MONODROMY GROUPS OF REAL ENRIQUES SURFACES

A DISSERTATION SUBMITTED TO
THE DEPARTMENT OF MATHEMATICS
AND THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

By
Sultan Erdoğan Demir
September, 2012

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Assoc. Prof. Dr. Alexander Degtyarev (Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Prof. Dr. Alexander Klyachko

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Prof. Dr. Yıldıray Ozan

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Prof. Dr. A. Sinan Sertöz

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Prof. Dr. Bilal Tanatar

Approved for the Graduate School of Engineering and Science:

Prof. Dr. Levent Onural
Director of the Graduate School

ABSTRACT

MONODROMY GROUPS OF REAL ENRIQUES SURFACES

Sultan Erdoğan Demir
P.h.D. in Mathematics
Supervisor: Assoc. Prof. Dr. Alexander Degtyarev
September, 2012

In this thesis, we compute the monodromy groups of real Enriques surfaces. The principal tools are the deformation classification of such surfaces and a modified version of Donaldson's trick, relating real Enriques surfaces and real rational surfaces.

Keywords: real Enriques surface, deformation, monodromy group.

ÖZET

GERÇEL ENRIQUES YÜZEYLERİNİN MONODROMİ GRUPLARI

Sultan Erdoğan Demir
Matematik, Doktora
Tez Yöneticisi: Doç. Dr. Alexander Degtyarev
Eylül, 2012

Bu tezde gerçel Enriques yüzeylerinin monodromi gruplarını hesapladık. Kullanılan temel araçlar, bu yüzeylerin deformasyon sınıflandırmaları ve gerçel Enriques yüzeyleri ile gerçel rasyonel yüzeyleri ilişkilendiren Donaldson metodunun modifiye versiyonudur.

Anahtar sözcükler: gerçel Enriques yüzeyi, deformasyon, monodromi grubu.

Acknowledgement

I would like to express my sincere gratitude to my supervisor Prof. Alex Degtyarev for his excellent guidance, valuable suggestions, continuous encouragement and infinite patience.

I would like to thank the rest of my thesis committee Alexander Klyachko, Yıldıray Ozan, Sinan Sertöz, and Bilal Tanatar, for accepting to read and review this thesis.

The financial support of TÜBİTAK is gratefully acknowledged.

I wish to thank my best friends Aslı Güçlükan, Aslı Pekcan and Seçil Gergün for helping me get through the difficult times, for all the emotional support they provided and the enjoyable time that we had.

It is a pleasure to express my gratitude to all my student colleagues for providing a stimulating and fun environment which I have strongly needed.

Last but not the least, I would like to thank my family for their encouragement, support and endless love. I am especially very grateful to my husband Sait Demir who has taken the load off my shoulder. Without his encouragement, understanding and persistent confidence in me, it would have been impossible for me to finish this work.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Federigo Enriques, one of the founders of the theory of algebraic surfaces, constructed some Enriques surfaces to give first examples of irrational algebraic surfaces on which there are no regular differential forms. Enriques surfaces are important in the theory of surfaces, both algebraic and analytic. They form one of the four classes of Kodaira dimension 0. From the algebraic point of view, Enriques surfaces are irrational and have no holomorphic differential forms. From the topological point of view, they are the simplest examples of smooth 4-manifolds which have even intersection form and whose signature is not divisible by 16.

An Enriques surface is a complex analytic surface with fundamental group $\mathbb{Z}_2$ and having a $K3$-surface as its universal cover. The orbit space of any fixed point free holomorphic involution on a $K3$-surface is an Enriques surface. An Enriques surface is called *real* if it is supplied with an anti-holomorphic involution, called *complex conjugation*. The *real part* of a real Enriques surface E is the fixed point set of its complex conjugation, and is denoted by $E_{\mathbb{R}}$.

The complex Enriques surfaces form a single deformation family. They are all diffeomorphic to each other. Nikulin started the topological study of real Enriques surfaces [19]. Degtyarev and Kharlamov completed the topological classification of the real parts [5]. They also gave a more refined classification of the so called *half decomposition.* $E_{\mathbb{R}}$ splits naturally into disjoint union of two halves, denoted by $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \sqcup E_{\mathbb{R}}^{(2)}$. The half decomposition is a deformation invariant of the surface. Both of the classifications are finite.

Finally, the classification of real Enriques surfaces up to deformation was given by Degtyarev, Itenberg and Kharlamov in [3] where one can find a complete list of deformation classes, the invariants necessary to distinguish them, and detailed explanations of the invariants. They proved that the deformation class of a real Enriques surface is determined by the topology of its complex conjugation involution.

Deformation classification can be regarded as the study of the set of connected components, (*i.e.,* π_0) of the moduli space. In this thesis, we attempt to understand its fundamental group (*i.e.,* π_1). More precisely, for each connected component of the moduli space, we study the canonical representation of its fundamental group in G, where G is the group of permutations of the components of the real part of the surfaces in that component of the moduli space. In other words, we discuss the *monodromy groups* of real Enriques surfaces, *i.e.,* the subgroups of G realized by 'auto-deformations' and/or automorphisms of the surfaces.

The similar question for various families of $K3$-surfaces has been extensively covered in the literature. The monodromy groups have been studied for nonsingular plane sextics by Itenberg [13] and for nonsingular surfaces of degree four in $\mathbb{RP}^3$ by Kharlamov [14]-[16] and Moriceau [18].

A real Enriques surface is said to be of *hyperbolic, parabolic,* or *elliptic type* if the minimal Euler characteristic of the components of $E_{\mathbb{R}}$ is negative, zero, or positive, respectively. In the deformation classification, hyperbolic and parabolic cases are treated geometrically (based on Donaldson's trick [11]) whereas the elliptic cases are treated arithmetically (calculations using the global Torelli theorem for $K3$-surfaces *cf.* [1]). There also is a crucial difference between the approaches to surfaces of hyperbolic and parabolic types. In the former case, natural *complex* models of the so called complex *DPN*-pairs are constructed, and a real structure descends to the model by naturality. In the latter case, it is difficult to study complex *DPN*-pairs systematically and *real* models of *real DPN*-pairs are constructed from the very beginning. We study the surfaces of hyperbolic and parabolic types in this work. Thus, we deal with an equivariant version of Donaldson's trick for Enriques surfaces modified by Degtyarev and Kharlamov [7], which transforms a real Enriques surface to a real rational surface with a nonsingular real anti-bicanonical curve on it.

We analyze this construction and adopt it to the study of the monodromy groups. In particular, we discuss the conditions necessary for an additional automorphism of the real rational surface to define an automorphism of the resulting real Enriques surface. The principal result of this thesis can be roughly stated as follows (for the exact statements see Theorems 5.2.1, 5.2.2, 5.2.3, 5.2.4 and 5.2.5): *For any real Enriques surface of hyperbolic type and for the real Enriques surfaces of parabolic type with* $E_{\mathbb{R}}^{(1)} = S_1$ *or* $2V_2$*, with some exceptions listed explicitly in each statement, any permutation of homeomorphic components of each half of* $E_{\mathbb{R}}$ *can be realized by deformations and/or automorphisms.* The part of this work concerning the real Enriques surfaces of hyperbolic type is published in [9].

The exceptions deserve a separate discussion. In most cases, the nonrealizable permutations are prohibited by a purely topological invariant, the so-called Pontrjagin-Viro form (see [2] and remarks following the relevant statements). There are, however, a few surfaces, those with $E_{\mathbb{R}}^{(1)} = V_3 \sqcup ...$, for which the Pontrjagin-Viro form is not well defined but the spherical components of $E_{\mathbb{R}}^{(1)}$ cannot be permuted. The question whether these permutations are realizable by equivariant auto-homeomorphisms of the surface remains open. Calculation of monodromy groups for the remaining parabolic cases and elliptic cases is a subject of future study as it seems to require completely different means.

Organization of the thesis is as follows: In Chapter 2, we remind some properties of real Enriques surfaces. In Chapter 3, we recall some real rational surfaces, real curves on them, and a few results related to their classification up to rigid isotopy. In Chapter 4, we describe (modified) Donaldson's trick and the resulting correspondence theorem, the construction backwards, and recall some results concerning specific families of real Enriques surfaces. In Chapter 5, a few necessary conditions for lifting automorphisms are discussed and the main result is stated and proved in five theorems.

Chapter 2

Real Enriques Surfaces

2.1 Notation and Conventions

By a *variety*, we mean a compact complex analytic manifold. Unless stated otherwise, by a *surface* we mean a variety of complex dimension 2.

Throughout the text, we identify the 2-homology and 2-cohomology groups of a closed smooth 4-manifold X via Poincaré duality isomorphism. Recall that both $H_2(X)/\operatorname{Tors}$ and $H^2(X)/\operatorname{Tors}$ are unimodular lattices, the pairing being induced by the intersection index.

Let X be a topological space. We denote $\mathbb{Z}$-Betti number of X by $b_i(X) = \operatorname{rk} H_i(X)$ and $\mathbb{Z}_2$-Betti number of X by $\beta_i(X) = \dim H_i(X; \mathbb{Z}_2)$. The corresponding total Betti numbers are denoted by $b_*(X) = \Sigma_{i \geqslant 0}\, b_i(X)$ and $\beta_*(X) = \Sigma_{i \geqslant 0}\, \beta_i(X)$.

By a *real variety* (a real surface, a real curve) we mean a pair (X, conj), where X is a complex variety and $\operatorname{conj} : X \to X$ an anti-holomorphic involution, called the *complex conjugation* or the *real structure*. The *real part* of X is $\operatorname{Fix}\operatorname{conj}$, the fixed point set of conj, and is denoted by $X_{\mathbb{R}}$.

A real curve C with the real structure $\operatorname{conj} : C \to C$ is said to be of *type* I, if

C/conj is orientable; otherwise it is of *type* II.

For any real variety X, one has the following *Smith inequality*:

$$\beta_*(X_{\mathbb{R}}) \leqslant \beta_*(X), \quad \text{and} \quad \beta_*(X_{\mathbb{R}}) = \beta_*(X) \pmod 2.$$

It follows that $\beta_*(X_{\mathbb{R}}) = \beta_*(X) - 2d$ for some nonnegative integer d. In this case X is called an $(M-d)$-*variety*. If X is a complex surface with a real structure, it is called an $(M-d)$-*surface*.

To describe the topological type of a closed (topological) 2-manifold M we use the notation $M_1 \sqcup M_2 \sqcup \ldots$, where $M_1, M_2, \ldots$ are the connected components of M, each component being either $S = S^2$, or $S_g = \sharp_g(S^1 \times S^1)$, or $V_p = \sharp_p \mathbb{RP}^2$. ($p$ and g are positive integers.)

2.2 Real Enriques Surfaces

Definition 2.2.1. An analytic surface X is called a $K3$-surface if $\pi_1(X) = 0$ and $c_1(X) = 0$.

Definition 2.2.2. An analytic surface E is called an Enriques surface if $\pi_1(E) = \mathbb{Z}_2$ and the universal covering X of E is a $K3$-surface.

Note that the classical definition of an Enriques surface is the requirement that $c_1(E) \neq 0$ and $2c_1(E) = 0$, and the relation to $K3$-surfaces above follows from the standard classification. Note also that all Enriques surfaces are algebraic.

In order to picture an Enriques surface see the following example:

Example 2.2.1 (See [3])**.** Let $s : \mathbb{P}^1 \to \mathbb{P}^1$ be a holomorphic involution and $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a nonsingular curve of bidegree $(4,4)$ such that $(s \times s)(C) = C$. Let X be the double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over C. In this setting, X is a $K3$-surface. If the fixed point set $\operatorname{Fix} s \times s$ does not intersect the curve C then $s \times s$ lifts to a fixed point free holomorphic involution $t : X \to X$ and the orbit space X/t is an Enriques surface.

All complex $K3$-surfaces form a single deformation family; they are all diffeomorphic to a degree 4 surface in $\mathbb{P}^3$. Similarly, all complex Enriques surfaces form a single deformation family and are all diffeomorphic to each other.

Definition 2.2.3. A real Enriques surface is an Enriques surface E supplied with an anti-holomorphic involution $\operatorname{conj} : E \to E$, called *complex conjugation.* The fixed point set $E_{\mathbb{R}} = \operatorname{Fix} \operatorname{conj}$ is called the *real part* of E.

A real Enriques surface E is a smooth 4-manifold, its real part $E_{\mathbb{R}}$ is a closed 2-manifold with finitely many components.

Two real Enriques surfaces are said to have the same *deformation type* if they can be included into a continuous one-parameter family of real Enriques surfaces, or, equivalently, if they belong to the same connected component of the *moduli space of real Enriques surfaces*. Contrary to the complex case, the moduli space of real Enriques surfaces is not connected. There are more than 200 distinct deformation types.

Fix a real Enriques surface E with real part $E_{\mathbb{R}}$ and denote by $p : X \to E$ its universal covering and by $\tau : X \to X$, the deck translation of p, called the Enriques involution.

Theorem 2.2.1 (See [6]). *There are exactly two liftings $t^{(1)}, t^{(2)} : X \to X$ of* conj *to X, which are both involutions. They are anti-holomorphic, commute with each other and with τ, and their composition is τ. Both the real parts $X_{\mathbb{R}}^{(i)} = \operatorname{Fix} t^{(i)}$, $i = 1, 2$, and their images $E_{\mathbb{R}}^{(i)} = p(X_{\mathbb{R}}^{(i)})$ in E are disjoint, and $E_{\mathbb{R}}^{(1)} \sqcup E_{\mathbb{R}}^{(2)} = E_{\mathbb{R}}$.*

Due to this theorem, $E_{\mathbb{R}}$ canonically decomposes into two disjoint parts, called *halves*. Both the halves $E_{\mathbb{R}}^{(1)}$ and $E_{\mathbb{R}}^{(2)}$ consist of whole components of $E_{\mathbb{R}}$, and $X_{\mathbb{R}}^{(i)}$ is an unbranched double covering of $E_{\mathbb{R}}^{(i)}$, $i = 1, 2$. This decomposition is a deformation invariant of pair (E, conj). We use the notation $E_{\mathbb{R}} = \{\text{half } E_{\mathbb{R}}^{(1)}\} \sqcup \{\text{half } E_{\mathbb{R}}^{(2)}\}$ for the half decomposition.

The study of real Enriques surfaces is equivalent to the study of real $K3$-surfaces equipped with a fixed point free holomorphic involution commuting with the real structure.

Recall that the topology of a real structure or, more generally, of a Klein action (*i.e.*, a finite group action on a complex analytic variety by both holomorphic and anti-holomorphic maps) is invariant under equivariant deformations. It is proved that for real Enriques surfaces the converse also holds. The deformation type of a real Enriques surface E is determined by the topology of its real structure [3]. Moreover, the latter is determined by the induced $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action in the homology $H_2(X;\mathbb{Z})$ of the covering $K3$-surface.

A *topological type* of real surfaces is a class of surfaces with homeomorphic real parts. To describe the topological types of the real part the notion of *topological Morse simplification* is used. A topological Morse simplification is a Morse transformation of the topological type which decreases the total Betti number. Therefore, a topological Morse simplification is either removing a spherical component ($S \to \emptyset$) or contracting a handle ($S_{g+1} \to S_g$ or $V_{p+2} \to V_p$). The complex deformation type of surfaces being fixed, a topological type is called *extremal* if it cannot be obtained from another one (in the same complex deformation type) by a topological Morse simplification.

A real Enriques surface E with the maximal total $\mathbb{Z}_2$-Betti number $\beta_*(E_{\mathbb{R}}) = 16$ is called an $M-$*surface.* A topological invariant, so called *Pontrjagin-Viro form*, is well defined on real Enriques M-surfaces. It defines a decomposition of each half $E_{\mathbb{R}}^{(i)}$ into two *quarters*, called *complex separation.* Denote by $E_{\mathbb{R}} = \{(Q_1^{(1)}) \sqcup (Q_2^{(1)})\} \sqcup \{(Q_1^{(2)}) \sqcup (Q_2^{(2)})\}$, the decomposition of the real part into quarters. Details on Pontrjagin-Viro form can be found in [2].

A real Enriques surface E (or a half $E_{\mathbb{R}}^{(i)}$) is said to be of *type* I_0 or I_u if $[E_{\mathbb{R}}]$ (respectively, $[E_{\mathbb{R}}^{(i)}]$) equals 0 or $w_2(E)$ (Stiefel-Whitney class) in $H_2(E;\mathbb{Z}_2)$, respectively; otherwise it is said to be of *type* II.

Chapter 3

Some Surfaces and Curves

In the proof of our results we need certain natural (in fact, anti-canonical or anti-bicanonical) models of some rational surfaces (resulting from Donaldson's trick, see Section 4.1). In this chapter, we recall the basic definitions and facts about them, and give a brief description of their properties and related results; details and further references can be found in [3].

3.1 *DPN*-pairs

Definition 3.1.1. A nonsingular algebraic surface admitting a nonempty nonsingular anti-bicanonical curve (*i.e.,* curve in the class $|-2K|$, where K is the canonical class), is called a *DPN-surface*.

Most *DPN*-surfaces are rational. Recall that a $(-d)$*-curve* is a nonsingular rational curve with self intersection $-d$, where d is a positive integer.

Definition 3.1.2. A pair (Y, B), where Y is a DPN-surface and $B \in |-2K_Y|$ is a nonsingular curve, is called a *DPN-pair*. A DPN-pair (Y, B) is called *unnodal* if Y is unnodal (does not contain a (-2)-curve), *rational* if Y is rational, and *real* if both Y and B are real. The *degree* of a rational DPN-pair (Y, B) is the degree of Y, *i.e.,* K^2.

Theorem 3.1.1 (See [3])**.** *If (Y, B) is a rational DPN-pair, the double covering X of Y ramified along B is a $K3$-surface.*

A *DPN*-surface contains finitely many (-4)-curves.

Definition 3.1.3. A rational *DPN*-surface Y of degree d that has r many (-4)-curves is called a *(g, r)-surface*, where $g = d + r + 1$.

