefined in Eq. 8. Now we apply boundary condition (1),

$$\textit{Boundary condition (1):} \quad u = \frac{R^2}{4\mu}\frac{dP}{dx} + 0 + C_2 = 0 \quad \rightarrow \quad C_2 = -\frac{R^2}{4\mu}\frac{dP}{dx}$$

Finally, Eq. 7 becomes

$$\textit{Axial velocity:} \qquad u = \frac{1}{4\mu}\frac{dP}{dx}(r^2 - R^2) \qquad \textbf{(9)}$$

The axial velocity profile is thus in the shape of a paraboloid, as sketched in Fig. 9–74.

Step 6 *Verify the results*. You can verify that all the differential equations and boundary conditions are satisfied.

We calculate some other properties of fully developed laminar pipe flow as well. For example, the maximum axial velocity obviously occurs at the centerline of the pipe (Fig. 9–74). Setting $r = 0$ in Eq. 9 yields

$$\textit{Maximum axial velocity:} \qquad u_{\max} = -\frac{R^2}{4\mu}\frac{dP}{dx} \qquad \textbf{(10)}$$

The volume flow rate through the pipe is found by integrating Eq. 9 through a cross-section of the pipe,

$$\dot{V} = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} ur\,dr\,d\theta = \frac{2\pi}{4\mu}\frac{dP}{dx}\int_{r=0}^{R}(r^2 - R^2)r\,dr = -\frac{\pi R^4}{8\mu}\frac{dP}{dx} \qquad \textbf{(11)}$$

Since volume flow rate is also equal to the average axial velocity times cross-sectional area, we easily determine the average axial velocity V:

Average axial velocity: $$V = \frac{\dot{V}}{A} = \frac{(-\pi R^4/8\mu)\,(dP/dx)}{\pi R^2} = -\frac{R^2}{8\mu}\frac{dP}{dx} \tag{12}$$

Comparing Eqs. 10 and 12 we see that for fully developed laminar pipe flow, the average axial velocity is equal to exactly half of the maximum axial velocity.

To calculate the viscous shear force per unit surface area acting on the pipe wall, we consider a differential fluid element adjacent to the bottom portion of the pipe wall (Fig. 9–75). Pressure stresses and mathematically positive viscous stresses are shown. From Eq. 9–63 (modified for our coordinate system), we write the viscous stress tensor as

$$\tau_{ij} = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rx} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta x} \\ \tau_{xr} & \tau_{x\theta} & \tau_{xx} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mu\dfrac{\partial u}{\partial r} \\ 0 & 0 & 0 \\ \mu\dfrac{\partial u}{\partial r} & 0 & 0 \end{pmatrix} \tag{13}$$

We use Eq. 9 for u, and set $r = R$ at the pipe wall; component τ_{rx} of Eq. 13 reduces to

Viscous shear stress at the pipe wall: $$\tau_{rx} = \mu\frac{du}{dr} = \frac{R}{2}\frac{dP}{dx} \tag{14}$$

For flow from left to right, dP/dx is negative, so the viscous shear stress on the bottom of the fluid element at the wall is in the direction opposite to that indicated in Fig. 9–75. (This agrees with our intuition since the pipe wall exerts a retarding force on the fluid.) The shear force per unit area on the *wall* is equal and opposite to this; hence,

Viscous shear force per unit area acting on the wall: $$\frac{\vec{F}}{A} = -\frac{R}{2}\frac{dP}{dx}\vec{i} \tag{15}$$

