101- 101E-103-103E

MATHEMATICS FINAL EXAM — KEY 14 August 2013

QUESTION 1 The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	Score
Signature:	Electronic Post(e-mail) address:	Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

Evaluate the following integrals

[12 pt] a)

$$\int \sqrt{1+e^x}dx$$

[13 pt] b)

$$\int \frac{\sin x+\cos x}{\tan x+\cot x}dx$$

a) $\displaystyle\int \sqrt{1+e^x}\,dx = \left\{ \begin{array}{l} 1+e^x = u^2 \\ e^x dx = 2u\,du \end{array} \right\} = \int \sqrt{u^2}.\ \frac{2u\,du}{u^2-1}$

$\displaystyle = 2\int \frac{u^2\,du}{u^2-1} = 2\int \frac{u^2-1+1}{u^2-1}\,du = 2\left\{ \int\left(1+\frac{1}{u^2-1}\right)du \right\}$

$\displaystyle \frac{1}{(u+1)(u-1)} = \frac{A}{u-1} + \frac{B}{u+1}$ $\quad A=\frac{1}{2},\ B=-\frac{1}{2}$

$\displaystyle = 2\left\{ \int\left(1+\frac{1/2}{u-1} - \frac{1/2}{u+1}\right)du \right.$

$\displaystyle = 2\left\{ u + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| \right\} + C$

$\displaystyle = 2u + \ln\left|\frac{u-1}{u+1}\right| + C = 2\sqrt{1+e^x} + \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C$

b) $\displaystyle\int \frac{\sin x+\cos x}{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}}\,dx = \int \frac{\sin x+\cos x}{\frac{1}{\cos x\sin x}}\,dx$

$\displaystyle = \int (\sin^2 x\cos x + \cos^2 x\sin x)\,dx = \int \underbrace{(\sin x)}_{u}{}^2\,\underbrace{\cos x\,dx}_{du} + \int \underbrace{(\cos x)}_{v}{}^2\,\underbrace{\sin x\,dx}_{-dv}$

$\displaystyle = \frac{u^3}{3} - \frac{v^3}{3} + C = \frac{1}{3}(\sin x)^3 - \frac{1}{3}(\cos x)^3 + C$

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QUESTION 2 The blanks below will be filled by students. (Except the score)

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[12 pt] a) Suppose that y is a differentiable function of x satisfying the equation

$$\int_0^{x+y} e^{-t^2}\,dt = xy.$$

Find $\left.\frac{dy}{dx}\right|_{(x,y)=(0,0)}$.

[13 pt] b) The region bounded by the curve $y = \ln x$, the lines $x = 4$ and $y = 1$ is revolved about the y-axis to genarate a solid. Find the volume of this solid.

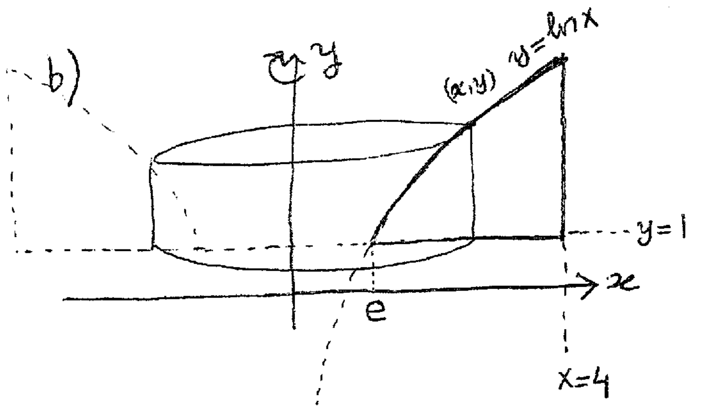
a) Differentiation wrt x gives

$$e^{-(x+y)^2} \cdot \frac{d}{dx}(x+y) = 1 \cdot y + x \cdot \frac{dy}{dx}$$

$$e^{-(x+y)^2} \cdot \left(1 + \frac{dy}{dx}\right) = y + x\frac{dy}{dx}$$

$x=y=0$: $\quad e^{-(0+0)^2} \cdot \left(1 + \left.\frac{dy}{dx}\right|_{(0,0)}\right) = 0 + 0 \cdot \left.\frac{dy}{dx}\right|_{(0,0)} \Rightarrow \left.\frac{dy}{dx}\right|_{(0,0)} = -1$ //

b)