Lemma 3.1.1 (See [3])**.** *Let Y be a (g, r)-surface. Then $g \geq 1$ and any nonsingular curve $B \in |-2K_Y|$ is one of the following topological types:*

1. *$B \cong S_g \sqcup rS$ if $g > 1$;*
2. *$B \cong S_1 \sqcup rS$ or rS if $g = 1$ and $r > 0$;*
3. *$B \cong 2S_1$ or S_1 if $g = 1$ and $r = 0$.*

Definition 3.1.4. A real curve $B \subset Y$ with $B_{\mathbb{R}} = \varnothing$ is said to be *not linked with* $Y_{\mathbb{R}}$ if for any path $\gamma : [0, 1] \to Y \setminus B$ with $\gamma(0), \gamma(1) \in Y_{\mathbb{R}}$, the loop $\gamma^{-1} \cdot \mathrm{conj}_Y \gamma$ is $\mathbb{Z}/2$-homologous to zero in $Y \setminus B$.

Definition 3.1.5. Let Y be a real surface with $H_1(Y) = 0$. An *admissible branch curve* on Y is a nonsingular real curve $B \subset Y$ such that $[B] = 0$ in $H_2(Y)$, the real part $B_{\mathbb{R}}$ is empty and B is not linked with $Y_{\mathbb{R}}$. An *admissible DPN-pair* is a real rational DPN-pair (Y, B) with B an admissible branch curve.

Donaldson's trick (see Section 4.1) and inverse Donaldson's trick (see Section 4.2) give correspondence between the deformation classes of real Enriques surfaces with distinguished nonempty half and the deformation classes of unnodal admissible *DPN*-pairs.

3.2 Geometrically ruled rational surfaces

Definition 3.2.1. A *geometrically ruled rational surface* is a relatively minimal conic bundle over $\mathbb{P}^1$.

We use the notation Σ_a, for the geometrically ruled rational surface that has a section of square $(-a)$, where a is a nonnegative integer. Such a section is unique when $a > 0$, it is called the exceptional section and is denoted by E_0. All these surfaces, except Σ_1, are minimal. $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and Σ_1 is the plane $\mathbb{P}^2$ blown up at one point. The classes of the exceptional section E_0 and of a generic section is denoted by e_0 and e_∞, respectively, so that $e_0^2 = -a$, $e_\infty^2 = a$, and $e_0 \cdot e_\infty = 0$. The class of the fiber (generatrix) will be denoted by l; one has $l^2 = 0$ and $l \cdot e_0 = l \cdot e_\infty = 1$.

Any irreducible curve in Σ_a with $a \geq 1$, either is E_0 or belongs to $|xl + ye_\infty|$, for some nonnegative integers x and y. If $a = 0$ then $e_0 = e_\infty$. Thus, if l_1 denotes $e_0 = e_\infty$ and l_2 denotes l then any irreducible curve in Σ_0 belongs to $|xl_1 + yl_2|$, for some nonnegative integers x and y.

Up to isomorphism, there exist four real structures on $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$: one structure with $(\Sigma_0)_\mathbb{R} = S_1$ (standard), one structure with $(\Sigma_0)_\mathbb{R} = S$, and two structures with $(\Sigma_0)_\mathbb{R} = \varnothing$ (which are $c_0 \times c_1$ and $c_1 \times c_1$, where c_0 is the usual complex conjugation on $\mathbb{P}^1$ with $\mathbb{P}^1_\mathbb{R} = S^1$ and c_1 is the quaternionic real structure with $\mathbb{P}^1_\mathbb{R} = \varnothing$). On Σ_a with $a \geq 2$ even, there exists two nonisomorphic real structures: one structure with $(\Sigma_a)_\mathbb{R} = S_1$ (standard) and one structure with $(\Sigma_a)_\mathbb{R} = \varnothing$. There is only one isomorphism class of real structures on Σ_a with $a \geq 2$ odd, with respect to which $(\Sigma_a)_\mathbb{R} = V_2$ (standard).

3.3 $(2, r)$-Surfaces

Let Y be a $(2, r)$-surface, $r \geq 1$. Then from Lemma 3.1.1, any nonsingular anti-bicanonical curve B is of the form $B \cong S_2 \sqcup rS$, $r \geq 1$. The anti-bicanonical system defines a degree 2 map $\varphi : Y \to \mathbb{P}^3$ which takes Y onto a quadric cone in $\mathbb{P}^3$. The map φ lifts to a degree 2 map $\widetilde{\varphi} : Y \to \Sigma_2$ whose branch locus is a curve $U \in |2e_\infty + 2l|$. The exceptional section E_0 of Σ_2 and U intersect as follows:

(1) two points of transversal intersection if $r = 1$;

(2) one point of simple tangency if $r = 2$;

(3) one singular point of type $\mathbf{A}_{r-2}$ of U if $r \geq 3$.

The pull-back $\widetilde{\varphi}^{-1}(E_0)$ consists of the fixed components of $|-2K_Y|$ and, possibly, several (-1)-curves. Figure 3.1 shows its Dynkin graph. The components of the proper transform of E_0 are $\widetilde{E_0}$.

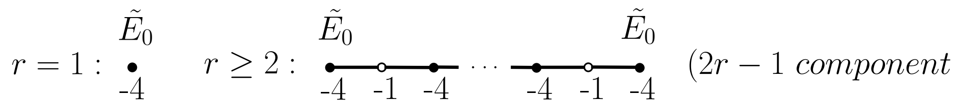


Figure 3.1: Dynkin graph of $\widetilde{\varphi}^{-1}(E_0)$

Let $Q = \widetilde{\varphi}(B)$, where B is a nonsingular curve in $|-2K_Y|$. Then Q consists of E_0 and a generic section $F \in |e_\infty|$. The curve $U + F$ has at most simple singular points. If Y is unnodal, then U is transversal to F and singularities of U, if any, are at its intersection with E_0.

Conversely, if $U \in |2e_\infty + 2l|$ a curve and $F \in |e_\infty|$ is a nonsingular section such that $U + F + E_0$ has at most simple singular points, then the DPN-double (Y, B) (*i.e.*, the resolution of singularities of the double covering of Σ_2 branched over U, where the rational components of B correspond to E_0 and the irrational component of B corresponds to F) of $(\Sigma_2; U, F + E_0)$ is a $(2, r)$-surface, $r \geq 1$, and the composition $Y \to \Sigma_2 \to \mathbb{P}^3$ is the anti-bicanonical map.

Theorem 3.3.1 (See [3])**.** *Let Y be a $(2, r)$ surface described as above. If Y is real then its model $\widetilde{\varphi} : Y \to \Sigma_2$ is real with respect to a standard real structure on Σ_2. If, in addition, Y contains a nonsingular real curve $B \in |-2K_Y|$ with $B_{\mathbb{R}} = \varnothing$ and r is odd, then the two branches of U intersecting E_0 are conjugate to each other.*

The deformation classification of real $(2, r)$-surfaces is reduced to the rigid isotopy classification of suitable pairs (see Section 3.6.5).

3.4 $(3, 2)$-Surfaces

In this section, we briefly consider two different models of $(3, 2)$-surfaces that we use in the proof of main results. Details and further explanations can be found in [3].

3.4.1 Model (I)

Let (Y, B) be an unnodal real rational DPN-pair of degree 0 with $B \cong S_3 \sqcup 2S$ such that $Y_{\mathbb{R}}$ is connected. Suppose that $B_{\mathbb{R}} = \varnothing$ and the rational components of B are real. Then Y blows down over $\mathbb{R}$ to Σ_0 (with the real structure $c_0 \times c_1$, see Section 3.2). The image of B is the transversal union of smooth components C', C'' and C''', where $C', C'' \in |l_2|$ are two distinct real generatrices and $C''' \in |4l_1 + 2l_2|$. Denote it by $Q = C' + C'' + C'''$.

3.4.2 Model (II)

Let (Y, B) be an unnodal real rational DPN-pair of degree 0 with $B \cong S_3 \sqcup 2S$ and $B_{\mathbb{R}} = \varnothing$. Let X be the covering $K3$-surface. Suppose that $[B] = 0$ in $H_2(X)$. Then there is a regular degree 2 map $\phi : Y \to \Sigma_4$ branched over a nonsingular real curve $U \in |2e_\infty|$. The irrational component of B is mapped to a real curve $F \in |e_\infty|$ and each rational component is mapped isomorphically to the exceptional section E_0 of Σ_4. The rational components of B are conjugate in this model. B is an admissible branch curve if and only if $U_{\mathbb{R}}$ is contained in a connected component of $(\Sigma_4)_{\mathbb{R}} \backslash ((E_0)_{\mathbb{R}} \sqcup F_{\mathbb{R}})$.

3.5 DPN-pairs with $\widetilde{B} \cong 2S_1$ or $\widetilde{B} \cong S_1 \sqcup rS$, $r > 0$

Let $(\widetilde{Y}, \widetilde{B})$ be a DPN-pair with $\widetilde{B} \cong 2S_1$ or $\widetilde{B} \cong S_1 \sqcup rS$, $r > 0$. Then $|-K|$ or the moving part of $|-2K|$ is an elliptic pencil $\widetilde{f} : \widetilde{Y} \to \mathbb{P}^1$ without multiple fibers if $\widetilde{B} \cong 2S_1$ or $\widetilde{B} \cong S_1 \sqcup rS$, $r > 0$, respectively. The genus 1 components of $\widetilde{B}$ are fibers of $\widetilde{f}$. If $\widetilde{B} \cong S_1 \sqcup rS$, $r > 0$, then the rational components of $\widetilde{B}$ belong to a single fiber of $\widetilde{f}$. Let $f : Y \to \mathbb{P}^1$ be the associated relatively minimal pencil, obtained by contracting all the (-1)-curves in the fibers of $\widetilde{f}$ (simply called the minimal pencil of $\widetilde{Y}$). If $\widetilde{B} \cong 2S_1$ then the pencil $\widetilde{f}$ is relatively minimal. Assume that $(\widetilde{Y}, \widetilde{B})$ is real. Then the pencils $\widetilde{f} : \widetilde{Y} \to \mathbb{P}^1$ and $f : Y \to \mathbb{P}^1$ are also real. Let B denote the image of $\widetilde{B}$ in Y. Then B is also a real anti-bicanonical curve.

The pencil $f : Y \to \mathbb{P}^1$ is one of the following real fibrations:

(A) Y is the double covering of Σ_0 with the standard real structure branched over a nonsingular real curve $U \in |2l_1 + 4l_2|$. The real part $U_{\mathbb{R}}$ consists of four ovals and $Y_{\mathbb{R}}$ covers their interior. The fibers of f are mapped to the generatrices in $|l_1|$.

(B) Y is the minimal resolution of the double covering of Σ_2 with the standard real structure branched over the disjoint union of the exceptional section E_0 and a real curve $C \in |3e_\infty|$ with at most simple singularities. The fibers of f are mapped to the generatrices of Σ_2.

(C) Y is the minimal resolution of the double covering of $\mathbb{P}^2$ branched over a real quartic U with at most simple singularities. The fibers of f are mapped to the lines through a fixed point $O \in \mathbb{P}^2 \setminus U$.

Y admits model (A) if and only if it is minimal over $\mathbb{R}$. Model (B) exists if and only if Y contains a real (-1)-curve. This (-1)-curve is mapped to the exceptional section E_0 of Σ_2. Model (C) exists if and only if Y contains two conjugate (-1)-curves. These (-1)-curves are mapped to the fixed point O. There are elliptic pencils admitting both models (B) and (C). In that case model (C) is used if and only if Y does not contain a real (-1)-curve.

In all the models Y is the minimal resolution of the double covering of a real surface Z branched over a real curve U. Let P denote the image of $\widetilde{B}$ in Z. The minimal pencil f defines a ruling (rational pencil) on Z. The two distinguished fibers of f containing B, denote by F' and G', are mapped to the distinguished fibers of the ruling of Z, denote by F and G. Then $P = F + G$.

Denote by $c : Z \to Z$ the real structure on Z. Consider the double covering $X \to \widetilde{Y}$ branched over $\widetilde{B}$ obtained as the fiberwise product of $\widetilde{Y} \to Z$ and the double covering of Z branched over P. Since $Z_{\mathbb{R}} \neq \varnothing$, the real structure c on Z lifts to two real structures on Y and four real structures in X. Denote the latter by $c^{\pm\pm}$. Then $U_{\mathbb{R}}$ divides $Z_{\mathbb{R}}$ into two parts, denoted by $Z^{\pm U} = Z^{\pm}$, with common boundary $U_{\mathbb{R}}$. They are the images of the fixed point sets of the two lifts of c to Y.

Similarly, $P_{\mathbb{R}}$ divides $Z_{\mathbb{R}}$ into two parts, denoted by $Z^{\pm P}$, with common boundary $P_{\mathbb{R}}$. Let $Z^{\epsilon\delta} = Z^{\epsilon U} \cap Z^{\delta P}$ for $\epsilon, \delta = \pm$. Since $Z_{\mathbb{R}}$, U, and P are nonempty, $c^{\pm\pm}$ are involutions on X. There is a natural one-to-one correspondence between $c^{\pm\pm}$ and the regions $Z^{\pm\pm}$ so that $\operatorname{Fix} c^{\pm\pm}$ projects onto $Z^{\pm\pm}$. Denote by q the deck translation of the covering $X \to \widetilde{Y}$, and by $p : X \to X$ the lift of the deck translation of the covering $Y \to Z$ such that $\operatorname{Fix} p$ projects onto U. Index $c^{\pm\pm}$ so that $c^{\pm\delta} = p \circ c^{\mp\delta}$, $c^{\epsilon\pm} = q \circ c^{\epsilon\mp}$ for $\epsilon, \delta = \pm$, and c^{++} is fixed point free. Then $Z^{++} = \varnothing$, *i.e.,* $U_{\mathbb{R}} \subset Z^{-P}$ and $P_{\mathbb{R}} \subset Z^{-U}$, and the real structure on $\widetilde{Y}$ is the descend of c^{+-}. Therefore, the projection $\widetilde{Y} \to Z$ establishes a one-to-one correspondence between the components of $\widetilde{Y}_{\mathbb{R}}$ and those of $Z^{+} = Z^{+-}$.

3.6 Rigid Isotopies

A *rigid homotopy* of real algebraic curves on W is a path Q_s of real curves on W such that each member of the path consists of a fixed number of smooth components and have at most type **A** singular points.

Theorem 3.6.1 (See [3]). *Let Q_1 and Q_2 be two real anti-bicanonical curves on a real rational surface W with at most simple singularities. If Q_1 and Q_2 can be connected by a rigid homotopy equisingular in a neighborhood of $W_{\mathbb{R}}$, then the DPN-resolutions of (W, Q_1) and (W, Q_2) (resolutions of singularities of Q_i's so that the resulting pairs are DPN-pairs) are deformation equivalent in the class of real DPN-pairs.*

An *isotopy* is a homotopy from one embedding of a manifold M into a manifold N to another embedding such that, at every time, it is an embedding. An isotopy in the class of nonsingular (or, more generally, equisingular, in some appropriate sense) embeddings of analytic varieties is called *rigid*. We are mainly dealing with rigid isotopies of nonsingular curves on rational surfaces. Clearly, such an isotopy is merely a path on the space of nonsingular curves.

An obvious rigid isotopy invariant of a real curve C on a real surface Z is its *real scheme, i.e.,* the topological type of the pair $(Z_{\mathbb{R}}, C_{\mathbb{R}})$.

The deformation classification of real Enriques surfaces and hence the monodromy problem of those leads to a variety of auxiliary classification problems for curves on surfaces and surfaces in projective spaces. Below we recall the basic definitions and facts about them, and give a brief account of the related results. Details and further references can be found, *e.g.*, in [3].

3.6.1 Real Schemes

The real point set $C_{\mathbb{R}}$ of a nonsingular curve C in $\mathbb{P}^2_{\mathbb{R}}$ is a collection of circles A embedded in $\mathbb{P}^2_{\mathbb{R}}$, two- or one-sidedly. In the former case the component is called an *oval*. Any oval divides $\mathbb{P}^2_{\mathbb{R}}$ into two parts; the *interior* of the oval, homeomorphic to a disk and the *exterior* of the oval, homeomorphic to the Möbius band. The real point set of a nonsingular curve of even degree consists of ovals only. The real point set of a nonsingular curve of odd degree contains exactly one one-sided component. The relation *to be in the interior of* defines a partial order on the set of ovals, and the collection A equipped with this partial order determines the *real scheme* of C. The following notation is used to describe real schemes: If a real scheme has a single component, it is denoted by $\langle J \rangle$, if the component is one-sided, or by $\langle 1 \rangle$, if it is an oval. The empty real scheme is denoted by $\langle 0 \rangle$. If $\langle \mathcal{A} \rangle$ stands for a collection of ovals, the collection obtained from it by adding a new oval surrounding all the old ones is denoted by $\langle 1\langle \mathcal{A} \rangle\rangle$. If a real scheme splits into two subschemes $\langle \mathcal{A}_1 \rangle$, $\langle \mathcal{A}_2 \rangle$ so that no oval of $\langle \mathcal{A}_1 \rangle$ (respectively, $\langle \mathcal{A}_2 \rangle$) surrounds an oval of $\langle \mathcal{A}_2 \rangle$ (respectively, $\langle \mathcal{A}_1 \rangle$), it is denoted by $\langle \mathcal{A}_1 \sqcup \mathcal{A}_2 \rangle$. If a real scheme contains n disjoint copies of $\langle 1 \rangle$ it is denoted by $\langle n \rangle$.