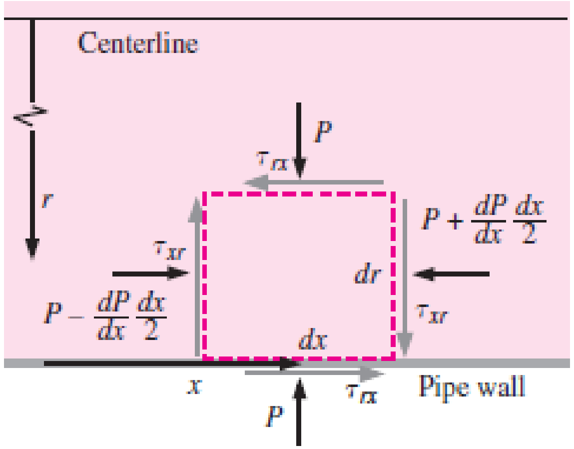


FIGURE 9–75
Pressure and viscous shear stresses acting on a differential fluid element whose bottom face is in contact with the pipe wall.

The direction of this force again agrees with our intuition; namely, the fluid tries to pull the bottom wall to the right, due to friction, when dP/dx is negative.

Discussion Since $du/dr = 0$ at the centerline of the pipe, $\tau_{rx} = 0$ there. You are encouraged to try to obtain Eq. 15 by using a control volume approach instead, taking your control volume as the fluid in the pipe between any two x-locations, x_1 and x_2 (Fig. 9–76). You should get the same answer. (*Hint*: Since the flow is fully developed, the axial velocity profile at location 1 is identical to that at location 2.) Note that when the volume flow rate through the pipe exceeds a critical value, instabilities in the flow occur, and the solution presented here is no longer valid. Specifically, flow in the pipe becomes *turbulent* rather than laminar; turbulent pipe flow is discussed in more detail in Chap. 8. This problem is also solved in Chap. 8 using an alternative approach.

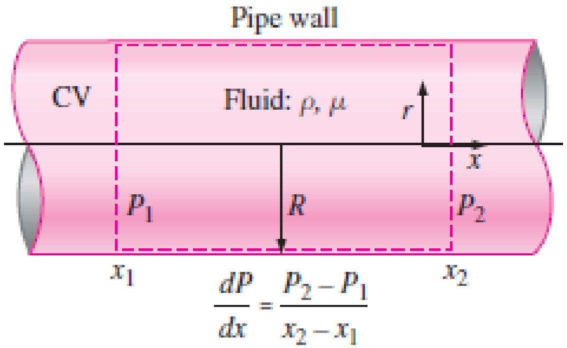


FIGURE 9–76
Control volume used to obtain Eq. 15 of Example 9–18 by an alternative method.

So far, all our Navier–Stokes solutions have been for steady flow. You can imagine how much more complicated the solutions must get if the flow is allowed to be unsteady, and the time derivative term in the Navier–Stokes equation does not disappear. Nevertheless, there are some unsteady flow problems that can be solved analytically. We present one of these in Example 9–19.

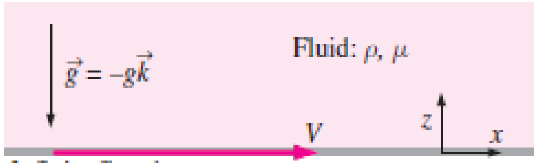


FIGURE 9–77
Geometry and setup for Example 9–19; the y-coordinate is into the page.

EXAMPLE 9–19 Sudden Motion of an Infinite Flat Plate

Consider a viscous Newtonian fluid on top of an infinite flat plate lying in the xy-plane at $z = 0$ (Fig. 9–77). The fluid is at rest until time $t = 0$, when the plate suddenly starts moving at speed V in the x-direction. Gravity acts in the $-z$-direction. Determine the pressure and velocity fields.

SOLUTION The velocity and pressure fields are to be calculated for the case of fluid on top of an infinite flat plate that suddenly starts moving.

Assumptions **1** The wall is infinite in the x- and y-directions; thus, nothing is special about any particular x- or y-location. **2** The flow is *parallel* everywhere ($w = 0$). **3** Pressure $P =$ constant with respect to x. In other words, there is no applied pressure gradient pushing the flow in the x-direction; flow occurs due to viscous stresses caused by the moving plate. **4** The fluid is incompressible and Newtonian with constant properties, and the flow is laminar. **5** The velocity field is two-dimensional in the xz-plane; therefore, $v = 0$, and all partial derivatives with respect to y are zero. **6** Gravity acts in the $-z$-direction.