Using Shell Method

$$V = \int_a^b 2\pi \left(\text{Shell Radius}\right)\left(\text{Shell Height}\right) dx$$

$$= \int_e^4 2\pi \cdot x \cdot (\ln x - 1)\, dx$$

$u = \ln x - 1 \Rightarrow du = \frac{dx}{x}$

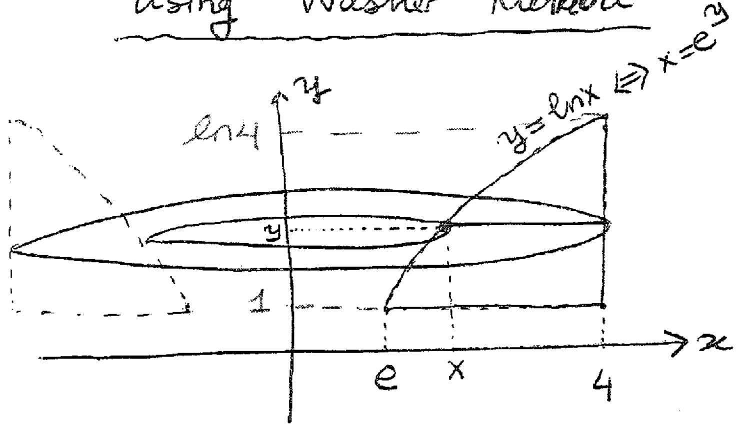
$dv = x\,dx \Rightarrow v = \frac{x^2}{2}$

$$= 2\pi\left\{ (\ln x - 1) \cdot \frac{x^2}{2}\Big|_e^4 - \int_e^4 \frac{x^2}{2} \cdot \frac{dx}{x} \right\}$$

$$= 2\pi\left\{ (\ln 4 - 1) \cdot \frac{4^2}{2} - 0 - \frac{1}{2} \cdot \frac{x^2}{2}\Big|_e^4 \right\}$$

$$= 2\pi\left\{ 8\ln 4 + \frac{e^2}{4} - 12 \right\}$$

Using Washer Method



$R(y) = 4$
$r(y) = e^{y}$

$$V = \int_a^b \pi \left[\Big(\underbrace{\text{outer radius}}_{R(y)} \Big)^2 - \Big(\underbrace{\text{inner radius}}_{r(y)} \Big)^2 \right] dy$$

$$= \int_1^{\ln 4} \pi \left[(4)^2 - (e^{y})^2 \right] dy$$

$$= \pi \int_1^{\ln 4} \left(16 - e^{2y}\right) dy = \pi \left(16y - \frac{1}{2} e^{2y}\right)\Big|_1^{\ln 4}$$

$$= \pi \left(16 \ln 4 - \frac{1}{2} e^{2\ln 4} - 16 + \frac{1}{2} e^2\right)$$

$$= \pi \left(16 \ln 4 - 8 - 16 + \frac{1}{2} e^2\right)$$

$$= 2\pi \left(8 \ln 4 - 12 + \frac{e^2}{4}\right)$$

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QUESTION 3 The blanks below will be filled by students. (Except the score)

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[13 pt] a) Evaluate the following improper integral

$$\int_0^\infty \frac{dx}{4x^2+9}$$

[12 pt] b)Evaluate the following limit:

$$\lim_{x\to 0^+} \frac{\sin x - \cos\sqrt{x} + 1}{x^2}$$

a) $$I=\int_0^\infty \frac{dx}{4x^2+9} = \lim_{b\to\infty}\int_0^b \frac{dx}{4x^2+9}$$

$$\int \frac{dx}{4x^2+9} = \int\left\{\begin{array}{l} x=\frac{3}{2}\tan\theta \\ dx=\frac{3}{2}\sec^2\theta\, d\theta \end{array}\right\} = \int \frac{\frac{3}{2}\sec^2\theta\, d\theta}{4\cdot\frac{9}{4}\tan^2\theta+9}$$

$$= \int \frac{\frac{3}{2}\sec^2\theta\, d\theta}{9(1+\tan^2\theta)} = \frac{1}{6}\int d\theta = \frac{1}{6}\theta + c = \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)$$

$$I = \lim_{b\to\infty} \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)\Big|_0^b = \frac{1}{6}\left\{\lim_{b\to\infty}\tan^{-1}\left(\frac{2b}{3}\right) - \tan^{-1}(0)\right\}$$