Theorem 3.6.2 (See [17])**.** *A nonsingular real quartic C in $\mathbb{P}^2$ is determined up to rigid isotopy by its real scheme. There are six rigid isotopy classes, with real schemes $\langle \alpha \rangle$, $\alpha = 0, ..., 4$ and $\langle 1\langle 1 \rangle\rangle$. The M-quartic $\langle 4 \rangle$ and the nest $\langle 1\langle 1 \rangle\rangle$ are of type I; the other quartics are of type II.*

Lemma 3.6.1 (See [3])**.** *Let C be a nonsingular real quartic with the real scheme $\langle \alpha \rangle$, $\alpha = 2, 3, 4$ in $\mathbb{P}^2$. Then any permutation of the ovals of C can be realized by a rigid isotopy.*

3.6.2 Cubic sections on a quadratic cone

Let $C \in |ne_\infty|$ be a nonsingular real curve in $Z = \Sigma_2$ with its standard real structure. Each connected component of $C_\mathbb{R}$ is either an oval or homologous to $(E_0)_\mathbb{R}$. The latters, together with $(E_0)_\mathbb{R}$, divide $Z_\mathbb{R}$ into several connected components $Z_1, ..., Z_k$. Fixing an orientation of the real part of a real generatrix of Σ_2 determines an order of the components Z_i, and the real scheme of C can be described via $\langle\ \mathcal{A}_1|...|\mathcal{A}_k\rangle$, where $|$ stands for a component homologous to $(E_0)_\mathbb{R}$ and $\mathcal{A}_i$ encodes the arrangement of the ovals in Z_i (similar to the case of plane curves), for each $i \in \{1, 2, ..., k\}$.

Theorem 3.6.3 (See [3])**.** *A nonsingular real curve $C \in |3e_\infty|$ on Σ_2 is determined up to rigid isotopy by its real scheme. There are 11 rigid isotopy classes, with real schemes $\langle\alpha|0\rangle$, $1 \leq \alpha \leq 4$, $\langle 0|\alpha\rangle$, $1 \leq \alpha \leq 4$, $\langle 0|0\rangle$, $\langle 1|1\rangle$, and $\langle|||\rangle$.*

By analyzing the proof of Theorem 3.6.3, one can easily see that the curves with real schemes $\langle\alpha|0\rangle$ and $\langle 0|\alpha\rangle$, $1 \leq \alpha \leq 4$, are isomorphic up to a real automorphism of Σ_2. Furthermore, a stronger statement holds.

Refinement 3.6.1 (of Theorem 3.6.3)**.** *Any two pairs (U,O), where the real scheme of U is $\langle\alpha|0\rangle$ with $0 \leq \alpha \leq 3$ and O is a distinguished oval of U, are rigidly isotopic. For an alternative proof of Theorem 3.6.3 and the last assertion, one can use the theory of the trigonal curves, see [4].* $\square$

3.6.3 Real root schemes

Let $Z = \Sigma_k$, $k \geq 0$, with the standard real structure. Since we use Σ_0, Σ_2 and Σ_4 in this work we will consider only the cases when k is even. For k odd and further details, see [3]. Consider a real curve $U \in |2e_\infty + pl|$, $p \geq 0$, and a real curve $Q = E_0 \cup F$, where E_0 is the exceptional section and $F \in |e_\infty|$ is a generic real section of Z. The complement $Z_\mathbb{R} \setminus Q_\mathbb{R}$ consists of two connected orientable components. Fix one of them and let Z^- denote its closure. Fix an orientation of $F_\mathbb{R} \subset \partial Z^-$. Assume that U does not contain any generatrix of Z, is transversal to F and $U_{\mathbb{R}}$ lies entirely in Z^-. Fix an auxiliary real generatrix L of Z transversal to $U \cup E_0$. Consider a real coordinate system (x, y) in the affine part $Z \setminus (E_0 \cup L)$ whose x-axis is F. Choose the positive direction of the y-axis so that the upper half-plane lies in Z^-. In these coordinates U has equation $a(x)y^2+b(x)y+c(x) = 0$, where $a(x)$, $b(x)$ and $c(x)$ are real polynomials of degree p, $p+k$ and $p+2k$, respectively. Let $\Delta(x) = b^2(x) - 4a(x)c(x)$ and let $\mu(x)$ and $\nu(x)$ denote the multiplicity of a point $x \in F$ in $a(x)$ and $\Delta(x)$, respectively. Consider the sets

$$\mathcal{A}_{\mathbb{R}} = \{x \in F_{\mathbb{R}} \mid \mu(x) \geq 1\}, \quad \mathcal{A} = \{x \in F \mid \mu(x) \geq 1\},$$

$$\mathcal{D}_{\mathbb{R}} = \{x \in F_{\mathbb{R}} \mid \Delta(x) \geq 0\}, \quad \mathcal{D}_r = \{x \in F \mid \nu(x) \geq r\},\ r \geq 1, \quad \mathcal{D} = \mathcal{D}_2 \cup \mathcal{D}_{\mathbb{R}}.$$

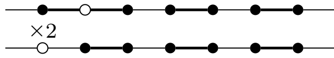
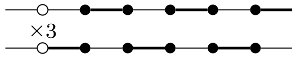
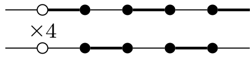
The multiplicity functions μ and ν are invariant under complex conjugation. Identify F with the base $B \cong \mathbb{P}^1$ of the ruling of Z. Thus, $B_{\mathbb{R}}$ receives an orientation, $\mathcal{A}$ and $\mathcal{D}$ can be regarded as subsets of B, and, μ and ν are functions defined on B.

The *root marking* of (U, Q) is the triple $(\mathcal{B}, \mathcal{D}, \mathcal{A})$ equipped with the complex conjugation in B, the orientation of $B_{\mathbb{R}}$, and the multiplicity functions μ and ν. An *isotopy* of root markings is an equivariant isotopy of triples $(\mathcal{B}, \mathcal{D}, \mathcal{A})$ followed by a continuous change of the orientation of $B_{\mathbb{R}}$, μ, and ν restricted to $\mathcal{D}$. A *root scheme* is an equivalence class of root markings up to isotopy. The *real root marking* of (U, Q) is the triple $(\mathcal{B}_{\mathbb{R}}, \mathcal{D}_{\mathbb{R}}, \mathcal{A}_{\mathbb{R}})$ equipped with the orientation of $B_{\mathbb{R}}$, and the multiplicity functions μ and ν. A *real root scheme* is an equivalence class of real root markings up to isotopy.

Theorem 3.6.4 (See [3])**.** *Let $Z = \Sigma_4$ (with the standard real structure), let $U \in |2e_\infty|$ be a nonsingular real curve on Z, let $F \in |e_\infty|$ be a generic real section transversal to U, and let E_0 be the exceptional section. If $U_{\mathbb{R}}$ belongs to the closure of one of the two components of $Z_{\mathbb{R}} \setminus ((E_0)_{\mathbb{R}} \cup F_{\mathbb{R}})$, then, up to rigid isotopy and automorphism of Z, the pair (U, F) is determined by its real root scheme or, equivalently, by the real scheme of U. The latter consists either of $a = 0, ..., 4$ ovals (*i.e.*, components bounding disks) or of two components isotopic to $F_{\mathbb{R}}$.*

Theorem 3.6.5 (See [3]). *Let Z be Σ_0 with the standard real structure. Then up to rigid isotopy and automorphism of Z there is a unique nonsingular real M-curve $U \in |2l_1 + 4l_2|$ on Z.*

Table 3.1: Real root schemes of some curves $U \in |2e_\infty + pl|$ on Σ_{2k}

Real root scheme	$\{E_{\mathbb{R}}^{(1)}\} \sqcup \{E_{\mathbb{R}}^{(2)}\}$
	$\{(V_4 \sqcup S) \sqcup (\varnothing)\} \sqcup \{(2S) \sqcup (2S)\}$
	$\{(V_3 \sqcup V_1) \sqcup (\varnothing)\} \sqcup \{(2S) \sqcup (2S)\}$
	$\{(V_3 \sqcup S) \sqcup (\varnothing)\} \sqcup \{(V_1 \sqcup S) \sqcup (2S)\}$
	$\{(V_3 \sqcup S) \sqcup (V_1)\} \sqcup \{(2S) \sqcup (S)\}$
	$\{(V_3 \sqcup S) \sqcup (S)\} \sqcup \{(V_1 \sqcup S) \sqcup (S)\}$
	$\{(V_3 \sqcup V_1 \sqcup S) \sqcup (S)\} \sqcup \{(S) \sqcup (S)\}$

Comments: The first column indicates the real root schemes of pairs (U, F) and the second column indicates the quarter decomposition of the real part $E_{\mathbb{R}}$ of the real Enriques surfaces obtained from $(\Sigma_{2k}; U, E_0 \cup F)$. For the first row $p = 0$ and $k = 2$ and for the others $p = 2$ and $k = 1$. In the schemes, $\bullet$ represents a real root of Δ and $\circ$ represents a real root of a (necessary 2-fold), that corresponds to the real intersection point of U and E_0. The number over a $\circ$ -vertex indicates the multiplicity of the corresponding root in Δ (when greater than 1). The segments $\bullet\!\!-\!\!\bullet$ correspond to ovals of $U_{\mathbb{R}}$. Only extremal root schemes are listed; the others are obtained by removing one or several segments $\bullet\!\!-\!\!\bullet$.

3.6.4 Dividing curves

Let U, F, G be as in Section 3.6.3. Then U is a dividing curve if and only if one of the followings holds:

(1) Δ has no imaginary roots of odd multiplicity, or

(2) $\mathcal{D}_{\mathbb{R}} = B_{\mathbb{R}}$, *i.e.*, the projection $U_{\mathbb{R}} \to B_{\mathbb{R}}$ is generically two-to-one.

Suppose that U is a dividing curve. Denote by U^+ and U^-, the components of $U \backslash U_{\mathbb{R}}$, and by F^+ and F^-, the components of $F \backslash F_{\mathbb{R}}$. Suppose also that $U \circ F = 0$ mod 4 (*i.e.*, $2k + p = 0$ mod 4). Then Card($U^+ \cap F^+$) - Card($U^- \cap F^+$) mod 4 is independent of the choice of $U^\pm$ and $F^\pm$. Denote this number by $\mathcal{P}$. In fact, $\mathcal{P} \in 2\mathbb{Z}/4\mathbb{Z}$.

3.6.5 Suitable pairs

On Σ_2: Let $U \in |2e_\infty + 2l|$ be a reduced (not containing any multiple component) real curve on Σ_2 with the standard real structure. Assume that U is nonsingular outside of E_0 and does not contain E_0 as a component. Then U and E_0 intersect with multiplicity 2. The *grade* of U is said to be 1 if it intersects E_0 transversally at two points, 2 if it is tangent to E_0 at one point, and r if it has a single singular point of type $\mathbf{A}_{r-2}$, $r \geq 3$, on E_0. A curve U as above is called *suitable* if either its grade is even, or its grade is odd and the two branches of U at E_0 are conjugate to each other. A pair (U, F) is called a *suitable pair* if U is a suitable curve and $F \in |e_\infty|$ is a nonsingular real section transversal to U such that $U_\mathbb{R}$ belongs to the closure of a single connected component of $(\Sigma_2)_\mathbb{R} \setminus ((E_0)_\mathbb{R} \cup F_\mathbb{R})$. The *grade* of a suitable pair (U, F) is the grade of U. The condition that $U_\mathbb{R}$ should belong to the closure of a single connected component of $(\Sigma_2)_\mathbb{R} \setminus ((E_0)_\mathbb{R} \cup F_\mathbb{R})$ guarantees that the real *DPN*-double (Y, B) of $(\Sigma_2; U, E_0 \cup F)$, where (U, F) is a suitable pair, corresponds to a real Enriques surface by inverse Donaldson's trick (see Section 4.2).

All the pairs (U, F) satisfying the hypothesis of the following theorem are suitable.

Theorem 3.6.6 (See [3])**.** *Let $Z = \Sigma_2$ (with the standard real structure), let $U \in |2e_\infty + 2l|$ be a reduced real curve on Z, nonsingular outside the exceptional section E_0 and not containing E_0 as a component, and let $F \in |e_\infty|$ be a generic real section transversal to U. If $U_\mathbb{R}$ belongs to the closure of a single connected component of $Z_\mathbb{R} \setminus ((E_0)_\mathbb{R} \cup F_\mathbb{R})$, then, up to rigid isotopy and automorphism of Z, the pair (U, F) is determined by its real root scheme or, equivalently, by the type of the singular point of U (if any) and the topology of the pair $(Z_\mathbb{R}, U_\mathbb{R} \cup (E_0)_\mathbb{R})$.*

In Table 3.1, we list the extremal real root schemes of some pairs (U, F) mentioned in Theorem 3.6.4 and Theorem 3.6.6 that are used in the proof of the main results. The complete lists can be found in [3].

Each real root marking gives rise to a connected family of pairs (U, Q) such that there is a bijection between the ovals of each curve U and the segments of the real root marking. Recall that these curves are defined by explicit equations.

Refinement 3.6.2 (of Theorems 3.6.4, 3.6.5, and 3.6.6)**.** *Theorems 3.6.4, 3.6.5, and 3.6.6 can be refined as follows:*

(1) *Each isotopy of real root markings is followed by a rigid isotopy of curves that is consistent with the bijection between ovals and segments.*

(2) *Any symmetry of a real root marking (not necessarily preserving the orientation of $B_{\mathbb{R}}$) is induced by an automorphism of Σ_{2k}, $k \geq 0$, preserving appropriate pairs (U, F) and consistent with the bijection between ovals and segments.*

□

On Σ_0 or $\mathbb{P}^2$: Let Z be either $\mathbb{P}^2$, or Σ_0 with the standard real structure. A *suitable pair* on Z is a pair (U, P) of signed real curves (see Section 3.5 for signing), where $P = F + G$ and

(A) $U \in |2l_1 + 4l_2|$ is an M-curve and $F,\ G \in |l_1|$ are real lines, if $Z = \Sigma_0$, or
(C) $U \in |4l|$ and $F,\ G \in |l|$ are real lines, if $Z = \mathbb{P}^2$

such that

(1) U is nonsingular;
(2) F and G are transversal to U;
(3) U and P are signed so that $Z^{++} = Z^{+U} \cap Z^{+P} = \varnothing$.

All M-curves on Σ_0 are dividing, and the condition (3) above implies that $U_{\mathbb{R}}$ belongs to the closure of one of the two components of $Z_{\mathbb{R}} \setminus (F_{\mathbb{R}} \cup G_{\mathbb{R}})$. Therefore the suitable pairs on Σ_0 satisfy the conditions of the following theorem.

Theorem 3.6.7 (See [3])**.** *Let $Z = \Sigma_0$ (with the standard real structure), $U \in |2l_1 + 4l_2|$ a nonsingular real M-curve, and $F, G \in |l_1|$ two real generatrices so that U is transversal to F and G and $U_{\mathbb{R}}$ belongs to the closure of one of the two components of $Z_{\mathbb{R}} \setminus (F_{\mathbb{R}} \cup G_{\mathbb{R}})$. Then up to rigid isotopy and automorphism of Z the triple $(U; F, G)$ is determined by its real root scheme, and by the value of $\mathcal{P} = 0, 2$.*

3.7 Ramified complex scheme

Let $U \subset \mathbb{P}^2$ be a real nonsingular curve with even degree and G a real line such that $G_{\mathbb{R}}$ belongs to the nonorientable part of $\mathbb{P}^2_{\mathbb{R}} \setminus U_{\mathbb{R}}$. If U is of type I, one can sign the ovals of $U_{\mathbb{R}}$ as follows. Fix a complex orientation of $G_{\mathbb{R}}$ and assign to an oval O of $U_{\mathbb{R}}$ the sign $+$ if the complex orientations of O and $G_{\mathbb{R}}$ induce opposite orientations on the interior of O, and the sign $-$ otherwise. The signs of ovals depend on the orientation of $G_{\mathbb{R}}$ and are defined up to simultaneous change. We always make the choice of the orientation of $G_{\mathbb{R}}$ in such a way that the number of ovals marked with a $+$ sign is not less than the number of ovals marked with a $-$ sign.

Let (U, G) be as above and assume that G has at most simple tangency points with U. The *ramified complex scheme* of (U, G) is the real scheme of U equipped with the following additional structures:

(1) each oval of U is marked with as many asterisks $(*)$ as it has tangency points with G;

(2) if U is of type I, the ovals are marked with the signs $\pm$ defined above.

The suitable pairs on $\mathbb{P}^2$ satisfies the conditions of the following theorem.

Theorem 3.7.1 (See [3]). *Let U be a nonsingular real quartic $\mathbb{P}^2$ and F, G a pair of real lines transversal to U, and $U_{\mathbb{R}}$ belongs to the closure of one of the two components of $\mathbb{P}_{\mathbb{R}} \setminus (F_{\mathbb{R}} \cup G_{\mathbb{R}})$. Then the triple (U, F, G) is determined up to rigid isotopy by the ramified complex scheme of (U, G).*

Chapter 4

Reduction to DPN-pairs

4.1 Donaldson's trick

At present, we know the classification of real Enriques surfaces up to deformation equivalence (which is the strongest equivalence relation from the topological point of view). In the deformation classification, the equivariant version of Donaldson's trick is used. It employs the hyper-Kähler structure to change the complex structure of the covering $K3$-surface X so that $t^{(1)}$ is holomorphic, and $t^{(2)}$ and τ are anti-holomorphic, where $t^{(1)}$, $t^{(2)}$ and τ are as in Theorem 2.2.1. Furthermore, $Y = \widetilde{X}/t^{(1)}$ is a real rational surface, where the real structure is the common descent of τ and $t^{(2)}$, and $B \cong \operatorname{Fix} t^{(1)}$ is a real nonsingular anti-bicanonical curve on Y. As a result, the problem about real Enriques surfaces is reduced to the study of real nonsingular anti-bicanonical curves on real rational surfaces.

Theorem 4.1.1 (See [7])**.** *Donaldson's trick establishes a one-to-one correspondence between the set of deformation classes of real Enriques surfaces with distinguished nonempty half (*i.e.*, pairs $(E, E_{\mathbb{R}}^{(1)})$ with $E_{\mathbb{R}}^{(1)} \neq \varnothing$) and the set of deformation classes of pairs (Y, B), where Y is a real rational surface and $B \subset Y$ is a nonsingular real curve such that*

(1) B is anti-bicanonical,

(2) the real point set of B is empty, and

(3) B is not linked with the real point set $Y_{\mathbb{R}}$ of Y.
One has $E_{\mathbb{R}}^{(2)} = Y_{\mathbb{R}}$ and $E_{\mathbb{R}}^{(1)} = B/t^{(2)}$.