Analysis To obtain the velocity and pressure fields, we follow the step-by-step procedure outlined in Fig. 9–52.

Step 1 *Lay out the problem and the geometry*. (See Fig. 9–77.)

Step 2 *List assumptions and boundary conditions*. We have listed six assumptions. The boundary conditions are: (1) At $t = 0$, $u = 0$ everywhere (no flow until the plate starts moving); (2) at $z = 0$, $u = V$ for all values of x and y (no-slip condition at the plate); (3) as $z \rightarrow \infty$, $u = 0$ (far from the plate, the effect of the moving plate is not felt); and (4) at $z = 0$, $P = P_{\text{wall}}$ (the pressure at the wall is constant at any x- or y-location along the plate).

Step 3 *Write out and simplify the differential equations*. We start with the incompressible continuity equation in Cartesian coordinates (Eq. 9–61a),

$$\frac{\partial u}{\partial x} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{\text{assumption 5}} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{\text{assumption 2}} = 0 \quad \rightarrow \quad \frac{\partial u}{\partial x} = 0 \qquad \textbf{(1)}$$

Equation 1 tells us that u is not a function of x. Furthermore, since u is not a function of y (assumption 5), we conclude that u is at most a function of z and t, *Result of continuity:* $$u = u(z, t) \text{ only} \qquad (2)$$

The *y*-momentum equation reduces to

$$\frac{\partial P}{\partial y} = 0 \qquad (3)$$

by assumptions 5 and 6 (all terms with v, the *y*-component of velocity, vanish, and gravity does not act in the *y*-direction). Equation 3 simply tells us that pressure is not a function of *y*; hence,

Result of y-momentum: $$P = P(z, t) \text{ only} \qquad (4)$$

Similarly the *z*-momentum equation reduces to

$$\frac{\partial P}{\partial z} = -\rho g \qquad (5)$$

We now simplify the *x*-momentum equation (Eq. 9–61b) as far as possible.

$$\rho\left(\frac{\partial u}{\partial t} + \underbrace{u\frac{\partial u}{\partial x}}_{\text{continuity}} + \underbrace{v\frac{\partial u}{\partial y}}_{\text{assumption 5}} + \underbrace{w\frac{\partial u}{\partial z}}_{\text{assumption 2}}\right) = \underbrace{-\frac{\partial P}{\partial x}}_{\text{assumption 3}} + \underbrace{\rho g_x}_{\text{assumption 6}}$$

$$+ \mu\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{continuity}} + \underbrace{\frac{\partial^2 u}{\partial y^2}}_{\text{assumption 5}} + \frac{\partial^2 u}{\partial z^2}\right) \quad \rightarrow \quad \rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial z^2} \qquad (6)$$

It is convenient to combine the viscosity and density into the kinematic viscosity, defined as $\nu = \mu/\rho$. Equation 6 reduces to the well-known **one-dimensional diffusion equation** (Fig. 9–78),

Result of x-momentum: $$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial z^2} \qquad (7)$$

Step 4 *Solve the differential equations.* Continuity and *y*-momentum have already been "solved," resulting in Eqs. 2 and 4, respectively. Equation 5 (*z*-momentum) is integrated once, resulting in