$$= \frac{1}{6}\cdot\left(\frac{\pi}{2}-0\right) = \frac{\pi}{12}$$

b) $$\lim_{x\to 0^+}\frac{\sin x - \cos\sqrt{x}+1}{x^2} \overset{0/0}{=} \lim_{x\to 0^+}\frac{\cos x + \frac{1}{2\sqrt{x}}\sin\sqrt{x}}{2x}$$

$$= \lim_{x\to 0^+}\frac{2\sqrt{x}\cos x + \sin\sqrt{x}}{4x^{3/2}} \overset{0/0}{=} \lim_{x\to 0^+}\frac{2\cdot\frac{1}{2\sqrt{x}}\cdot\cos x + 2\sqrt{x}(-\sin x) + \frac{1}{2\sqrt{x}}\cos\sqrt{x}}{4\cdot\frac{3}{2}\cdot x^{1/2}}$$

$$= \lim_{x\to 0^+}\frac{2\cos x - 4x\sin x + \cos\sqrt{x}}{12x} = \frac{2-0+1}{0^+} = +\infty$$

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QUESTION 4 The blanks below will be filled by students. (Except the score)

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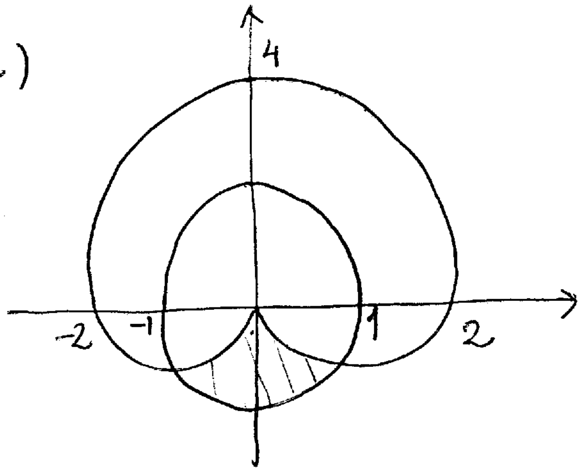
[15 pt] a) Find the area of the region which lies outside the curve $r = 2 + 2\sin\theta$ and inside the curve $r = 1$.

[10 pt] b) Find the length of the curve $r = \theta^2, \quad 0 \le \theta \le \sqrt{5}$.

(a)



$r = 2+2\sin\theta = 1 \to \sin\theta = -\frac{1}{2}$

$\Rightarrow \theta = \frac{7\pi}{6}, \quad \theta = \frac{11\pi}{6}$

$$A = \int_\alpha^\beta \frac{1}{2}\left[r_2^2 - r_1^2 \right] d\theta$$

$$= 2\cdot\int_{7\pi/6}^{3\pi/2} \frac{1}{2}\left[1^2 - (2+2\sin\theta)^2 \right] d\theta$$

$$= \int_{\frac{7\pi}{6}}^{3\pi/2} \left[1-(4+8\sin\theta+4\sin^2\theta\right] d\theta = \int_{\frac{7\pi}{6}}^{3\pi/2} \left[-3-8\sin\theta-4\cdot\frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= (-3\theta + 8\cos\theta - 2\theta + \sin 2\theta)\Big|_{7\pi/6}^{3\pi/2}$$

$$= -5\left(\frac{3\pi}{2}-\frac{7\pi}{6}\right) + 8\left(\cos\frac{3\pi}{2} - \cos\frac{7\pi}{6}\right) + \left(\sin 3\pi - \sin\frac{7\pi}{3}\right)$$

$$= -\frac{5\pi}{3} + 4\sqrt{3} - \frac{\sqrt{3}}{2} = \frac{7\sqrt{3}}{2} - \frac{5\pi}{3}$$

(b) $$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\sqrt{5}} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta.$$

$$= \int_0^{\sqrt{5}} \theta\sqrt{\theta^2+4}\, d\theta = \left\{ \begin{array}{l} u = \theta^2+4 \\ du = 2\theta d\theta \end{array} \right\} = \int_4^9 \sqrt{u}\cdot\frac{du}{2} = \frac{1}{2}\frac{u^{3/2}}{3/2}\Big|_{4=2^2}^{9=3^2}$$

$$= \frac{1}{3}\left(3^3 - 2^3\right) = 19/3 \quad //$$