In the above theorem, the first condition on B guarantees that the double covering X of Y branched over B is a $K3$-surface; and the other two conditions ensure the existence of a fixed point free lift of the real structure on Y to X. The statement deals with deformation classes rather than individual surfaces because the construction involves a certain choice (that of an invariant Kähler class).

4.2 Inverse Donaldson's trick

Since we want to construct deformation families of real Enriques surfaces with particular properties, we are using Donaldson's construction backwards. Strictly speaking, Donaldson's trick is not invertible. However, it establishes a bijection between the sets of deformation classes (see Theorem 4.1.1); thus, at the level of deformation classes one can speak about 'inverse Donaldson's trick'.

Before explaining the construction, recall some properties of $K3$-surfaces (see [1] for further details). All $K3$-surfaces are Kähler. All nontrivial holomorphic forms on a $K3$-surface are proportional to each other and trivialize the canonical bundle (*i.e.,* $K = 0$ for $K3$-surfaces). They are called fundamental holomorphic forms. Any fundamental holomorphic form ω satisfies the relations $\omega^2 = 0$, $\omega \cdot \bar{\omega} > 0$, and $d\omega = 0$. The converse also holds: given a $\mathbb{C}$-valued 2-form satisfying the above relations, there exists a unique complex structure in respect to which the form is holomorphic (and the resulting variety is necessarily a $K3$-surface).

Let a be a holomorphic involution of a $K3$-surface X equipped with the complex structure defined by a holomorphic form ω. Then, analyzing the behavior of ω in a neighborhood of a fixed point, one can easily see that, if the fixed point set $\operatorname{Fix} a$ of a is nonempty then it consists either only of isolated points or only of curves. If the fixed point set $\operatorname{Fix} a$ of a consists of only isolated points then $a^*\omega = \omega$. If $\operatorname{Fix} a$ is empty or consists of curves only then $a^*\omega = -\omega$.

Let conj be a real structure on X. Then $\mathrm{conj}^*\omega = \lambda\overline{\omega}$ for some $\lambda \in \mathbb{C}^*$. Clearly, w can be chosen (uniquely up to real factor) so that $\mathrm{conj}^*\omega = -\overline{\omega}$. We always assume this choice and we denote by $\operatorname{Re}\omega$ and $\operatorname{Im}\omega$ the real part $(\omega+\overline{\omega})/2$ and the imaginary part $(\omega-\overline{\omega})/2$ of ω, respectively.

Let Y be a real rational surface with a real nonsingular anti-bicanonical curve $B \subset Y$ such that $B_{\mathbb{R}} = \varnothing$ and B is not linked with the real point set $Y_{\mathbb{R}}$ of Y. Let X be the (real) double covering $K3$-surface branched over B, $\widetilde{p} : X \to Y$ the covering projection and $\phi : X \to X$ the deck translation of $\widetilde{p}$. Then ϕ is a holomorphic involution with nonempty fixed point set. There exist two liftings $c^{(1)}, c^{(2)} : X \to X$ of the real structure $\mathrm{conj} : Y \to Y$ to X, which are both anti-holomorphic involutions. They commute with each other and with ϕ, and their composition is ϕ. Because of the requirements on B, at least one of these involutions is fixed point free. Assume that it is $c^{(1)}$.

Pick a holomorphic 2-form μ with the real and imaginary parts $\operatorname{Re}\mu, \operatorname{Im}\mu$, respectively, and a fundamental Kähler form ν. Due to the Calabi-Yau theorem, there exists a unique Kähler-Einstein metric with fundamental class $[\nu]$, see [12]. After normalizing μ so that $(\operatorname{Re}\mu)^2 = (\operatorname{Im}\mu)^2 = \nu^2$, we get three complex structures on X given by the forms:

$$\mu = \operatorname{Re}\mu + i\operatorname{Im}\mu, \quad \widetilde{\mu} = \nu + i\operatorname{Re}\mu, \quad \text{and} \quad \operatorname{Im}\mu + i\nu.$$

In fact, $\operatorname{Re}\mu$, $\operatorname{Im}\mu$, and ν define a whole 2-sphere of complex structures on X, but we are only interested in the three above. Let $\widetilde{X}$ be the surface X equipped with the complex structure defined by $\widetilde{\mu}$. Since $c^{(1)}$ is an anti-holomorphic involution of X, the holomorphic form μ and the fundamental Kähler form ν can be chosen so that $(c^{(1)})^*\mu = -\overline{\mu}$ and $(c^{(1)})^*\nu = -\nu$. Then $(c^{(1)})^*\widetilde{\mu} = -\widetilde{\mu}$ and, hence, $c^{(1)}$ is holomorphic on $\widetilde{X}$. Since ϕ is a holomorphic involution of X commuting with $c^{(1)}$, $\phi^*\mu = -\mu$ and ν can be chosen ϕ^*-invariant so that $\phi^*\widetilde{\mu} = \overline{\widetilde{\mu}}$, *i.e.*, the involution ϕ is anti-holomorphic on $\widetilde{X}$. Then $E = \widetilde{X}/c^{(1)}$ is a real Enriques surface (the real structure being the common descent of ϕ and $c^{(2)}$) and the projection $p : \widetilde{X} \to E$ is a real double covering. Hence, we have $Y_{\mathbb{R}} = E^{(2)}_{\mathbb{R}}$ and $B/c^{(2)} = E^{(1)}_{\mathbb{R}}$. The maps $\phi = t^{(1)}$, $c^{(1)} = \tau$ and $c^{(2)} = t^{(2)}$, where $t^{(1)}$, $t^{(2)}$ and τ are as in Theorem 2.2.1.

Theorem 4.2.1 (See [3]). *Inverse Donaldson's trick establishes a surjective map* *from the set of deformation classes of unnodal admissible DPN-pairs to the set of deformation classes of real Enriques surfaces with distinguished nonempty half.*

4.3 Deformation classes

Definition 4.3.1. A *deformation* of complex surfaces is a proper analytic submersion $p : Z \to D^2$, where Z is a 3-dimensional analytic variety and $D^2 \subset \mathbb{C}$ a disk. If Z is real and p is equivariant, the deformation is called *real*. Two (real) surfaces X' and X'' are called *deformation equivalent* if they can be connected by a chain $X' = X_0, ..., X_k = X''$ so that X_i and X_{i-1} are isomorphic to (real) fibers of a (real) deformation.

Theorem 4.3.1 (See [3])**.** *With few exceptions listed below the deformation class of a real Enriques surface E with a distinguished half $E_{\mathbb{R}}^{(1)}$ is determined by the topology of its half decomposition. The exceptions are:*

(1) $M-$surfaces of parabolic and elliptic type, i.e., *those with $E_{\mathbb{R}} = 2V_2 \sqcup 4S$, $V_2 \sqcup 2V_1 \sqcup 3S$, or $4V_1 \sqcup 2S$; the additional invariant is the Pontrjagin-Viro form;*

(2) surfaces with $E_{\mathbb{R}} = 2V_1 \sqcup 4S$; the additional invariant is the integral complex separation;

(3) surfaces with a half $E_{\mathbb{R}}^{(1)} = 4S$ other than those mentioned in (1), (2)*; the additional invariants are the types, I_u, I_0, or II, of $E_{\mathbb{R}}^{(1)}$ in E and $X/t^{(2)}$;*

(4) surfaces with $E_{\mathbb{R}} = \{V_{10}\} \sqcup \{\varnothing\}$, $\{V_4 \sqcup S\} \sqcup \{\varnothing\}$, $\{V_2 \sqcup 4S\} \sqcup \{\varnothing\}$, and $\{2S\} \sqcup \{2S\}$; the additional invariant is the type, I_u or I_0, of $E_{\mathbb{R}}$ in E;

(5) surfaces with $E_{\mathbb{R}} = 2V_1 \sqcup 3S$; the additional invariant is the type, I_u or II, of $E_{\mathbb{R}}$ in E;

(6) surfaces with $E_{\mathbb{R}} = \{S_1\} \sqcup \{S_1\}$; the additional invariant is the linking coefficient form of $E_{\mathbb{R}}^{(1)}$.

The Pontrjagin-Viro form and the linking coefficient form are not introduced here. A complete list of deformation classes, as well as detailed explanations of these forms can be found in [3].

4.4 Real Enriques Surfaces with disconnected $E_{\mathbb{R}}^{(1)} = V_d \sqcup ...,\ d \geq 4$

The following theorem gives the deformation classification of real Enriques surfaces with disconnected half $E_{\mathbb{R}}^{(1)} = V_d \sqcup ...,\ d \geq 4$.

Theorem 4.4.1 (See [3]). *With one exception, a real Enriques surface with disconnected $E_{\mathbb{R}}^{(1)} = V_d \sqcup ...,\ d \geq 4$, is determined up to deformation by the topology of $(E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$. In the exceptional case $E_{\mathbb{R}} = \{V_4 \sqcup S\} \sqcup \{\varnothing\}$ there are two deformation classes which differ by the type, I_u or I_0, of $E_{\mathbb{R}}$. The topological types of $(E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ are the extremal types listed below and all their derivatives $(E_{\mathbb{R}}^{(1)}, \cdot)$ obtained from the extremal ones by sequences of topological Morse simplifications of $E_{\mathbb{R}}^{(2)}$:*

$$\begin{array}{ll}
E_{\mathbb{R}}^{(1)} = V_{11} \sqcup V_1; & E_{\mathbb{R}}^{(2)} = \varnothing; \\
E_{\mathbb{R}}^{(1)} = V_9 \sqcup V_1; & E_{\mathbb{R}}^{(2)} = \varnothing; \\
E_{\mathbb{R}}^{(1)} = V_7 \sqcup V_1; & E_{\mathbb{R}}^{(2)} = \varnothing; \\
E_{\mathbb{R}}^{(1)} = V_5 \sqcup V_1; & E_{\mathbb{R}}^{(2)} = S; \\
E_{\mathbb{R}}^{(1)} = V_5 \sqcup S; & E_{\mathbb{R}}^{(2)} = V_1; \\
E_{\mathbb{R}}^{(1)} = V_4 \sqcup V_1; & E_{\mathbb{R}}^{(2)} = V_1; \\
E_{\mathbb{R}}^{(1)} = V_5 \sqcup V_1 \sqcup S; & E_{\mathbb{R}}^{(2)} = \varnothing; \\
E_{\mathbb{R}}^{(1)} = V_4 \sqcup 2V_1; & E_{\mathbb{R}}^{(2)} = \varnothing; \\
E_{\mathbb{R}}^{(1)} = V_4 \sqcup S; & E_{\mathbb{R}}^{(2)} = V_2,\ 4S, or\ S_1.
\end{array}$$

For the monodromy problem, we need to consider only the following extremal types from the above list (as in the other cases there are no homeomorphic components):

(1) $E_{\mathbb{R}}^{(1)} = V_4 \sqcup 2V_1;\quad E_{\mathbb{R}}^{(2)} = \varnothing;$
(2) $E_{\mathbb{R}}^{(1)} = V_4 \sqcup S;\qquad E_{\mathbb{R}}^{(2)} = 4S.$

Below we give a brief account of the results regarding the first case.

Theorem 4.4.2 (See [3]). *Let Q_1 and Q_2 be two real curves on Σ_0 (with the real structure $c_0 \times c_1$) so that both are as in Model I (Section 3.4.1). Then the*

DPN-resolutions of (Σ_0, Q_1) *and* (Σ_0, Q_2) *are deformation equivalent in the class of admissible DPN-pairs.*

The above theorem is proved by making use of Theorem 3.6.1 and showing that Q_1 and Q_2 are rigidly homotopic. Thus, a generic rigid homotopy of Q_s defines a deformation of the *DPN*-resolutions (Y_s, B_s) of the pairs (Σ_0, Q_s), hence, a deformation of the covering $K3$-surfaces. Choosing a continuous family of invariant Kähler metrics leads to a deformation of the corresponding real Enriques surfaces obtained by inverse Donaldson's trick. Therefore, we have the following stronger result.

Refinement 4.4.1 (of Theorem 4.4.2)**.** *Let* Q *be a real curve on* Σ_0 *(with the real structure* $c_0 \times c_1$*) as in Model I. Then a generic rigid homotopy of* Q *defines a deformation of the appropriate real Enriques surfaces with* $E_{\mathbb{R}}^{(1)} = V_4 \sqcup 2V_1$. □

The results below are related to the second case.

Theorem 4.4.3 (See [3])**.** *Let* $F \in |e_\infty|$ *and* $U \in |2e_\infty|$ *be nonsingular real curves on* $Z = \Sigma_4$ *with standard real structure. Suppose that* $U_{\mathbb{R}}$ *is contained in a connected component of* $Z_{\mathbb{R}} \backslash ((E_0)_{\mathbb{R}} \sqcup F_{\mathbb{R}})$*. Then the DPN-double of* $(Z; U, E_0 \cup F)$ *is determined up to deformation in the class of admissible DPN-pairs by the real root scheme of the pair* (U, F)*.*

Proof of the above theorem is based on showing that a generic rigid isotopy of the pairs (U_s, F_s), where $U_s \in |2e_\infty|$ and $F_s \in |e_\infty|$ for each s, defines a deformation of the *DPN*-doubles (Y_s, B_s) of $(\Sigma_4; U_s, E_0 \cup F_s)$. A deformation of the (Y_s, B_s) defines a deformation of the covering $K3$-surfaces. Choosing a continuous family of invariant Kähler metrics gives a deformation of the corresponding real Enriques surfaces obtained by inverse Donaldson's trick, which implies the following stronger result.

Refinement 4.4.2 (of Theorem 4.4.3)**.** *Let* $F \in |e_\infty|$ *and* $U \in |2e_\infty|$ *be nonsingular real curves on* $Z = \Sigma_4$ *with standard real structure such that* $U_{\mathbb{R}}$ *is contained in a connected component of* $Z_{\mathbb{R}} \backslash ((E_0)_{\mathbb{R}} \sqcup F_{\mathbb{R}})$*. Then a generic rigid isotopy of pairs* (U, F) *defines a deformation of the appropriate real Enriques surfaces with* $E_{\mathbb{R}}^{(1)} = V_4 \sqcup S$ *and* $E_{\mathbb{R}}^{(2)} \neq \varnothing$*.* □

4.5 Real Enriques Surfaces with disconnected $E_{\mathbb{R}}^{(1)} = V_3 \sqcup ...$

The following theorem gives the deformation classification of real Enriques surfaces with disconnected half $E_{\mathbb{R}}^{(1)} = V_3 \sqcup ...$

Theorem 4.5.1 (See [3]). *A real Enriques surface with disconnected $E_{\mathbb{R}}^{(1)} = V_3 \sqcup ...$ is determined up to deformation by the topology of $(E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$. The topological types of $(E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ are the extremal types listed below and all their derivatives $(E_{\mathbb{R}}^{(1)}, \cdot)$ obtained from the extremal ones by sequences of topological Morse simplifications of $E_{\mathbb{R}}^{(2)}$:*

$$\begin{array}{ll}
E_{\mathbb{R}}^{(1)} = V_3 \sqcup V_1; & E_{\mathbb{R}}^{(2)} = V_2 \textit{ or } 4S; \\
E_{\mathbb{R}}^{(1)} = V_3 \sqcup S; & E_{\mathbb{R}}^{(2)} = V_3 \textit{ or } V_1 \sqcup 3S; \\
E_{\mathbb{R}}^{(1)} = V_3 \sqcup V_1 \sqcup S; & E_{\mathbb{R}}^{(2)} = 3S; \\
E_{\mathbb{R}}^{(1)} = V_3 \sqcup 2S; & E_{\mathbb{R}}^{(2)} = V_1 \sqcup 2S; \\
E_{\mathbb{R}}^{(1)} = V_3 \sqcup V_1 \sqcup 2S; & E_{\mathbb{R}}^{(2)} = 2S; \\
E_{\mathbb{R}}^{(1)} = V_3 \sqcup 3S; & E_{\mathbb{R}}^{(2)} = V_1 \sqcup S; \\
E_{\mathbb{R}}^{(1)} = V_3 \sqcup V_1 \sqcup 3S; & E_{\mathbb{R}}^{(2)} = S; \\
E_{\mathbb{R}}^{(1)} = V_3 \sqcup 4S; & E_{\mathbb{R}}^{(2)} = V_1; \\
E_{\mathbb{R}}^{(1)} = V_3 \sqcup V_1 \sqcup 4S; & E_{\mathbb{R}}^{(2)} = \varnothing.
\end{array}$$

The deformation classification of these surfaces is reduced to that of real $(2, r)$-surfaces, $r \geq 1$ with a real nonsingular anti-bicanonical curve $B \cong S_2 \sqcup rS$ (see Section 3.3) and, hence, to the rigid isotopy classification of suitable pairs (see Section 3.6.5).

Lemma 4.5.1 (See [3]). *There is a natural surjective map from the set of rigid isotopy classes of suitable pairs of grade r onto the set of deformation classes of real Enriques surfaces with $E_{\mathbb{R}}^{(1)} = V_3 \sqcup \frac{r}{2}S$, if r is even, or $E_{\mathbb{R}}^{(1)} = V_3 \sqcup V_1 \sqcup \frac{r-1}{2}S$, if r is odd.*

Proof of the above lemma is based on showing that a generic rigid isotopy of suitable pairs (U_s, F_s) defines a deformation of the DPN-doubles (Y_s, B_s) of

$(\Sigma_2; U_s, E_0 \cup F_s)$, so a deformation of the covering $K3$-surfaces. Then it remains to choose a continuous family of invariant Kähler metrics, to obtain a deformation of the corresponding real Enriques surfaces obtained by inverse Donaldson's trick which implies the following stronger result.

Refinement 4.5.1 (of Theorem 4.5.1). *A generic rigid isotopy of suitable pairs (U, F) of grade r defines a deformation of the real Enriques surfaces with $E_{\mathbb{R}}^{(1)} = V_3 \sqcup \frac{r}{2}S$, if r is even, or $E_{\mathbb{R}}^{(1)} = V_3 \sqcup V_1 \sqcup \frac{r-1}{2}S$, if r is odd.* □

4.6 Real Enriques Surfaces with $E_{\mathbb{R}}^{(1)} = S_1$

The following theorem gives the deformation classification of real Enriques surfaces with $E_{\mathbb{R}}^{(1)} = S_1$.