$$P = -\rho g z + f(t) \qquad (8)$$

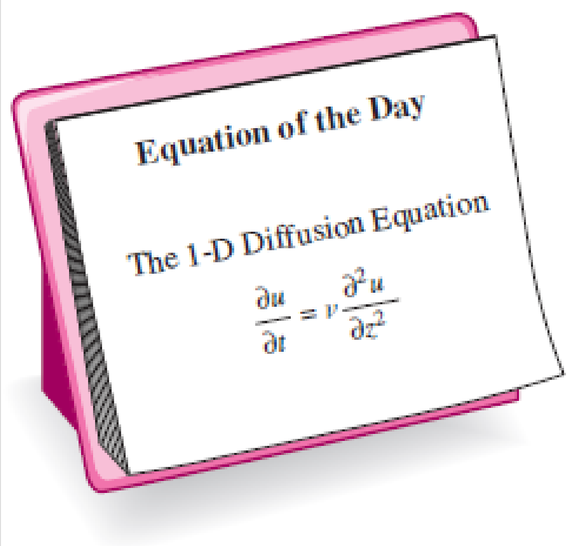


FIGURE 9–78
The one-dimensional diffusion equation is *linear*, but it is a *partial differential equation* (PDE). It occurs in many fields of science and engineering.

where we have added a function of time instead of a constant of integration since P is a function of two variables, z and t (see Eq. 4). Equation 7 (x-momentum) is a linear partial differential equation whose solution is obtained by combining the two independent variables z and t into one independent variable. The result is called a **similarity solution**, the details of which are beyond the scope of this text. Note that the one-dimensional diffusion equation occurs in many other fields of engineering, such as diffusion of species (mass diffusion) and diffusion of heat (conduction); details about the solution can be found in books on these subjects. The solution of Eq. 7 is intimately tied to the boundary condition that the plate is impulsively started, and the result is

Integration of x-momentum:
$$u = C_1\left[1 - \text{erf}\left(\frac{z}{2\sqrt{\nu t}}\right)\right] \qquad \textbf{(9)}$$

where **erf** in Eq. 9 is the **error function** (Çengel, 2003), defined as

Error function:
$$\text{erf}(\xi) = \frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-\eta^2} d\eta \qquad \textbf{(10)}$$

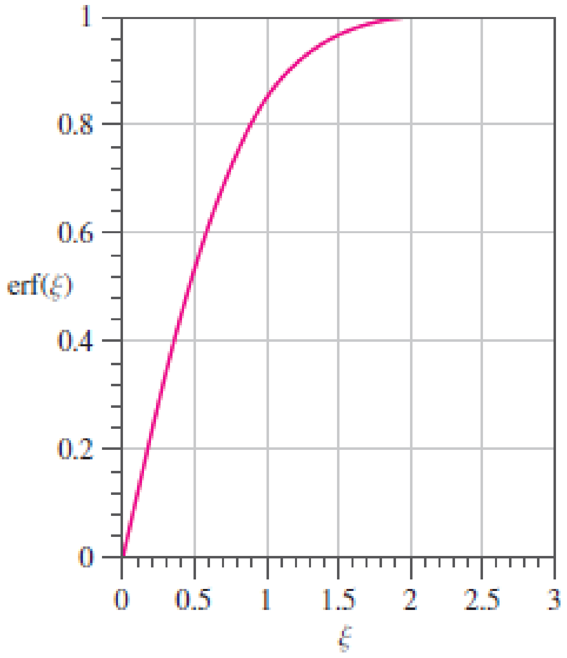


FIGURE 9–79
The error function ranges from 0 at $\xi = 0$ to 1 as $\xi \to \infty$.

The error function is commonly used in probability theory and is plotted in Fig. 9–79. Tables of the error function can be found in many reference books, and some calculators and spreadsheets can calculate the error function directly. It is also provided as a function in the EES software that comes with this text.

