MATHEMATICS 101E | FINAL | 7 August 2012

QUESTION 1 The blanks below will be filled by students. (Except the score)

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For the solution of this question please use only the front face and if necessary the back face of this page.

Calculate the following integrals.

[7 pts] i) $\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2} dx$ [8 pts] ii) $\int \frac{\sqrt{x^2 - 25}}{x} dx$ [10 pts] iii) $\int x \arctan x \, dx$

i) $\frac{5x^3-3x^2+2x-1}{x^2(1+x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{1+x^2}$

$B=-1$, $A=2$, $C=3$ (0.5), $D=-2$ (0.5)

$\int \frac{5x^3-3x^2+2x-1}{x^4+x^2} dx = \int \frac{2}{x}dx - \int \frac{1}{x^2}dx + \int \frac{3x}{1+x^2}dx - \int \frac{2dx}{1+x^2}$

$= 2\ln|x|$ (1) $+ \frac{1}{x}$ (1) $+ \frac{3}{2}\ln(1+x^2)$ (1) $- 2\tan^{-1}x$ (1) $+ C$ (1)

ii) $\int \frac{\sqrt{x^2-25}}{x} dx$ $\left[\begin{array}{l} x = 5\sec t \\ dx = 5\sec t \tan t \, dt \end{array} \right.$ (2)

$= \int \frac{\sqrt{25(\sec^2 t - 1)}}{5\sec t} 5\sec t \tan t \, dt = 5\int \frac{\tan^2 t}{\sec^2 t - 1} dt$ (2) $= 5\int \sec^2 t \, dt - \int 5 dt$

$= 5\tan t$ (1) $- 5t$ (1) $+ C = 5\sqrt{\sec^2 t - 1} - 5t + C$

$= 5\sqrt{\left(\frac{x}{5}\right)^2 - 1} - 5\sec^{-1}\frac{x}{5} + C$ (2)

iii) $\int x \arctan x \, dx$ $\left[\begin{array}{l} \arctan x = u \Rightarrow \frac{1}{1+x^2}dx = du \\ x dx = dv \Rightarrow v = \frac{x^2}{2} \end{array} \right]$ (3)

$= uv - \int v \, du$

$= \frac{x^2}{2}\tan^{-1}x - $ (2) $\int \frac{x^2}{2} \cdot \frac{dx}{1+x^2} = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$ (2)

$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2}$ (1) $+ \frac{\tan^{-1}x}{2}$ (1) $+ C$

$= -\frac{x}{2} + \left(1 + \frac{x^2}{2}\right)\tan^{-1}x + C$ (1)

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QUESTION 2 — The blanks below will be filled by students. (Except the score)

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[15 pts] a) Investigate the convergence of the following integrals

$$i)\ \int_1^{\infty} \sin\left(\frac{1}{x}\right)dx \qquad ii)\ \int_0^{\frac{\pi}{2}} \frac{\tan\theta}{\sqrt{\sec\theta}}\, d\theta$$

[10 pts] a) Evaluate the limit

$$\lim_{x\to 0} \frac{\sin^{-1}x + \int_0^x \ln(t^2+1)dt}{e^x - 1}.$$

a) Let $g(x) = \frac{1}{x}$ (1)

By limit comparison test, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = 1$ (3) $(\neq 0, \pm\infty)$ then $\int_1^\infty f(x)dx$ and $\int_1^\infty g(x)dx$ both converges or diverges. (2) Since $\int_1^\infty \frac{1}{x^{p=1}}dx$ is divergent, $\int_1^\infty f(x)dx = \int_1^\infty \sin\frac{1}{x}dx$ **diverges** (2)

b) $\int_0^{\pi/2} \frac{\tan\theta}{\sqrt{\sec\theta}} d\theta = \lim_{b\to\frac{\pi}{2}^-} \int_0^b \frac{\sin\theta}{\sqrt{\cos\theta}} d\theta$ (2) $\left[u = \cos\theta \Rightarrow du = -\sin\theta d\theta;\ \theta = 0 \Rightarrow u = 1;\ \theta = b \Rightarrow u = \cos b \right]$ (2)

$= \lim_{b\to\frac{\pi}{2}^-} \int_{u=1}^{\cos b} \frac{-du}{\sqrt{u}} = \lim_{b\to\frac{\pi}{2}^-} -2\sqrt{u}\Big|_{u=1}^{\cos b}$ (1) $= \lim_{b\to\frac{\pi}{2}^-} -2\left[\sqrt{\cos b} - \sqrt{1}\right] = 2$ (1)

Then the given improper integral of the second kind is **Convergent** (1)

c) $\lim_{x\to 0} \frac{\sin^{-1}x + \int_0^x \ln(t^2+1)dt}{e^x - 1} = \frac{0}{0}$ (L'hop) (1)

$= \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}} + \ln(x^2+1)}{e^x}$ (2) (2) (2) $= \frac{1+\ln 1}{e^0} = \frac{1}{1} = \boxed{1}$ (3)

QUESTION 3 The blanks below will be filled by students. (Except the score)