Theorem 4.6.1 (See [3]). *With the exception of the few cases listed below a real Enriques surface with $E_{\mathbb{R}}^{(1)} = S_1$ is determined up to deformation by the topology of $(E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$. The exceptional cases are:*

(1) surfaces with $E_{\mathbb{R}} = \{S_1\} \sqcup \{4S\}$: there are two deformation classes, which differ by the type, I_u or I_0, of $E_{\mathbb{R}}$ or, equivalently, by the type I_u or I_0, of $E_{\mathbb{R}}^{(2)}$ in $X/t^{(1)}$;

(2) surfaces with $E_{\mathbb{R}} = \{S_1\} \sqcup \{S_1\}$: there are two deformation classes, which differ by the linking coefficient form of $E_{\mathbb{R}}^{(1)}$;

(3) surfaces with $E_{\mathbb{R}} = \{S_1\}$: there are two deformation classes, which differ by the types of $E_{\mathbb{R}}$ in E and $X/t^{(2)}$; the pair of types takes values (I_u, I_0) and (I_0, I_u).

The topological types of $(E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ are the extremal types listed below and, with the exception of $\{S_1\} \sqcup \{5S\}$, all their derivatives $(E_{\mathbb{R}}^{(1)}, .)$ obtained from the extremal types by sequences of topological Morse simplifications of $E_{\mathbb{R}}^{(2)}$:

$E_{\mathbb{R}}^{(1)} = S_1;\quad E_{\mathbb{R}}^{(2)} = V_{10},\ V_4 \sqcup S,\ 2V_2,\ V_2 \sqcup 4S,$ *or* $S_1;$

The deformation classification of these surfaces is reduced to that of *DPN*-pairs $(\widetilde{Y}, \widetilde{B} \cong 2S_1)$ where the genus 1 components of $\widetilde{B}$ are conjugate. In the notation of Section 3.5, $(\widetilde{Y}, \widetilde{B})$ is the *DPN*-pair resulting from Donaldson's trick, X is the covering $K3$-surface of $\widetilde{Y}$ branched over $\widetilde{B}$ and the maps $q = t^{(1)}$, $c^{+-} = t^{(2)}$ and $\tau = c^{++}$, where $t^{(1)}$, $t^{(2)}$ and τ are as in Theorem 2.2.1.

Theorem 4.6.2 (See [3]). *If $E_{\mathbb{R}}^{(1)} = S_1$ and $\widetilde{Y}$ is unnodal, $\widetilde{Y}$ admits one of the following models:*

(1) model (B), if $E_{\mathbb{R}}^{(2)}$ is nonorientable;

(2) model (A) or (C) with U an M-curve, if $E_{\mathbb{R}}^{(2)} = 4S$;

(3) model (C) otherwise.

In all the cases, the branch curve U is nonsingular and the distinguished fibers F and G are conjugate and transversal to U; in model (C) the part $Z^+ \subset \mathbb{P}^2_{\mathbb{R}}$ is orientable.

Theorem 4.6.3 (See [3]). *The set of deformation classes of real Enriques surfaces E with $E_{\mathbb{R}}^{(1)} = S_1$ is the image under a natural surjective map from the union of the sets of rigid isotopy classes of the following objects:*

(A) nonsingular real M-curves $U \in |2l_1 + 4l_2|$ on $Z = \Sigma_0$ with $Z_{\mathbb{R}} = S_1$;

(B) nonsingular real curves in $|3e_\infty|$ on $Z = \Sigma_2$ with $Z_{\mathbb{R}} = S_1$;

(C) triples (U, O, ϵ), where U is a nonsingular real quartic in $Z = \mathbb{P}^2$, signed so that $Z^- = Z^{-U}$ is nonorientable, O is a point in $Z^- \setminus U$, and ϵ a choice of sign of P such that $Z^{++} = \varnothing$.

In case (C) the condition $Z^{++} = \varnothing$ implies $Z^{+P} = \varnothing$ whenever $U_{\mathbb{R}} \neq \varnothing$. Thus, ϵ matters only if $U_{\mathbb{R}} = \varnothing$

Proof of this theorem is based on showing that a generic rigid isotopy of the objects in (A)-(C) defines a deformation of the *DPN*-pairs $(\widetilde{Y}, \widetilde{B})$ which are obtained from the corresponding models (A)-(C), so a deformation of the covering $K3$-surfaces. Choosing a continuous family of invariant Kähler metrics gives a deformation of the corresponding real Enriques surfaces obtained via inverse Donaldson's trick. Hence we have the following result.

Refinement 4.6.1 (of Theorem 4.6.3)**.** *A generic rigid isotopy of the objects in (A)-(C) defines a deformation of the appropriate real Enriques surfaces E with $E_{\mathbb{R}}^{(1)} = S_1$.*

(A) nonsingular real M-curves $U \in |2l_1 + 4l_2|$ on $Z = \Sigma_0$ with $Z_{\mathbb{R}} = S_1$;

(B) nonsingular real curves in $|3e_\infty|$ on $Z = \Sigma_2$ with $Z_{\mathbb{R}} = S_1$;

(C) triples (U, O, ϵ), where U is a nonsingular real quartic in $Z = \mathbb{P}^2$, signed so that $Z^- = Z^{-U}$ is nonorientable, O is a point in $Z^- \setminus U$, and ϵ a choice of sign of P such that $Z^{++} = \varnothing$.

□

4.7 Real Enriques Surfaces with $E_{\mathbb{R}}^{(1)} = 2V_2$

The following theorem gives the deformation classification of real Enriques surfaces with $E_{\mathbb{R}}^{(1)} = 2V_2$.

Theorem 4.7.1 (See [3])**.** *With one exception below a real Enriques surface with $E_{\mathbb{R}}^{(1)} = 2V_2$ is determined up to deformation by the topology of $(E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$. The exceptional case is:*

- *M-surfaces with $E_{\mathbb{R}} = 2V_2 \sqcup 4S$: a surface is determined by the decomposition $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \sqcup E_{\mathbb{R}}^{(2)}$, the complex separation, and the value of the Pontrjagin-Viro form on the characteristic class of a component V_2.*

The topological types of $(E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ are the extremal types listed below and, all their derivatives $(E_{\mathbb{R}}^{(1)}, .)$ obtained from the extremal types by sequences of topological Morse simplifications of $E_{\mathbb{R}}^{(2)}$:

$$E_{\mathbb{R}}^{(1)} = 2V_2; \quad E_{\mathbb{R}}^{(2)} = 4S, \text{ or } S_1.$$

The deformation classification of these surfaces is reduced to that of *DPN*-pairs $(\widetilde{Y}, \widetilde{B} \cong 2S_1)$ where the genus 1 components of $\widetilde{B}$ are real.

Theorem 4.7.2 (See [3])**.** *If* $E_{\mathbb{R}}^{(1)} = 2V_2$ *and* $\widetilde{Y}$ *is unnodal, then* $\widetilde{Y}$ *admits model (A) or (C) so that both the distinguished fibers* F *and* G *(images of genus 1 components of* $\widetilde{B}$ *in* Σ_0 *or* $\mathbb{P}^2$*, respectively) are real and transversal to the branch curve* U*. In model (C) the part* $Z^+ \subset \mathbb{P}^2_{\mathbb{R}}$ *covered by* $\widetilde{Y}_{\mathbb{R}}$ *is orientable.*

Theorem 4.7.3 (See [3])**.** *The set of deformation classes of real Enriques surfaces* E *with disconnected* $E_{\mathbb{R}}^{(1)} = V_2 \sqcup ...$ *is the image under a natural surjective map from the union of the sets of equivalence classes of suitable pairs* (U, P) *on* $Z = \Sigma_0$ *or* $\mathbb{P}^2$*, considered up to rigid isotopy and real automorphism of* Z*.*

Proof of this theorem is based on showing that a generic rigid isotopy of suitable pairs (U, P) on $Z = \Sigma_0$ or $\mathbb{P}^2$ defines a deformation of the DPN-pairs $(\widetilde{Y}, \widetilde{B})$ which are obtained from the corresponding models (A) and (C), so a deformation of the covering $K3$-surfaces. Choosing a continuous family of invariant Kähler metrics gives a deformation of the corresponding real Enriques surfaces obtained via inverse Donaldson's trick. Hence we have the following result.

Refinement 4.7.1 (of Theorem 4.7.3)**.** *A generic rigid isotopy of suitable pairs* (U, P) *on* $Z = \Sigma_0$ *or* $\mathbb{P}^2$ *defines a deformation of the corresponding real Enriques surfaces* E *with disconnected* $E_{\mathbb{R}}^{(1)} = V_2 \sqcup ...$ $\square$

Chapter 5

Main Results

5.1 Lifting Involutions

Let Z be a simply connected surface and $\pi : Y \to Z$ a branched double covering with the branch divisor C. Then, any involution $a : Z \to Z$ preserving C as a divisor admits two lifts to Y, which commute with each other and with the deck translation of the covering. If $\operatorname{Fix} a \neq \varnothing$, then both lifts are also involutions. Any fixed point of a in $Z \setminus C$ has two pull-backs on Y. One of the lifts fixes these two points and the other one interchanges them.

In this section we will use the notation of Section 4.2.

Lemma 5.1.1. *Let $Z = \Sigma_4$ (with the standard real structure), and $U \in |2e_\infty|$ a nonsingular real curve. Let $a : Z \to Z$ be an involution preserving U and such that $\operatorname{Fix} a \cap U \neq \varnothing$. Then a lifts to four distinct involutions on the covering $K3$-surface X and at least one of the four lifts defines an automorphism of an appropriate real Enriques surface obtained from X by inverse Donaldson's trick.*

Proof: For a nonsingular real curve $F \in |e_\infty|$ in Z, if $U_{\mathbb{R}}$ is contained in a connected component of $Z_{\mathbb{R}} \setminus ((E_0)_{\mathbb{R}} \sqcup F_{\mathbb{R}})$ then the *DPN*-double (Y, B) of $(Z; U, E_0 \cup F)$ is as follows: Y is a real unnodal $(3,2)$-surface, and, B is an admissible branch curve with two rational components which are conjugate to each other and $[B] = 0$ in $H_2(X)$ where X is the covering $K3$-surface of Y branched over B (see Model II in Section 3.4.1). Any point $p \in \operatorname{Fix} a \cap U$ has a unique pullback $\widetilde{p} \in Y$ which is a fixed point of both lifts of a to Y. Any point $p' \in \operatorname{Fix} a \setminus U$, in a small neighborhood of p, has two pullbacks p_1 and p_2 in Y. If a_1 is the lift of a to Y that permutes p_1 and p_2, then $\widetilde{p}$ is an isolated fixed point of a_1 (note that we do **not** assert that **all** fixed points of a_1 are isolated). Choose $F \in |e_\infty|$ and the point $p \in \operatorname{Fix} a \cap U$ in such a way that B is a_1-invariant and $\widetilde{p} \notin B$. Let X be the double covering of Y branched over B and let a_2 be the lift of a_1 to X that fixes the two pullbacks of $\widetilde{p}$. Then, the pullbacks of $\widetilde{p}$ are isolated fixed points of a_2. Since X is a $K3$-surface, $\operatorname{Fix} a_2$ consists of isolated points only, and $(a_2)^*\mu = \mu$. We can choose for ν a generic fundamental Kähler form preserved by ϕ, $c^{(1)}$, $c^{(2)}$, and a_2. Then, we have $(a_2)^*\widetilde{\mu} = \widetilde{\mu}$, *i.e.*, a_2 is also holomorphic on $\widetilde{X}$. With the projection $p : \widetilde{X} \to E$, a_2 defines an automorphism $\widetilde{a}$ of E.. $\square$

Lemma 5.1.2. *Let $Z = \Sigma_2$ (with the standard real structure), let $U \in |2e_\infty + 2l|$ be a suitable curve on Z, and let $a : Z \to Z$ be an involution preserving U such that $\operatorname{Fix} a \cap U \neq \varnothing$. Then, a lifts to four distinct involutions on the covering $K3$-surface X and at least one of the four lifts defines an automorphism of an appropriate real Enriques surface obtained from X by inverse Donaldson's trick.*

Proof: For a nonsingular real section $F \in |e_\infty|$ in Z, if (U, F) is a suitable pair then the DPN-double of $(Z; U, E_0 \cup F)$ is (Y, B) where Y is a $(2, r)$-surface and B is an admissible branch curve on Y (see Section 3.3). Thus, for any such curve F, we can make choices of the points $p \in \operatorname{Fix} a \cap U$ and $p' \in \operatorname{Fix} a \setminus U$, and the lift a_1 of a to Y in the same way that we did in the proof of Lemma 5.1.1 so that $\widetilde{p} \in Y$ will be an isolated fixed point of a_1. Choose $F \in |e_\infty|$ and the point $p \in \operatorname{Fix} a \cap U$ in such a way that B is a_1-invariant and does not contain $\widetilde{p}$. Then the result follows by making the same choices for the rest as in the proof of Lemma 5.1.1. $\square$

Lemma 5.1.3. *Let Z be a real quadric cone in $\mathbb{P}^3$, let $C' \subset Z$ be a nonsingular real cubic section disjoint from the vertex, and let $a : Z \to Z$ be an involution preserving C' and such that $\operatorname{Fix} a \cap C' \neq \varnothing$. Then a lifts to four distinct involutions on the covering $K3$-surface X and at least one of the four lifts defines an* *automorphism of an appropriate real Enriques surface obtained from X by inverse Donaldson's trick.*

Proof: According to Section 3.5, the minimal resolution of the double covering of Z, branched at the vertex and over C', is a surface Y admitting an elliptic fibration as in model (B). The pullback $\tilde{p} \in Y$ of any point $p \in \operatorname{Fix} a \cap C'$ is a fixed point of both lifts of a to Y. Let $p' \in \operatorname{Fix} a \setminus C'$ be in a small neighborhood of p. Then p' has two pullbacks p_1 and p_2 in Y. Let a_1 be the lift of a to Y that permutes p_1 and p_2. Then $\tilde{p}$ is an isolated fixed point of a_1. We can lift a_1 and $\tilde{p}$ on Y to $\widetilde{Y}$ and denote them by a_1' and $\tilde{p}'$, where $Y \to \mathbb{P}^1$ is the minimal pencil of $\widetilde{Y} \to \mathbb{P}^1$. The lift $\tilde{p}'$ is an isolated fixed point of a_1'. Pick an a_1'-invariant admissible branch curve $\widetilde{B} \subset \widetilde{Y}$ with $\tilde{p}' \notin \widetilde{B}$. Denote by X the double covering of $\widetilde{Y}$ branched over $\widetilde{B}$ and by a_2, the lift of a_1' to X that fixes the two pullbacks of $\tilde{p}'$. Then the pullbacks of $\tilde{p}'$ are isolated fixed points of a_2. Since X is a $K3$-surface, $\operatorname{Fix} a_2$ consists of isolated points only, and $(a_2)^*\mu = \mu$. Making a similar choice for the Kähler form as in the proof of Lemma 5.1.1 gives the result. $\square$

Lemma 5.1.4. *Let $Z = \Sigma_0$ (with the standard real structure), $U \in |2l_1 + 4l_2|$ a nonsingular real curve on Z, and $a : Z \to Z$ be an involution preserving U such that $\operatorname{Fix} a \cap U \neq \varnothing$. Then a lifts to four distinct involutions on the covering $K3$-surface X and at least one of the four lifts defines an automorphism of an appropriate real Enriques surface obtained from X by inverse Donaldson's trick.*

Proof: Proof is very similar to the previous ones. Choose the lift of a to Y and the lift of that to the covering $K3$-surface X in such a way that it has isolated fixed points on X. Choose a generic fundamental Kähler form preserved by ϕ, $c^{(1)}$, $c^{(2)}$, and the lift of a so that the lift of a is also holomorphic with respect to the new complex structure. Then the projection to Enriques surface defines an automorphism of the Enriques surface. $\square$

Lemma 5.1.5. *Let U be a nonsingular real quartic on $Z = \mathbb{P}^2$, and F and G be a pair of real lines transversal to U such that $U_{\mathbb{R}}$ belongs to the closure of one of the two components of $Z_{\mathbb{R}} \setminus (F_{\mathbb{R}} \cup G_{\mathbb{R}})$. Let $a : Z \to Z$ be an involution preserving U and the pair (F, G) such that $\operatorname{Fix} a \cap U \neq \varnothing$. Then a lifts to four distinct* *involutions on the covering $K3$-surface X and at least one of the four lifts defines an automorphism of an appropriate real Enriques surface obtained from X by inverse Donaldson's trick.*

Proof: Proof is similar to the previous ones. It is based on choosing one of the four lifts that has isolated fixed points on X and choosing a generic fundamental Kähler form preserved by ϕ, $c^{(1)}$, $c^{(2)}$ and the lift of a, so that the lift of a is also holomorphic with respect to the new complex structure. Then its projection defines an automorphism of the corresponding Enriques surface. □

5.2 Main Theorems

The following theorem is obtained during author's master study. For the sake of completeness, the result is added to the thesis. See [8] for a proof.

Theorem 5.2.1. *With one exception, any permutation of the homeomorphic components of the half $E_{\mathbb{R}}^{(2)}$ of a real Enriques surface with a distinguished half $E_{\mathbb{R}}^{(1)} = V_{d+2}$, $d \geq 1$, can be realized by deformations and automorphisms. In the exceptional case $E_{\mathbb{R}} = \{V_3\} \sqcup \{V_1 \sqcup 4S\}$, the realized group is $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset S_4$.*

Remark 5.2.1. In the exceptional case, E is an M-surface. Pontrjagin-Viro form is well defined. The decomposition of the second half into quarters is $E_{\mathbb{R}}^{(2)} = (V_1 \sqcup 2S) \sqcup (2S)$. Obviously, one cannot permute the spheres belonging to different quarters (even topologically), and Theorem 5.2.1 states that a permutation of the spherical components can be realized if and only if it preserves the quarter decomposition.