Step 5 *Apply boundary conditions.* We begin with Eq. 8 for pressure. Boundary condition (4) requires that $P = P_{wall}$ at $z = 0$ for all times, and Eq. 8 becomes

Boundary condition (4):
$$P = 0 + f(t) = P_{wall} \quad \rightarrow \quad f(t) = P_{wall}$$

In other words, the arbitrary function of time, $f(t)$, turns out not to be a function of time at all, but merely a constant. Thus,

Final result for pressure field:
$$P = P_{wall} - \rho g z \qquad \textbf{(11)}$$

which is simply hydrostatic pressure. We conclude that *hydrostatic pressure acts independently of the flow*. Boundary conditions (1) and (3) from step 2 have already been applied in order to obtain the solution of the x-momentum equation in step 4. Since erf(0) = 0, the second boundary condition yields

Boundary condition (2): $u = C_1(1 - 0) = V \quad \rightarrow \quad C_1 = V$

and Eq. 9 becomes

Final result for velocity field: $$u = V\left[1 - \text{erf}\left(\frac{z}{2\sqrt{\nu t}}\right)\right] \qquad \textbf{(12)}$$

Several velocity profiles are plotted in Fig. 9–80 for the specific case of water at room temperature ($\nu = 1.004 \times 10^{-6}$ m²/s) with $V = 1.0$ m/s. At $t = 0$, there is no flow. As time goes on, the motion of the plate is felt farther and farther into the fluid, as expected. Notice how long it takes for viscous diffusion to penetrate into the fluid—after 15 min of flow, the effect of the moving plate is not felt beyond about 10 cm above the plate!

We define normalized variables u^* and z^* as

Normalized variables: $$u^* = \frac{u}{V} \quad \text{and} \quad z^* = \frac{z}{2\sqrt{\nu t}}$$

Then we rewrite Eq. 12 in terms of nondimensional parameters:

Normalized velocity field: $$u^* = 1 - \text{erf}(z^*) \qquad \textbf{(13)}$$

The combination of unity minus the error function occurs often in engineering and is given the special name **complementary error function** and symbol **erfc**. Thus Eq. 13 can also be written as

Alternative form of the velocity field: $$u^* = \text{erfc}(z^*) \qquad \textbf{(14)}$$

The beauty of the normalization is that this *one equation* for u^* as a function of z^* is valid for *any* fluid (with kinematic viscosity ν) above a plate moving at *any* speed V and at *any* location z in the fluid at *any* time t! The normalized velocity profile of Eq. 13 is sketched in Fig. 9–81. All the profiles of Fig. 9–80 collapse into the single profile of Fig. 9–81; such a profile is called a **similarity profile**.

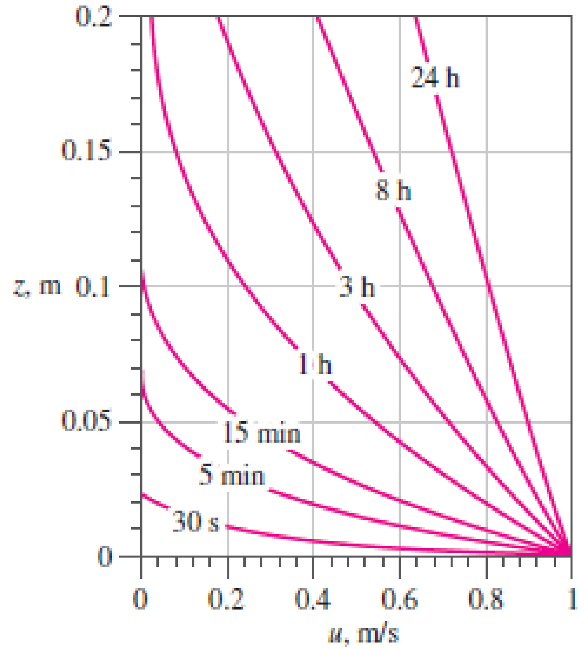


FIGURE 9–80
Velocity profiles of Example 9–19: flow of water above an impulsively started infinite plate; $\nu = 1.004 \times 10^{-6}$ m²/s and $V = 1.0$ m/s.

Step 6 *Verify the results*. You can verify that all the differential equations and boundary conditions are satisfied.