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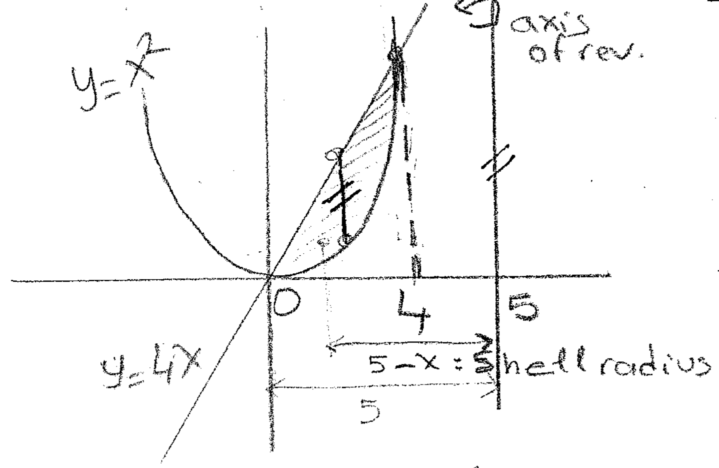
a) The region bounded by $\mathbf{y = x^2}$ and $\mathbf{y = 4x}$ is revolved about the line $\mathbf{x = 5}$. Write down the definite integral that calculates the volume of the region

[5 pts] i) by using Shell method
[5 pts] ii) by using Washer method
[5 pts] iii) Calculate the volume by evaluating one of the integrals above.

[10pts] b) Find the area of the region bounded by $\mathbf{y = \frac{1}{2}x}$ and $\mathbf{y^2 = 8 - x}$.

a)



$y=x^2$, $y=4x$: $x^2=4x$

$x(x-4)=0$

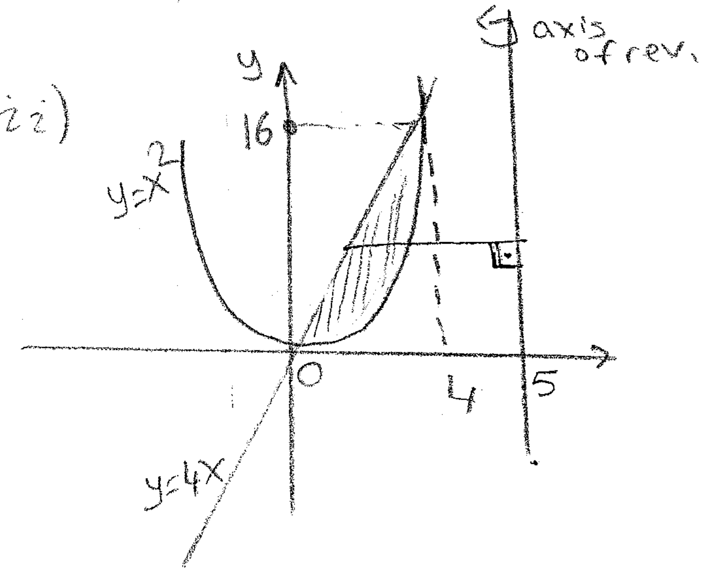
$x=0$, $x=4$: intersection points.

By cylindrical shells method,

i) $V_{(x=5)} = 2\pi \int (\text{shell radius} \cdot \text{shell height} \cdot \text{shell thickness})$

$= 2\pi \int_{x=0}^{4} (5-x) \cdot (4x - x^2)\, dx$ ① ② ①

ii)



By washer method, we have

$R(y) = 5 - \frac{y}{4}$; outer radius ①

$r(y) = 5 - \sqrt{y}$; inner radius ①

$V_{(x=5)} = \pi \int_y [R^2(y) - r^2(y)]\, dy$

$V_{(x=5)} = \pi \int_{y=0}^{16} \left[\left(5 - \frac{y}{4}\right)^2 - (5 - \sqrt{y})^2\right] dy$ ② ①

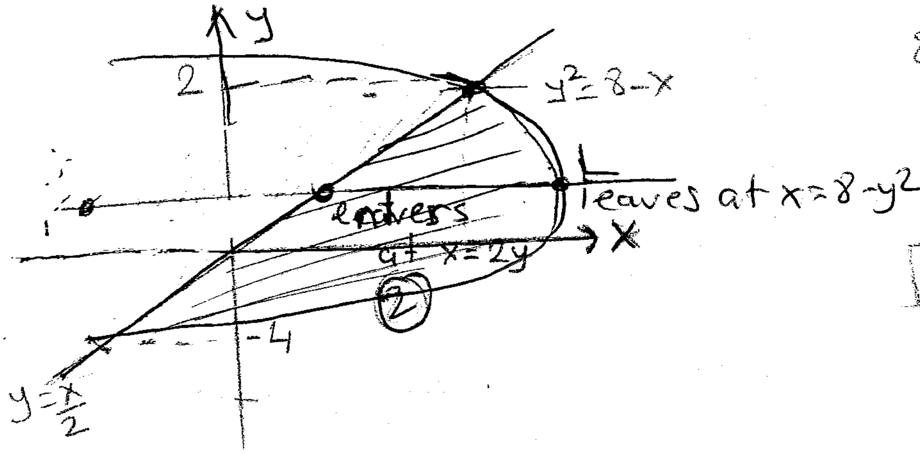
iii) By cylindrical shells,

$V_{(x=5)} = 2\pi \int_{x=0}^{4} (x^3 - 9x^2 + 20x)\, dx = 2\pi \left[\frac{x^4}{4} - 3x^3 + 10x^2\right]_0^4 = \boxed{64\pi}$ ③ ②