Theorem 5.2.2. *With one exception, any permutation of homeomorphic components of both halves of a real Enriques surface with a disconnected half $E_{\mathbb{R}}^{(1)} = V_d \sqcup ...$, $d \geq 4$, is realizable by deformations and automorphisms if and only if it preserves the half decomposition. In the exceptional case $E_{\mathbb{R}} = \{V_4 \sqcup S\} \sqcup \{4S\}$, the realized group is $D_8 \subset S_4$.*

Remark 5.2.2. In the exceptional case, E is an M-surface on which the Pontrjagin-Viro form is well defined. The quarter decomposition of $E_{\mathbb{R}}^{(2)}$ is

$(2S) \sqcup (2S)$. A permutation of the spherical components is not realizable if it does not preserve the quarter decomposition. Theorem 5.2.2 states that a permutation of the components can be realized if and only if it preserves the quarter decomposition.

Proof: The problem reduces to a question about appropriate (g, r)-surfaces, $g \geq 3$ and $r \geq 1$ (see [3] for the models of (g, r)-surfaces). We construct a particular surface (within each deformation class) that has a desired automorphism or 'auto-deformation'. Among the extremal types listed in Theorem 4.4.1, we need to consider only the following types and all their derivatives $(E_{\mathbb{R}}^{(1)}, \cdot)$ obtained from the extremal ones by sequences of topological Morse simplifications of $E_{\mathbb{R}}^{(2)}$:

(1) $E_{\mathbb{R}}^{(1)} = V_4 \sqcup 2V_1$; $\quad E_{\mathbb{R}}^{(2)} = \varnothing$;
(2) $E_{\mathbb{R}}^{(1)} = V_4 \sqcup S$; $\quad E_{\mathbb{R}}^{(2)} = 4S$.

Case (1): $E_{\mathbb{R}} = \{V_4 \sqcup 2V_1\} \sqcup \{\varnothing\}$: In this case, the homeomorphic components are in $E_{\mathbb{R}}^{(1)} = B/t^{(2)}$ so we need to deal with B. By Donaldson's trick, we obtain a DPN-pair (Y, B), where Y is a real $(3, 2)$-surface with empty real part and $B \cong S_3 \sqcup 2S$ is an admissible branch curve on Y such that the rational components of B are real. According to Model (I) in Section 3.4.1, Y blows down to $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with the real structure $c_0 \times c_1$. The image of B is $Q = C_1' + C_1'' + C_1'''$ where $C', C'' \in |l_2|$ are two distinct real generatrices and $C''' \in |4l_1 + 2l_2|$. By Theorem 4.4.2, there is only one rigid homotopy class of such curves $Q \subset \Sigma_0$ and if Q' is rigidly homotopic to Q then the DPN-resolutions of the pairs (Σ_0, Q') and (Σ_0, Q) are deformation equivalent in the class of admissible DPN-pairs.

Using Refinement 4.4.1, it suffices to connect Q with itself by a path that realizes the permutation of C' and C'', and such that the members Q_s of the path split into sums $C_s' + C_s'' + C_s'''$ of distinct real smooth irreducible curves such that $C_s', C_s'' \in |l_2|$ and $C_s''' \in |4l_1 + 2l_2|$. Identify the real part of the base $\mathbb{P}^1_{\mathbb{R}} \cong S^1 = \mathbb{R}/2\pi$. Let $Q = A_0 + A_\pi + A$, where A_α is the generatrix of Σ_0 over α. Then, the family $\{Q_t\} = \{A_t + A_{\pi+t} + A; t \in [0, \pi]\}$, defines a path that realizes the permutation of the generatrices A_0 and A_π.

Case (2): We consider the 3 subcases:

$$E_{\mathbb{R}} = \{V_4 \sqcup S\} \sqcup \{iS\}, \quad i = 2, 3, 4.$$

The corresponding *DPN*-pair resulting from Donaldson's trick is (Y, B), where Y is a real $(3,2)$-surface with $Y_{\mathbb{R}} = iS$, and $B \cong S_3 \sqcup 2S$ is an admissible branch curve such that $[B] = 0$ in $H_2(X)$ where X is the covering *K3*-surface. According to Model (II) in Section 3.4.2, there is a real regular degree 2 map $\phi : Y \to \Sigma_4$ (with the standard real structure), branched over a nonsingular curve $U \in |2e_\infty|$. The irrational component of B is mapped to a real curve $F \in |e_\infty|$ and each rational component is mapped isomorphically to the exceptional section E_0 of Σ_4. By Theorem 3.6.4, up to rigid isotopy and automorphism of Σ_4, the pair (U, F) is determined by its real root scheme. By Theorem 4.4.3, the real *DPN*-double of $(\Sigma_4; U, E_0 + F)$ is determined up to deformation in the class of admissible *DPN*-pairs by the real root scheme of the pair (U, F). In view of Refinement 4.4.2 and Lemma 5.1.1, it suffices to realize permutations of certain ovals of U by rigid isotopies of the pair (U, F) and/or involutive automorphisms of Σ_4 preserving U. In the latter case, the set of fixed points should have nonempty intersection with the branch locus U.

The real root scheme of (U, F) is a disjoint union of i segments (*cf.* the first row of Table 3.1 for $i = 4$), and it has a representative (real root marking) with the desired symmetry group (*i.e.*, S_2, S_3 and D_8 for $i = 2, 3$ and 4 respectively), generated by rotations and reflections of $F_{\mathbb{R}} \cong S^1$. By Refinement 3.6.2, these symmetries realize permutation of the corresponding ovals. Furthermore, the fixed point set of a reflection symmetry consists of two distinct points on S^1, which correspond to two distinct real generatrices of Z. Since $U \in |2e_\infty|$, it intersects the set of the fixed points of the induced involution and one can apply Lemma 5.1.1. For $i = 4$, the reason, why other permutations are not allowed is explained in Remark 5.2.2. □

Theorem 5.2.3. *For the real Enriques surfaces with disconnected $E^{(1)}_{\mathbb{R}} = V_3 \sqcup ...$, none of the permutations of the components of the half $E^{(1)}_{\mathbb{R}}$ is realizable by deformations or automorphisms. With the exceptions listed below, any permutation of homeomorphic components of the half $E^{(2)}_{\mathbb{R}}$ can be realized by deformations and automorphisms. The exceptional cases are:*

(1) surfaces with $E_{\mathbb{R}} = \{V_3 \sqcup V_1\} \sqcup \{4S\}$ *: the realized group is* D_8*;*

(2) surfaces with $E_{\mathbb{R}} = \{V_3 \sqcup S\} \sqcup \{V_1 \sqcup 3S\}$ *: the realized group is* S_2*;*

(3) surfaces with $E_{\mathbb{R}} = \{V_3 \sqcup V_1 \sqcup S\} \sqcup \{3S\}$ *: the realized group is* S_2*;*

(4) surfaces with $E_{\mathbb{R}} = \{V_3 \sqcup 2S\} \sqcup \{V_1 \sqcup 2S\}$ *: the realized group is trivial.*

Remark 5.2.3. The exceptional surfaces are M-surfaces. Pontrjagin-Viro form is well defined on them. The quarter decompositions of those surfaces are as follows:

(1) $E_{\mathbb{R}} = \{(V_3 \sqcup V_1) \sqcup (\varnothing)\} \sqcup \{(2S) \sqcup (2S)\}$;

(2) $E_{\mathbb{R}} = \{(V_3 \sqcup S) \sqcup (\varnothing)\} \sqcup \{(V_1 \sqcup S) \sqcup (2S)\}$;

(3) $E_{\mathbb{R}} = \{(V_3 \sqcup S) \sqcup (V_1)\} \sqcup \{(2S) \sqcup (S)\}$;

(4) $E_{\mathbb{R}} = \{(V_3 \sqcup S) \sqcup (S)\} \sqcup \{(V_1 \sqcup S) \sqcup (S)\}$.

One cannot permute homeomorphic components without preserving the quarter decomposition. The above theorem states that a permutation of homeomorphic components of $E_{\mathbb{R}}^{(2)}$ can be realized if and only if it preserves the quarter decomposition.

Proof: For these surfaces, the DPN-pair resulting from Donaldson's trick is (Y, B), where Y is a real unnodal $(2, r)$-surface and $B \cong S_2 \sqcup rS$ is an admissible branch curve on Y, $r \geq 1$.

We start by proving the first part of the theorem. In view of Theorem 4.5.1, we need to consider only the real Enriques surfaces with $E_{\mathbb{R}}^{(1)} = V_3 \sqcup mV_1 \sqcup nS$, $m = 0$ or 1 and $n = 2, 3$ or 4. By Donaldson's trick we obtain real $(2, r)$-surfaces with admissible branch curves $B \cong S_2 \sqcup rS$, where $r = m + 2n$, (as $E_{\mathbb{R}}^{(1)} = B/t^{(2)}$). According to the models of $(2, r)$-surfaces (Section 3.3), the Dynkin graph (Figure 3.1) of the pullback of the exceptional section E_0 contains $m + 2n$ copies of (-4)-curves that correspond to the spherical components of B. Since the map φ is anti-bicanonical, our model is canonical and both the Dynkin graph and the corresponding Coxeter diagram on the covering $K3$-surface are rigid. The only map that can realize a permutation of the spherical components of B is the deck translation of the covering φ which changes the order of the curves in the Dynkin graph. But since the spherical components permuted by the deck translation are identified by the map $t^{(2)}$ on the covering $K3$-surface and $E^{(1)}_{\mathbb{R}} = B/t^{(2)}$, the result follows.

Proof of the second part is based on suitable pairs. Theorem 3.6.6 states that, up to rigid isotopy and automorphism of Σ_2, a suitable pair (U, F) is determined by its real root scheme. In view of Refinement 4.5.1 and Lemma 5.1.2, it is enough to realize the permutations of certain ovals by rigid isotopies of the pair (U, F) and/or involutive automorphisms of Σ_2 preserving U, where in the latter case the set of fixed points should intersect U. Proof is very similar to that of Theorem 5.2.2, case 2. Among the extremal types listed in Theorem 4.5.1, we need to consider only the following types and all their derivatives $(E^{(1)}_{\mathbb{R}}, \cdot)$ obtained from the extremal ones by sequences of topological Morse simplifications of $E^{(2)}_{\mathbb{R}}$:

(1) $E^{(1)}_{\mathbb{R}} = V_3 \sqcup V_1$; $\quad E^{(2)}_{\mathbb{R}} = 4S$;
(2) $E^{(1)}_{\mathbb{R}} = V_3 \sqcup S$; $\quad E^{(2)}_{\mathbb{R}} = V_1 \sqcup 3S$;
(3) $E^{(1)}_{\mathbb{R}} = V_3 \sqcup V_1 \sqcup S$; $\quad E^{(2)}_{\mathbb{R}} = 3S$;
(4) $E^{(1)}_{\mathbb{R}} = V_3 \sqcup 2S$; $\quad E^{(2)}_{\mathbb{R}} = V_1 \sqcup 2S$;
(5) $E^{(1)}_{\mathbb{R}} = V_3 \sqcup V_1 \sqcup 2S$; $\quad E^{(2)}_{\mathbb{R}} = 2S$.

The extremal real root schemes of the pairs (U, F) for these cases are listed in Table 3.1; the others are obtained from the extremal ones by removing several segments not containing a $\circ$ -vertex.

Case (1): We have three subcases:

$$E_{\mathbb{R}} = \{V_3 \sqcup V_1\} \sqcup \{iS\}, \quad i = 2, 3, 4.$$

Corresponding surface is a $(2, 1)$-surface. According to the models of $(2, r)$-surfaces and Theorem 3.6.6, $U \in |2e_\infty + 2l|$ is a nonsingular real curve which has two conjugate transversal intersection points with E_0. For each i, there is only one rigid isotopy class of the pair (U, F) which is determined by the real root scheme of the pair (See Table 3.1 for the extremal one). The real root scheme of the pair (U, F) consists of i disjoint closed arcs on $F_{\mathbb{R}} \cong S^1$ which correspond to spherical components of $E^{(2)}_{\mathbb{R}}$. The real root scheme of (U, F) has a representative (real root marking) with the desired symmetry group (*i.e.*, S_2, S_3 and

D_8 for $i = 2, 3$ and 4 respectively), generated by rotations and reflections of S^1. By Refinement 3.6.2, these symmetries realize permutation of the corresponding ovals. In the case of automorphisms, induced by reflection symmetries, we observe that the fixed point set consists of a pair of generatrices and intersects U; hence, Lemma 5.1.2 applies. For the case $i = 4$, Remark 5.2.3 tells us why other permutations are not realizable.

Case (2): We have two subcases:

$$E_{\mathbb{R}} = \{V_3 \sqcup S\} \sqcup \{V_1 \sqcup iS\}, \quad i = 2, 3.$$

Corresponding surface is a $(2, 2)$-surface. According to the models of $(2, r)$-surfaces and Theorem 3.6.6, $U \in |2e_\infty + 2l|$ is a nonsingular real curve which is tangent to E_0 at one point. For both $i = 2, 3$, there is only one rigid isotopy class of the pair (U, F) which is determined by the real root scheme of the pair (see Table 3.1 for the extremal one). The real root scheme of the pair (U, F) consists of $i + 1$ disjoint closed arcs on $F_{\mathbb{R}} \cong S^1$ one of which has a distinguished point that corresponds to the tangency of U and E_0. The real root scheme of (U, F) has a representative (real root marking) with the desired symmetry group S_2 for both cases generated by reflections of S^1. By Refinement 3.6.2, these symmetries realize permutation of the corresponding ovals. As in the first case, fixed point set of reflections intersects U; hence, Lemma 5.1.2 applies. For the case $i = 3$, Remark 5.2.3 tells us why other permutations are not realizable.

Case (3): We have two subcases:

$$E_{\mathbb{R}} = \{V_3 \sqcup V_1 \sqcup S\} \sqcup \{iS\}, \quad i = 2, 3.$$

Corresponding surface is a $(2, 3)$-surface. According to the models of $(2, r)$-surfaces and Theorem 3.6.6, $U \in |2e_\infty + 2l|$ has an $\mathbf{A}_1$ type singularity at E_0. For both i, there is only one rigid isotopy class of the pair (U, F) which is determined by the real root scheme of the pair (See Table 3.1 for the extremal one). The real root scheme of the pair (U, F) consists of i disjoint closed arcs on $F_{\mathbb{R}} \cong S^1$ and a distinguished point on their complement that corresponds to the singularity of U. The real root scheme of (U, F) has a representative (real root marking) with the desired symmetry group S_2 for both cases generated by reflections of S^1. By Refinement 3.6.2, these symmetries realize permutation of the corresponding ovals. As in the first case, fixed point set of reflections intersects U; hence, Lemma 5.1.2 applies. For the case $i = 3$, Remark 5.2.3 tells us why other permutations are not realizable.

Case (4): $E_{\mathbb{R}} = \{V_3 \sqcup 2S\} \sqcup \{V_1 \sqcup 2S\}$
The permutation of the spherical components of $E_{\mathbb{R}}^{(2)}$ is not realizable in this case as explained in Remark 5.2.3.

Case (5): $E_{\mathbb{R}} = \{V_3 \sqcup V_1 \sqcup 2S\} \sqcup \{2S\}$
Corresponding surface is a $(2,5)$-surface. According to the models of $(2,r)$-surfaces and Theorem 3.6.6, $U \in |2e_\infty + 2l|$ has an $\mathbf{A}_3$ type singularity at E_0. There is only one rigid isotopy class of the pair (U, F) which is determined by the real root scheme of the pair (See Table 3.1). The real root scheme of the pair (U, F) consists of 2 disjoint closed arcs on $F_{\mathbb{R}} \cong S^1$ and a distinguished point on their complement that corresponds to the singularity of U. The real root scheme of (U, F) has a representative (real root marking) with the desired symmetry group S_2 generated by a reflection of S^1. By Refinement 3.6.2, this symmetry realizes permutation of the corresponding ovals. As in the previous cases, fixed point set of the reflection intersects U; hence, Lemma 5.1.2 applies. $\square$

Theorem 5.2.4. *With two exceptions, any permutation of the homeomorphic components of the half $E_{\mathbb{R}}^{(2)}$ of a real Enriques surface with a distinguished half $E_{\mathbb{R}}^{(1)} = S_1$, can be realized by deformations and automorphisms. The exceptional cases are:*

(1) surfaces with $E_{\mathbb{R}} = \{S_1\} \sqcup \{V_2 \sqcup 4S\}$: the realized group is $\mathbb{Z}_2 \times \mathbb{Z}_2$;

(2) surfaces E of type I_u with $E_{\mathbb{R}} = \{S_1\} \sqcup \{4S\}$: the realized group is D_8.

Remark 5.2.4. The exceptional surfaces are M-surfaces. Corresponding quarter decompositions are as follows:

(1) $E_{\mathbb{R}} = \{(S_1) \sqcup (\varnothing)\} \sqcup \{(V_2 \sqcup 2S) \sqcup (2S)\}$;

(2) $E_{\mathbb{R}} = \{(S_1) \sqcup (\varnothing)\} \sqcup \{(2S) \sqcup (2S)\}$;

One cannot permute homeomorphic components without preserving the quarter decomposition. The above theorem states that a permutation of homeomorphic components of $E_{\mathbb{R}}^{(2)}$ can be realized if and only if it preserves the quarter decomposition.

Proof: Following the deformation classification, we construct a particular surface (within each deformation class) and 'auto-deformations' and/or automorphisms on it that realizes the monodromy groups. We proceed case by case:

Case (1): $E_{\mathbb{R}}^{(2)} = 2V_2$. This case will be treated in the next theorem.