Discussion The time required for momentum to diffuse into the fluid seems much longer than we would expect based on our intuition. This is because the solution presented here is valid only for laminar flow. It turns out that if the plate's speed is large enough, or if there are significant vibrations in the plate or disturbances in the fluid, the flow will become turbulent. In a turbulent flow, large eddies mix rapidly moving fluid near the wall with slowly moving fluid away from the wall. This mixing process occurs rather quickly, so that turbulent diffusion is usually orders of magnitude faster than laminar diffusion.

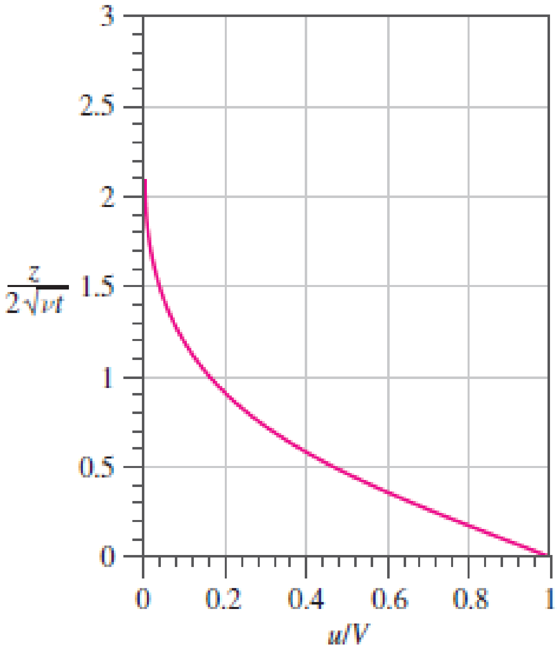


FIGURE 9–81
Normalized velocity profile of Example 9–19: laminar flow of a viscous fluid above an impulsively started infinite plate.

Examples 9–15 through 9–19 are for incompressible laminar flow. The same set of differential equations (incompressible continuity and Navier–Stokes) is valid for incompressible *turbulent* flow. However, turbulent flow solutions are much more complicated because the flow contains disordered, unsteady, three-dimensional eddies that mix the fluid. Furthermore, these eddies may range in size over several orders of magnitude. In a turbulent flow field, none of the terms in the equations can be ignored (with the exception of the gravity term in some cases), and thus solutions can be obtained only through numerical computations. Computational fluid dynamics (CFD) is discussed in Chap. 15.

Özet

- Giriş
- Kütlenin korunumu –Süreklilik denklemi
  - ✓ Diverjans teoremi ile türetme
  - ✓ Sonsuz küçük kontrol hacmi ile türetme
  - ✓ Süreklilik denkleminin alternatif formu
  - ✓ Silindirik koordinatlarda süreklilik denklemi
  - ✓ Süreklilik denkleminin özel halleri
- Akım Fonksiyonu
  - ✓ Kartezyen koordinatlarda akım fonksiyonu
  - ✓ Silindirik koordinatlarda akım fonksiyonu
  - ✓ Sıkıştırılabilir akım fonksiyonu
- Diferansiyel lineer momentum denklemi-Cauchy Denklemi
  - ✓ Diverjans teoremi ile türetme
  - ✓ Sonsuz küçük kontrol hacmi ile türetme
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- Navier-Stokes Denklemleri
  - ✓ Giriş
  - ✓ Newtonyen ve Newtonyen olmayan akışkanlar
  - ✓ Navier–Stokes denklemlerinin sıkıştırılamaz ve izotermal akışlar için türetimi
  - ✓ Süreklilik ve Navier–Stokes denklemlerinin kartezten ve silindirik koordinatlardaki ifadeleri
- Akş problemlerinin diferansiyel analizi
  - ✓ Bilinen hız alanından basınç alanı hesabı
  - ✓ Süreklilik ve Navier–Stokes denklemlerinin kesin çözümleri