OR, By washer method,

$V_{(x=5)} = \pi \int_{y=0}^{16} \left(\frac{y^2}{16} - \frac{7}{2}y + 10\sqrt{y}\right) dy = \pi \left[\frac{y^3}{48} - \frac{7y^2}{4} + \frac{20}{3}y^{3/2}\right]_0^{16} = \boxed{64\pi}\, u^3$ ③ ②

b)



$8 - y^2 = 2y$

$y^2 + 2y - 8 = 0$

$y \quad +4$
$y \quad -2$

$y_1 = 2$, $y_2 = -4$ (2)

$$A = \int_y \left[f(y) - g(y) \right] dy$$

$$= \int_{-4}^{2} \left(-2y + (8 - y^2) \right) dy = -y^2 + 8y - \frac{y^3}{3} \Big|_{-4}^{2}$$

(2) (2) (1)

$$= -4 + 16 - \frac{8}{3} + \left(16 + 32 - \frac{64}{3} \right) = +60 - \frac{72}{3} = 36$$

(1)

QUESTION 4 The blanks below will be filled by students. (Except the score)

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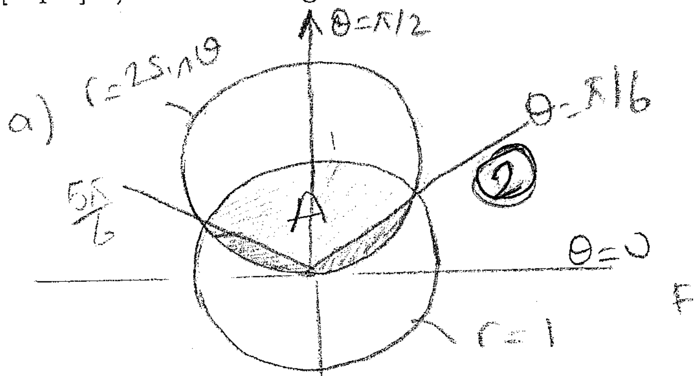
[15pts] a) Find the area of the region shared by the curves $\mathbf{r} = \mathbf{2}\sin\theta$ and $\mathbf{r} = \mathbf{1}$.

[10pts] b) Find the length of the curve whose equation is given by $\mathbf{r} = \sqrt{\mathbf{1} + \cos(\mathbf{2}\theta)}$ for $-\pi/2 \le \theta \le \pi/2$

a)



$r = 2\sin\theta$, $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $x^2+y^2=r^2$

$r^2 = 2y$

$x^2+y^2-2y=0$ or $\boxed{(x-0)^2+(y-1)^2=1}$

From $2\sin\theta = 1$, the intersection rays $\boxed{\theta = \frac{\pi}{6}}$ and (3) $\boxed{\theta = \pi - \frac{\pi}{6}}$

$A_1 = \frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/6} (2\sin\theta)^2 d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} 1^2 d\theta$ (2) (1) (2) (1)

$= \frac{1}{2}\int_0^{\pi/6} 4\left(\frac{1-\cos 2\theta}{2}\right) d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} d\theta$

$= \theta - \frac{\sin 2\theta}{2}\Big|_0^{\pi/6} + \frac{\theta}{2}\Big|_{\pi/6}^{\pi/2} = \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + \left(\frac{\pi}{4} - \frac{\pi}{12}\right) = \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$ (2) (1) (3)

$A_T = 2A_1 = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ units squared (1)

b) $l = \int \sqrt{r^2 + r'^2}\, d\theta = \int_{-\pi/2}^{\pi/2} \sqrt{1+\cos 2\theta + \left(\frac{-\sin 2\theta}{\sqrt{1+\cos 2\theta}}\right)^2}\, d\theta$ (2) (2)

$= \int_{-\pi/2}^{\pi/2} \sqrt{\frac{(1+\cos 2\theta)^2 + \sin^2 2\theta}{1+\cos 2\theta}}\, d\theta = \int_{-\pi/2}^{\pi/2} \sqrt{\frac{1+2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}{1+\cos 2\theta}}$

$= \int_{-\pi/2}^{\pi/2} \sqrt{\frac{2(1+\cos 2\theta)}{1+\cos 2\theta}}\, d\theta = \sqrt{2}\,\theta\Big|_{-\pi/2}^{\pi/2} = \sqrt{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$ (5)

$= \sqrt{2}\,\pi$ units (1)