Case (2): Derivatives of $E_{\mathbb{R}}^{(2)} = V_2 \sqcup 4S$:

Subcase (1): $E_{\mathbb{R}}^{(2)} = V_2 \sqcup iS$, $i = 2, 3, 4$.
The corresponding DPN-pair $(\widetilde{Y}, \widetilde{B})$ resulting from Donaldson's trick admits a minimal elliptic pencil $f : Y \to \mathbb{P}^1$ as in model (B) of Section 3.5. The curve $C \in |3e_\infty|$ is a real nonsingular curve in $\Sigma_2 \subset \mathbb{P}^3 \times \mathbb{P}^1$ such that $C_{\mathbb{R}}$ consists of i ovals and a component homologous to $(E_0)_{\mathbb{R}}$. Image of $\widetilde{B}$ consists of two conjugate generatrices of Σ_2 with empty real part. Equivalently, the anti-bicanonical system $|-2K_Y|$ defines a degree 2 map $\varphi : Y \to Z$ where Z is the irreducible singular quadric cone in $\mathbb{P}^3$. The branch locus of φ consists of the vertex V of Z and a nonsingular cubic section C' disjoint from V whose real part $C'_{\mathbb{R}}$ consists of i ovals and a component noncontractible in $Z_{\mathbb{R}} \setminus \{V\}$. The real part $Y_{\mathbb{R}}$ is the minimal resolution of the double covering of the domain D consisting of i disks bounded by the ovals of $C'_{\mathbb{R}}$ and of the part of $Z_{\mathbb{R}}$ bounded by the noncontractible component of $C'_{\mathbb{R}}$ and V. Rigid isotopy class of C' is induced by that of C. From Theorem 3.6.3 and Refinement 3.6.1, for each $i = 2, 3$ and 4, there is one rigid isotopy class of C' up to isomorphism.

Clearly, a rigid isotopy of C' in Z defines a deformation of Y, and an auto-involution of Z, preserving C' and having nonempty fixed point set, lifts to an involution on Y. Thus, in view of Refinement 4.6.1 and Lemma 5.1.3, it suffices to realize certain permutations of the ovals of a particular curve (in each rigid isotopy class) C' by rigid isotopies and/or involutive automorphisms of Z (in the latter case taking care that the fixed point set of the involution intersects C'). For each $i = 2, 3$ and 4, let C' and Z be constructed (due to S. Finashin, see [10]) as follows: Let Z be the quadric cone that is the double covering of the plane branched over L_3 and L_4 if $i = 2$, L_1 and L_3 if $i = 3$, and, L_1 and L_2 if $i = 4$ (see Figure 5.1). Let C' be the pull-back of the cubic curve, which is symmetric with respect to the line L, and is obtained by a perturbation of the lines P, Q and R

(dotted lines, see Figure 5.1). For $i = 2$, the symmetry of the cone with respect to the yz-plane permutes the ovals of C'. For $i = 3$, it suffices to permute one pair of ovals, see Refinement 3.6.1, and the symmetry of the cone with respect to the yz-plane does permute the opposite ovals of C'. For $i = 4$, the symmetries of the cone with respect to the yz-plane and xz-plane permutes the opposite ovals of C'. Fixed point set of each symmetry intersects C'. Thus, we obtain the groups S_2, S_3 and $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset S_4$ for $i = 2, 3$ and 4, respectively. For $i = 4$, the fact that other permutations cannot be realized is explained in Remark 5.2.4.

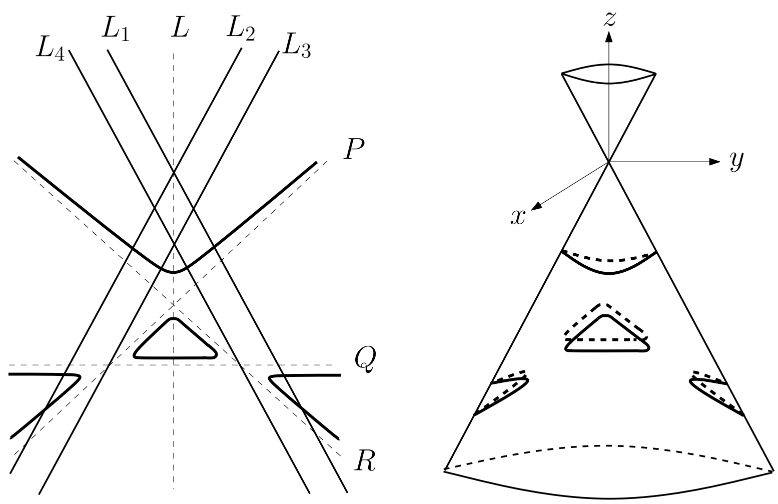


Figure 5.1: Elements of the construction of a quadric cone $Z \subset \mathbb{P}^3$ and a symmetric cubic section $C \subset Z$ (left), and an example of the maximal case (right)

Subcase (2): $E_{\mathbb{R}}^{(2)} = 4S$

There are two deformation classes of these surfaces obtained from model (A) and model (C) of Section 3.5 that differ by the type I_u or I_0 of $E_{\mathbb{R}}$, respectively:

(A): The corresponding DPN-pair $(\widetilde{Y}, \widetilde{B})$ resulting from Donaldson's trick admits a minimal elliptic pencil $f : Y \to \mathbb{P}^1$ where Y is the double covering of Σ_0 branched over a nonsingular real M-curve U of bi-degree $(4, 2)$. The image of $\widetilde{B}$ consists of two conjugate generatrices of bi-degree $(0, 1)$. The real part $U_{\mathbb{R}}$ has four ovals and $Y_{\mathbb{R}}$ covers their interior. From Theorem 3.6.5, U is unique up to rigid isotopy and automorphism of Σ_0. In view of Refinement 4.6.1 and Lemma 5.1.4, it is enough to realize the permutation of the certain ovals by rigid isotopies of U and/or involutive automorphisms of Σ_0 preserving U. Its real root scheme is a disjoint union of 4 segments, and it has a representative (real root marking) with the symmetry group D_8, generated by rotation and reflection symmetries of S^1. By Refinement 3.6.2, these symmetries realize permutation of the corresponding ovals. Furthermore, the fixed point set of a reflection symmetry consists of two distinct points on S^1, which correspond to two distinct real generatrices of Σ_0. Since $U \in |2l_1 + 4l_2|$, it intersects the set of the fixed points of the induced involution and one can apply Lemma 5.1.4. The reason, why other permutations are not allowed is explained in Remark 5.2.4.

(C): The resulting DPN-pair $(\widetilde{Y}, \widetilde{B})$ admits a minimal elliptic pencil $f : Y \to \mathbb{P}^1$ where Y is the minimal resolution of double covering of $\mathbb{P}^2$ branched over a nonsingular real quartic U. The image of $\widetilde{B}$ consists of two conjugate lines passing through a real point O in $P^2_{\mathbb{R}}$ such that none of the ovals of $U_{\mathbb{R}}$ surrounds O. The real part $U_{\mathbb{R}}$ has four unnested ovals and $Y_{\mathbb{R}}$ covers their interior. By Theorem 3.6.2, there is only one rigid isotopy class of U and by Lemma 3.6.1, any permutation of the ovals of U can be realized by a rigid isotopy. From Refinement 4.6.1, we conclude that the latter defines a deformation of $\widetilde{Y}$ and, via inverse Donaldson's trick, a deformation of the corresponding real Enriques surface that realizes the corresponding permutation of the spheres of $E^{(2)}_{\mathbb{R}}$.

Subcase (3): $E^{(2)}_{\mathbb{R}} = iS$, $i = 2, 3$.
The corresponding DPN-pair $(\widetilde{Y}, \widetilde{B})$ resulting from Donaldson's trick admits a minimal elliptic pencil $f : Y \to \mathbb{P}^1$ as in model (C) of Section 3.5. The real part $U_{\mathbb{R}}$ has i unnested ovals and $Y_{\mathbb{R}}$ covers their interior. By Theorem 3.6.2 and Lemma 3.6.1, there is only one rigid isotopy class of U and any permutation of the ovals of U can be realized by a rigid isotopy. Then by Refinement 4.6.1, the latter results in a deformation of the corresponding real Enriques surface that realizes the corresponding permutation of the spherical components of $E^{(2)}_{\mathbb{R}}$.

Case (3): $E^{(2)}_{\mathbb{R}} = S_1$.
The resulting DPN-pair $(\widetilde{Y}, \widetilde{B})$ admits a minimal elliptic pencil $f : Y \to \mathbb{P}^1$ as in model (C). The curve U is the nest with the real scheme $\langle 1\langle 1\rangle\rangle$. Since Z^- have one orientable and one nonorientable components there are two choices of the

position of the point O. These two choices produce two deformation classes with the same real part. The linking coefficient form is identically zero on precisely one of the halves [3]. So we have two deformation classes that differ by the order of the halves. Therefore the permutation of the halves is not realizable.

$\square$

Theorem 5.2.5. *With one exception, any permutation of the homeomorphic components of both halves of a real Enriques surface with a distinguished half* $E_{\mathbb{R}}^{(1)} = 2V_2$*, can be realized by deformations and automorphisms. The exceptional topological type is:*

- ◇ *surfaces with* $E_{\mathbb{R}} = \{2V_2\} \sqcup \{4S\}$.

Remark 5.2.5. In the exceptional case, there are four deformation classes with homeomorphic real parts and on each of them the Pontrjagin-Viro form is well defined. The quarter decompositions and the corresponding monodromy groups are as follows:

(1) $E_{\mathbb{R}} = \{(2V_2^0) \sqcup (\varnothing)\} \sqcup \{(2S) \sqcup (2S)\}$; $\mathbb{Z}_2 \times D_8$;
(2) $E_{\mathbb{R}} = \{(V_2^2 \sqcup V_2^{-2}) \sqcup (\varnothing)\} \sqcup \{(2S) \sqcup (2S)\}$; D_8;
(3) $E_{\mathbb{R}} = \{(V_2^0) \sqcup (V_2^0)\} \sqcup \{(2S) \sqcup (2S)\}$; $\mathbb{Z}_2 \times D_8$;
(4) $E_{\mathbb{R}} = \{(V_2^2) \sqcup (V_2^2)\} \sqcup \{(3S) \sqcup (S)\}$; $\mathbb{Z}_2 \times S_3$.

Upper index of the V_2 components in the first half is a topological invariant, so called *Brown invariant* (details can be found in [2]), depending on the choice of a quarter in the second half. One cannot permute homeomorphic components without preserving the quarter decomposition and the indices. The above theorem states that a permutation of homeomorphic components of both halves of $E_{\mathbb{R}}$ can be realized if and only if it preserves the quarter decomposition together with the indices.

Proof: The corresponding *DPN*-pairs $(\widetilde{Y}, \widetilde{B} \cong 2S_1)$ resulting from Donaldson's trick admit minimal elliptic pencils $f : Y \to \mathbb{P}^1$ of model (A) or (C). Images of genus 1 components of $\widetilde{B}$, F and G, are real fibers of Σ_0 or $\mathbb{P}^2$ for model (A) or (C), respectively. Both fibers are transversal to the nonsingular branch curve U.

Following the deformation classification, we construct a particular surface (within each deformation class) for each auto-deformation and automorphism that realizes the desired permutations of F and G, and the ovals of $U_{\mathbb{R}}$. We proceed case by case:

Case (1): $E_{\mathbb{R}} = \{(2V_2^0) \sqcup (\varnothing)\} \sqcup \{(2S) \sqcup (2S)\}$.

Case (2): $E_{\mathbb{R}} = \{(V_2^2 \sqcup V_2^{-2}) \sqcup (\varnothing)\} \sqcup \{(2S) \sqcup (2S)\}$.

We will treat Cases (1) and (2) together. The minimal pencil is of model (A). The real coordinate system we use here, as well as the set $D_{\mathbb{R}}$ and the polynomial $\Delta(x)$, are introduced in Section 3.6.3. The branch curve $U \in |2l_1 + 4l_2|$ is a real nonsingular curve on Σ_0 which is defined by an equation $a(x)y^2 + b(x)y + c(x) = 0$ that determines its real root marking where $a(x)$, $b(x)$ and $c(x)$ are real polynomials of degree 4. The real part $U_{\mathbb{R}}$ of the branch curve has four ovals. The polynomial $a(x)$ has two pairs of conjugate roots and $D_{\mathbb{R}}$ consists of four components.

Once we fix $a(x)$ and $\Delta(x)$, then $b(x)$ should satisfy the condition: $b^2(x_n) = \Delta(x_n)$ for all roots x_n of $a(x)$, $n = 1, 2, 3$ and 4. At each x_n we have two choices for $b(x_n)$, differing by sign, which are $\pm\sqrt{\Delta(x_n)}$. We have $b(\overline{x}_n) = \overline{b(x_n)}$. If we choose and fix $a(x)$ and $\Delta(x)$, and if $b(x)$ is subject to the above condition, then $c(x)$ is uniquely determined.

For example, take $a(x) = (x^2 + 1)(x^2 + 4)$ and the roots of $\Delta(x)$ to be the set $\{\pm1, \pm2, \pm3, \pm4\}$. Then the roots of $a(x)$ are $\pm i$ and $\pm 2i$, and $\Delta(x) = \pm(x^2 - 1)(x^2 - 4)(x^2 - 9)(x^2 - 16)$. Note that we have two choices for the sign of $\Delta(x)$. If we fix $\Delta(x)$, then up to rigid isotopy we can move $a(x)$ by a path $i \leftrightarrow 2i$. This move defines a bijection between the two choices for $b(i)$ and those for $b(2i)$. Hence, we have two coherent choices (respecting the bijection) and two incoherent ones, the two latter resulting in deformation equivalent curves. Thus, we have three deformation classes given by the following three bijections α, β and γ. The signs on the left-hand side and right-hand side are the signs of $b(i)$ and $b(2i)$, respectively.

$$\begin{aligned}\alpha &: \{+\} \longleftrightarrow \{-\}\\ \beta &: \{+\} \longleftrightarrow \{+\}\\ \gamma &: \{-\} \longleftrightarrow \{-\}\end{aligned}$$

The real root scheme of U of the above example is symmetric with respect to the y-axis, independent of the sign of $\Delta(x)$. The real generatrices $F,\ G \in |l_1|$ can be chosen to be symmetric with respect to the x-axis. Depending on the choice of the sign of $\Delta(x)$, the symmetry with respect to the y-axis realizes either the permutation of the ovals in different quarters (see Figure 5.2) or fixes the ovals of one quarter and permutes the ovals of the other one (see Figure 5.3). The symmetry with respect to the x-axis permutes the generatrices F and G and preserves the real root scheme.

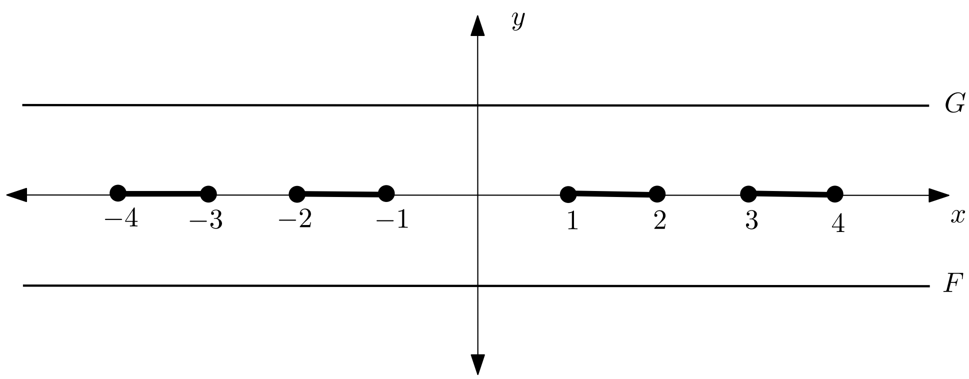


Figure 5.2: The real root scheme of U and the generatrices $F,\ G \in |l_1|$ for the $-$ choice of sign of $\Delta(x)$.

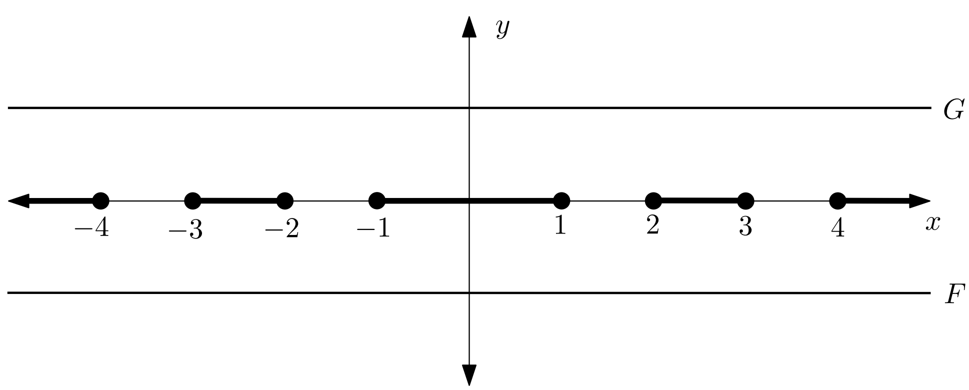


Figure 5.3: The real root scheme of U and the generatrices $F,\ G \in |l_1|$ for the $+$ choice of sign of $\Delta(x)$.

The deformation classes given by the bijections β and γ are equivalent up to automorphisms given by the symmetries with respect to x- and y-axes. Thus, up to deformation and automorphisms of Σ_0, there are 2 deformation classes which we denote by $[\alpha]$ and $[\beta]$. If we choose the $+$ sign for $\Delta(x)$, then $b(x_n) = \pm\sqrt{\Delta(x_n)}$ will be a real number so that it will be invariant under conjugation. So applying the symmetry with respect to the y-axis does not change the deformation classes $[\alpha]$ and $[\beta]$. If we choose the $-$ sign for $\Delta(x)$, then $b(x_n) = \pm\sqrt{\Delta(x_n)}$ will be purely imaginary so that $\overline{b(x_n)} = -b(x_n)$. So applying the symmetry with respect to the y-axis does not change the deformation class $[\alpha]$, whereas for the deformation class $[\beta]$ it should be composed with a symmetry with respect to x-axis in order to stay in the same deformation class. Thus, for the class $[\alpha]$ the symmetries with respect to the y-axis for both choices of sign of $\Delta(x)$ realizes the group D_8 of permutation of ovals of $U_{\mathbb{R}}$, and the symmetry with respect to x-axis realizes the group $\mathbb{Z}_2$ of the permutation of the generatrices F and G. Similarly, for the class $[\beta]$ the composition of the symmetries with respect to the y-axis and the x-axis for the $-$ choice of sign of $\Delta(x)$, and the symmetry with respect to the y-axis for the $+$ choice of sign of $\Delta(x)$ realizes the group D_8 of permutation of ovals of $U_{\mathbb{R}}$. The result follows from Refinement 4.7.1 and Lemma 5.1.4.

Case (3): $E_{\mathbb{R}} = \{(V_2^0) \sqcup (V_2^0)\} \sqcup \{(2S) \sqcup (2S)\}$.

Admitted model is (C). The branch curve U is of type I. The real part $U_{\mathbb{R}}$ has four ovals. Two of them are marked with a $+$ and the others with a $-$ in the complex scheme. From the existence of symmetric quartics or concretely by the perturbation $(2x^2 + y^2 - 1)(x^2 + 2y^2 - 1) + \epsilon = 0$, for $\epsilon > 0$ small enough, one can obtain a symmetric quartic U as in Figure 5.4. The opposite ovals of $U_{\mathbb{R}}$ are marked with the same sign. Composing the symmetries with respect to the lines L_1 and L_2 (for each symmetry we make choice of the symmetry axis) with the rigid isotopies of the pair (F, G) realize every permutation of the ovals of $U_{\mathbb{R}}$ that respect the marking and the permutation of F and G. The resulting group is $\mathbb{Z}_2 \times D_8$. Each symmetry has fixed points on U. The result follows from Refinement 4.7.1 and Lemma 5.1.5.

Case (4): $E_{\mathbb{R}} = \{(V_2^2) \sqcup (V_2^2)\} \sqcup \{(3S) \sqcup (S)\}$.

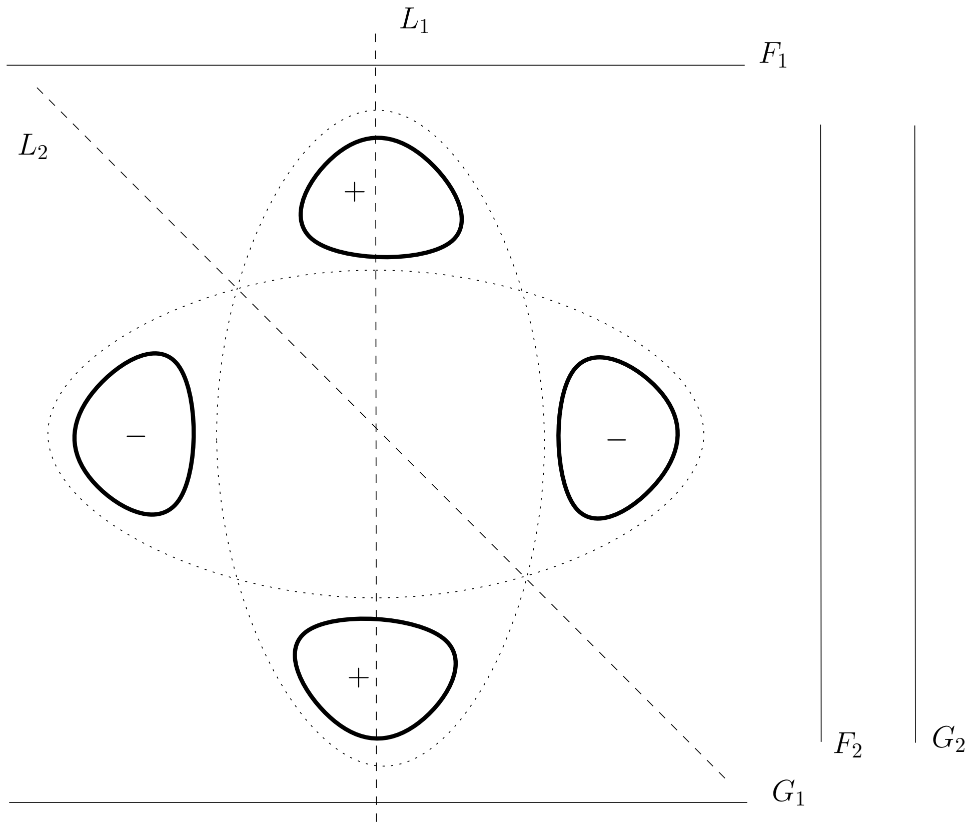


Figure 5.4: The quartic U is obtained by a perturbation of two dotted ellipses. Opposite ovals are marked with the same sign. U is symmetric with respect to the lines L_1 and L_2. Up to rigid isotopy, the pair $(F_{\mathbb{R}}, G_{\mathbb{R}})$ can be chosen as (F_1, G_1) or (F_2, G_2).

Resulting model is (C). The branch curve U is of type I. The real part $U_{\mathbb{R}}$ has four ovals. Three of them are marked with a $+$ and the other one with a $-$ in the complex scheme. From the existence of symmetric quartics or concretely by the perturbation of 4 double points in a symmetric non-convex position, one can obtain a symmetric quartic U as in Figure 5.5. The oval in the center is the one marked with a $-$ sign. Composing the symmetries with respect to the lines L_1 and L_2 (for each symmetry we make choice of the symmetry axis) with the rigid isotopies of the pair (F, G) realize every permutation of the ovals of $U_{\mathbb{R}}$ that respect the marking and the permutation of F and G. The resulting group is $\mathbb{Z}_2 \times S_3$. Each symmetry has fixed points on $U_{\mathbb{R}}$. The result follows from Refinement 4.7.1 and Lemma 5.1.5.

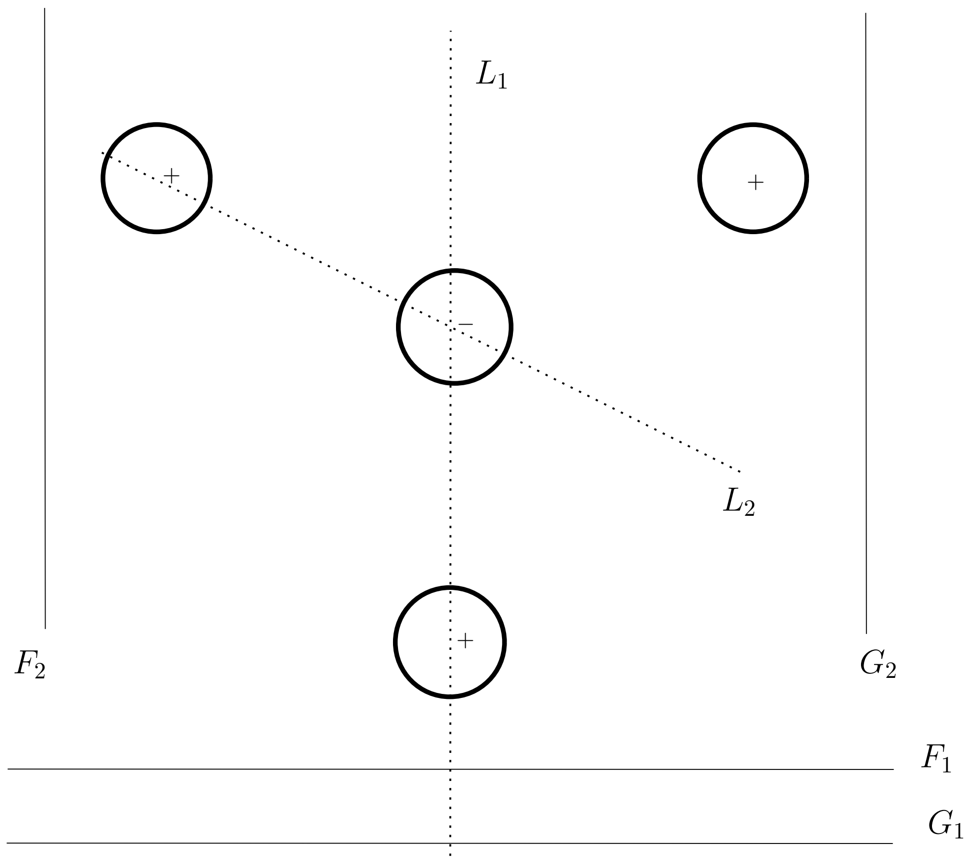


Figure 5.5: The quartic U is obtained by a perturbation of four double points. The oval in the center is marked with a $-$ sign, others are marked with a $+$ sign. U is symmetric with respect to the lines L_1 and L_2. Up to rigid isotopy, the pair $(F_{\mathbb{R}}, G_{\mathbb{R}})$ can be chosen as (F_1, G_1) or (F_2, G_2).

Case (5): $E_{\mathbb{R}} = \{2V_2\} \sqcup \{iS\}$, $i = 0, 1, 2,$ and 3.

The reduced model is C. The branch curve U is not of type I and is transversal to G so the ramified complex scheme of the pair (U, G) is determined by the real scheme of U which is $\langle i \rangle$, $i = 0, 1, 2,$ and 3. The real part $U_{\mathbb{R}}$ consists of i ovals. From the existence of symmetric quartics one can take a symmetric representative from the rigid isotopy class for each i (for $i = 3$ see Figure 5.6). Symmetries of U with respect to appropriate symmetry axes composed with the rigid isotopies of the pair (F, G) realize every permutation of the ovals of $U_{\mathbb{R}}$ and the permutation of F and G. Each symmetry has fixed points on U. Hence the result follows from Refinement 4.7.1 and Lemma 5.1.5.

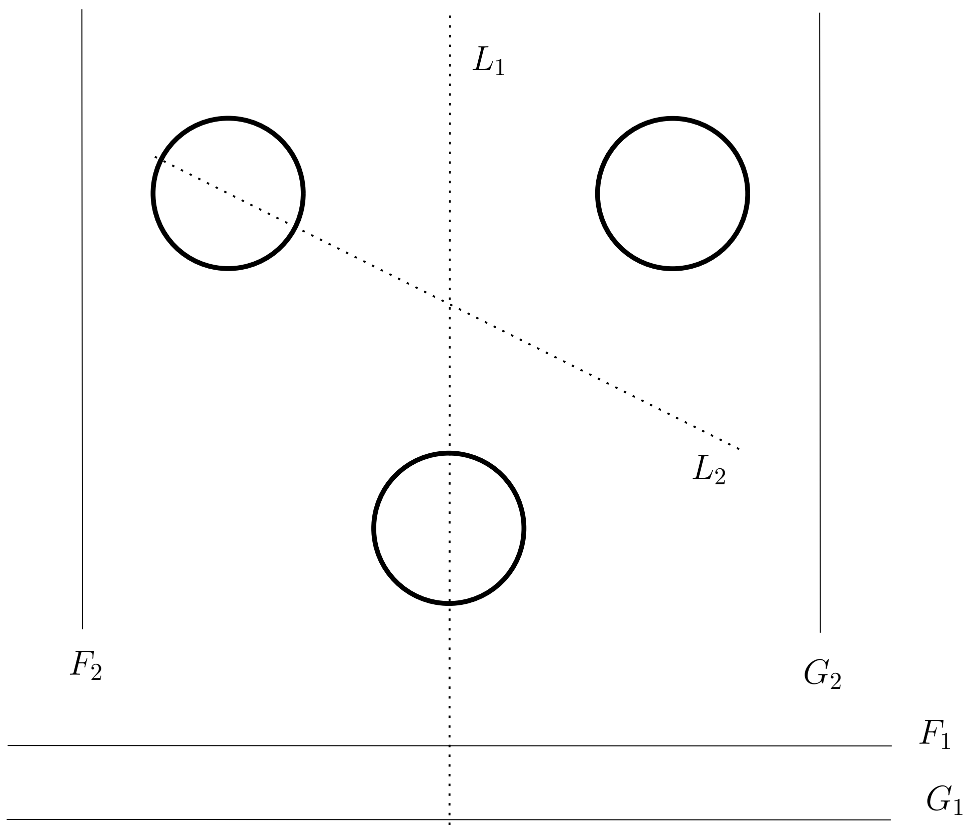


Figure 5.6: The quartic U is obtained by a perturbation of four double points. It is symmetric with respect to the lines L_1 and L_2. Up to rigid isotopy, the pair $(F_{\mathbb{R}}, G_{\mathbb{R}})$ can be chosen as (F_1, G_1) or (F_2, G_2).

Case (6): $E_{\mathbb{R}} = \{2V_2\} \sqcup \{S_1\}$.

The resulting model is (C). The branch curve U is the nested quartic with real scheme $\langle 1\langle 1\rangle\rangle$. It is of type I and both ovals of $U_{\mathbb{R}}$ are marked with $+$ in the complex scheme. From the existence of symmetric quartics or concretely by the perturbation $(2x^2 + y^2 - 1)(x^2 + 2y^2 - 1) = \epsilon$, for $\epsilon > 0$ small enough, one can obtain a symmetric quartic U as in Figure 5.7. Taking the symmetry with respect to the line L realizes the permutation of F and G. This automorphism has fixed points on $U_{\mathbb{R}}$. The result follows from Refinement 4.7.1 and Lemma 5.1.5.

$\square$

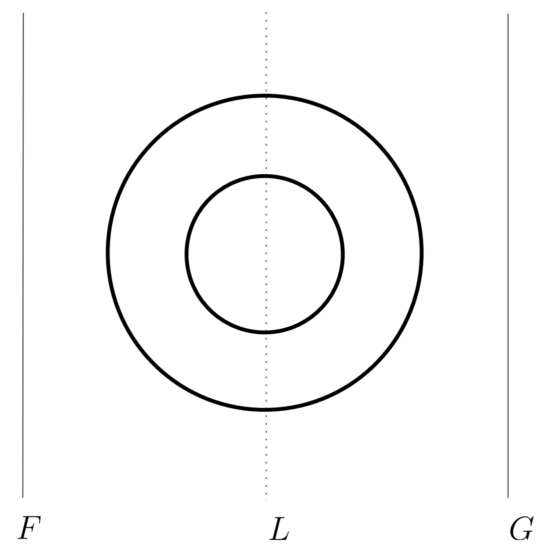


Figure 5.7: The quartic U is the nest that is obtained by the opposite perturbation of the two dotted ellipses in Figure 5.4. It is symmetric with respect to L.

Bibliography

[1] W. Barth, C. Peters and A. Van de Ven., "Compact complex surfaces", *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, Springer-Verlag, Berlin, 1984.

[2] A. Degtyarev, "On the Pontrjagin-Viro form", *Topology, ergodic theory, real algebraic geometry* , *Amer. Math. Soc. Transl. Ser.*, vol. 2, no. 202, pp. 71-94, 2001.

[3] A. Degtyarev, I. Itenberg and V. Kharlamov, "Real Enriques Surfaces",*Lecture Notes in Math.*, vol. 1746, Springer-Verlag, 2000.

[4] A. Degtyarev, I. Itenberg and V. Kharlamov, "On deformation types of real elliptic surfaces", *Amer. J. Math.*, vol. 130, no. 6, pp. 1561-1627, 2008.

[5] A. Degtyarev and V. Kharlamov, "Topological classification of real Enriques surfaces", *Topology*, vol. 35, no. 3, pp. 711-729, 1996.

[6] A. Degtyarev and V. Kharlamov, "Halves of a real Enriques surface", *Comment. Math. Helv.*, vol. 71, no. 4, pp. 628-663, 1996.

[7] A. Degtyarev and V. Kharlamov, "On the moduli space of real Enriques surfaces, *C. R. Acad. Sci. Paris Sr. I Math.*, vol. 324, no. 3, pp. 317-322, 1997.

[8] S. Erdoğan, "On the monodromy groups of real Enriques surfaces", *M.Sc. Thesis*, Bilkent University, 2003.

[9] S. Erdoğan Demir, "Monodromy groups of real Enriques surfaces", *Topology and its Applications*, vol. 159, no.10-11, pp. 2580-2591, 2012.

[10] S. Finashin, "Decomposability of quotients by complex conjugation for rational and Enriques surfaces", *Topology Appl.*, vol. 79, no. 2 , 121-28, 1997.

[11] S. K. Donaldson, "Yang-Mills invariants of four-manifolds", *Geometry of low-dimensional manifolds*, vol. 1, pp. 5-40, London Math. Soc. Lecture Note Ser., No. 150, Cambridge Univ. Press, Cambridge, 1990.

[12] N. Hitchin, "Compact four-dimensional Einstein manifolds", *J. Diff. Geom.*, vol. 9, pp. 435-441, 1974.

[13] I. V. Itenberg, "Groups of monodromy of non-singular curves of degree 6",*Real analytic and algebraic geometry (Trento, 1992)*, pp. 161-168, de Gruyter, Berlin, 1995.

[14] V. Kharlamov, "Topological types of real nonsingular surfaces of degree 4 in $\mathbb{RP}^3$ ", *Funktsional Anal. i Prilozhen*, vol. 10, no. 4, pp. 55-68, 1976 (Russian); English transl. in *Functional Anal. Appl.*, vol. 10, no. 4, pp. 295-305, 1976.

[15] V. Kharlamov, "Isotopic types of nonsingular surfaces of degree 4 in $\mathbb{RP}^3$", *Funktsional. Anal. i Prilozhen.*, vol. 12, no. 1, pp. 86-87, 1978; (Russian),*English transl. in Functional Anal. Appl.*, vol. 12, no. 1, 1978.

[16] V. Kharlamov, "On the classification of nonsingular surfaces of degree 4 in $\mathbb{RP}^3$ with respect to rigid isotopies", *Funktsional. Anal. i Prilozhen.*, vol. 18, no. 1, pp. 49-56, 1984, (Russian); *English transl. in Functional Anal. Appl.*, vol. 18, no. 1, pp. 39-45, 1984.

[17] F. Klein, " Über den Verlauf der Abel'schen Integrale bei den Curven vierten Grades", *Math. Ann.*, vol. 10, pp. 364-397, 1876.

[18] S. Moriceau, "Surfaces of degree 4 with a nondegenerate double point in $\mathbb{RP}^3$", *PhD thesis*, University of Rennes 1, 2004.

[19] V. V. Nikulin, "On the topological classification of real Enriques surfaces I", *Topology of real algebraic varieties and related topics*, pp. 187-201, Amer. Math. Soc. Transl. Ser. 2, vol. 173, Amer. Math. Soc., Providence, RI, 1996.